THE CAPELLI EIGENVALUE PROBLEM FOR LIE SUPERALGEBRAS

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ABSTRACT. For a finite dimensional unital complex simple Jordan superalgebra J, the Tits-Kantor-Koecher construction yields a 3-graded Lie superalgebra $\mathfrak{g}^\flat \cong \mathfrak{g}^\flat(-1) \oplus \mathfrak{g}^\flat(0) \oplus \mathfrak{g}^\flat(1)$, such that $\mathfrak{g}^\flat(-1) \cong J$. Set $V := \mathfrak{g}^\flat(-1)^*$ and $\mathfrak{g} := \mathfrak{g}^\flat(0)$. In most cases, the space $\mathscr{P}(V)$ of superpolynomials on V is a completely reducible and multiplicity-free representation of $\mathfrak{g}$, and there exists a direct sum decomposition $\mathscr{P}(V) := \bigoplus_{\lambda\in\Omega} V_\lambda$, where $(V_\lambda)_{\lambda\in\Omega}$ is a family of irreducible $\mathfrak{g}$-modules parametrized by a set of partitions Ω. In these cases, one can define a natural basis $(D_\lambda)_{\lambda\in\Omega}$ of "Capelli operators" for the algebra $\mathscr{PD}(V)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant superpolynomial differential operators on V. In this paper we complete the solution to the Capelli eigenvalue problem, which asks for the determination of the scalar $c_\mu(\lambda)$ by which D_μ acts on V_λ.

We associate a restricted root system Σ to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ that corresponds to J, which is either a deformed root system of type $\mathsf{A}(m,n)$ or a root system of type $\mathsf{Q}(n)$. We prove a necessary and sufficient condition on the structure of Σ for $\mathscr{P}(V)$ to be completely reducible and multiplicity-free. When Σ satisfies the latter condition we obtain an explicit formula for the eigenvalue $c_\mu(\lambda)$, in terms of Sergeev-Veselov's shifted super Jack polynomials when Σ is of type $\mathsf{A}(m,n)$, and Okounkov-Ivanov's factorial Schur Q-polynomials when Σ is of type $\mathsf{Q}(n)$. Along the way, we prove that the natural map from the centre of the enveloping algebra of $\mathfrak{g}$ into $\mathscr{PD}(V)^{\mathfrak{g}}$ is surjective in all cases except when $J \cong F$, where F is the 10-dimensional exceptional Jordan superalgebra.

1. Introduction and main results

Let J be a finite dimensional unital complex simple Jordan superalgebra (for the classification of these Jordan superalgebras see [16] and [3]). The Tits-Kantor-Koecher construction (see Appendix A) associates [1] to J a Lie superalgebra $\mathfrak{g}^\flat$ together with an imbedded $\mathfrak{sl}_2$-triple $\mathfrak{s} := \mathrm{Span}_{\mathbb{C}}\{h, e, f\}$ where

$$[h, e] = 2e,\ [e, f] = h,\ \text{and } [h, f] = -2f. \tag{1}$$

Following Kac (see [16] and [3]), we consider the grading of $\mathfrak{g}^\flat$ by the eigenspaces of $\mathrm{ad}(-\frac{1}{2}h)$. Then we obtain a "short grading"

$$\mathfrak{g}^\flat \cong \mathfrak{g}^\flat(-1) \oplus \mathfrak{g}^\flat(0) \oplus \mathfrak{g}^\flat(1),$$

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[1]We remark that $\mathfrak{g}^\flat$ is a slight modification of the simple Lie superalgebra that is constructed from J by the Kantor functor (see Remark A.3).

where $\mathfrak{g}^\flat(-1) \cong J$ and e is the identity element of J. Set $\mathfrak{g} := \mathfrak{g}^\flat(0)$ and $\mathfrak{k} := \mathrm{stab}_{\mathfrak{g}}(e)$. Then $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, and in fact $\mathfrak{k} = \mathfrak{g}^\Theta$ where $\Theta := \mathrm{Ad}_w$, for $w \in \mathrm{PSL}_2(\mathbb{C})$ representing the nontrivial element of the Weyl group, defined as in (55).

Set $V := \mathfrak{g}^\flat(-1)^*$, where $\mathfrak{g}^\flat(-1)^*$ denotes the dual of the $\mathfrak{g}$-module $\mathfrak{g}^\flat(-1)$. Let $\mathscr{P}(V)$ denote the superalgebra of superpolynomials on V. Note that there is a canonical $\mathfrak{g}$-module isomorphism $\mathscr{P}(V) \cong \mathcal{S}(J)$, where $\mathcal{S}(J)$ denotes the symmetric algebra of the $\mathbb{Z}/2$-graded vector space J. In most cases (see Theorem 1.4), the $\mathfrak{g}$-module $\mathscr{P}(V)$ is completely reducible and multiplicity-free, and the irreducible summands of $\mathscr{P}(V)$ are parametrized by a set of partitions Ω, i.e.,

$$\mathscr{P}(V) \cong \bigoplus_{\lambda \in \Omega} V_\lambda,$$

where the V_λ are mutually non-isomorphic irreducible finite dimensional $\mathfrak{g}$-modules. In these cases, to each λ one can associate a *Capelli operator* [2] $D_\lambda \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$, where $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ denotes the algebra of $\mathfrak{g}$-invariant superpolynomial differential operators on V. Indeed the family $(D_\lambda)_{\lambda \in \Omega}$ forms a basis of $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ (see Remark 1.7).

From Schur's Lemma it follows that each operator D_μ acts on V_λ by a scalar $c_\mu(\lambda)$. The problem of calculating this scalar (the *Capelli eigenvalue problem*) has a long history (see below). In this paper we complete the solution of this problem in the super setting.

For ordinary Jordan algebras (i.e., when $J_{\overline{1}} = \{0\}$), $\mathscr{P}(V)$ is a multiplicity-free representation of the reductive Lie algebra $\mathfrak{g}$ (see [26], [15]). In this case, the solution to the Capelli eigenvalue problem was given when D_λ corresponds to a one-dimensional representation by B. Kostant and the first author in [18], and later in full generality by the first author in [23]. Indeed in [23] the first author introduced a "universal" family of symmetric polynomials $\varphi_\mu^{(\rho)}(x)$ characterized by certain vanishing properties, and depending on an auxiliary vector $\rho = (\rho_1, \ldots, \rho_n)$. The main result of [23] is that $c_\mu(\lambda) = \varphi_\mu^{(r\delta)}(\lambda + r\delta)$, such that $\delta := (0, -1, \ldots, -n+1)$ where n is the rank of the symmetric space $(\mathfrak{g}, \mathfrak{k})$ associated to the Jordan algebra J, and r is half the multiplicity of restricted roots.

The polynomials $\varphi_\mu^{(r\delta)}(x)$ were studied by F. Knop and the first author in [17] for arbitrary r, who proved that they satisfy a system of difference equations, which are a discrete version of the Debiard-Sekiguchi system for Jack polynomials [8], [27]. Knop and the first author deduced that the top-degree terms of $\varphi_\mu^{(r\delta)}$ are proportional to the Jack polynomials $P_\mu^{(1/r)}$. For this reason, the $\varphi_\mu^{(r\delta)}(x)$ are sometimes referred to as Knop–Sahi polynomials, or shifted Jack polynomials. The supersymmetric analogue of these polynomials was constructed by Sergeev and Veselov [29]. The top-degree terms of the Sergeev-Veselov polynomials $SP_\lambda^*(x, y, \theta)$ are the super Jack polynomials. An analogous family of polynomials $Q_\lambda^*(x)$ whose top-degree terms are the Schur Q-polynomials was defined by Okounkov and Ivanov [14].

The study of the Capelli eigenvalue problem for Jordan superalgebras was initiated in [24], where it was solved in the cases $J \cong gl(m,n)_+$ and $J \cong osp(n,2m)_+$. These Jordan superalgebras correspond to symmetric pairs of types $(\mathfrak{gl} \times \mathfrak{gl}, \mathfrak{gl})$ and $(\mathfrak{gl}, \mathfrak{osp})$, respectively. Extending the results of Kostant and Sahi to these Jordan superalgebras, in [24] the first two authors showed that the eigenvalues of the Capelli operators are obtained by specialization of the polynomials SP_μ^* at $\theta = 1, \frac{1}{2}$. Later, in [1] the Capelli eigenvalue problem was considered for Jordan superalgebras of type $q(n)_+$, and it was shown that the eigenvalues $c_\mu(\lambda)$ are given by the polynomials Q_μ^*.

[2] The classical Capelli operator appears as a special case of the operators D_λ. For this reason, we call the D_λ the *Capelli operators*.

In this paper, we complete the project started in [24] and [1], and solve the Capelli eigenvalue problem for general unital simple Jordan superalgebras. The new phenomenon that arises in the present setting is the occurrence of certain deformations of the root system of the Lie superalgebra $\mathfrak{gl}(r|s)$, studied by Sergeev and Veselov [28], which we define below.

Let $r, s \geq 0$ be integers. We represent the roots of the root system $\mathsf{A}(r-1, s-1)$ by

$$\mathsf{R}_{r,s} := \{\underline{\varepsilon}_i - \underline{\varepsilon}_{i'}\}_{1\leq i\neq i'\leq r} \cup \{\underline{\delta}_j - \underline{\delta}_{j'}\}_{1\leq j\neq j'\leq s} \cup \{\pm(\underline{\varepsilon}_i - \underline{\delta}_j)\}_{1\leq i\leq r, 1\leq j\leq s}, \tag{2}$$

as a subset of the $(r+s)$-dimensional vector space $\mathsf{E}_{r,s} := \mathrm{Span}_{\mathbb{R}}\{\underline{\varepsilon}_i, \underline{\delta}_j : 1 \leq i \leq r, 1 \leq j \leq s\}$. Fix $\kappa \in \mathbb{R}$ (if $s > 0$, we assume $\kappa \neq 0$), and let $\langle \cdot, \cdot \rangle_\kappa$ be a (unique up to a scalar) nondegenerate symmetric bilinear form on $\mathsf{E}_{r,s}$ such that $\{\underline{\varepsilon}_i\}_{i=1}^r \cup \{\underline{\delta}_j\}_{j=1}^s$ is an orthogonal basis of $\mathsf{E}_{r,s}$ with respect to $\langle \cdot, \cdot \rangle_\kappa$ that satisfies

$$\langle \underline{\varepsilon}_i, \underline{\varepsilon}_i \rangle_\kappa = \langle \underline{\varepsilon}_j, \underline{\varepsilon}_j \rangle_\kappa = \kappa^{-1} \langle \underline{\delta}_{i'}, \underline{\delta}_{i'} \rangle_\kappa = \kappa^{-1} \langle \underline{\delta}_{j'}, \underline{\delta}_{j'} \rangle_\kappa \text{ for } 1 \leq i, j \leq r \text{ and } 1 \leq i', j' \leq s.$$

The deformed root system $\mathsf{A}_\kappa(r-1, s-1)$ is the subset $\mathsf{R}_{r,s}$ of the quadratic space $(\mathsf{E}_{r,s}, \langle \cdot, \cdot \rangle_\kappa)$. The root multiplicities of $\mathsf{A}_\kappa(r-1, s-1)$ are defined to be

$$\mathrm{mult}(\underline{\varepsilon}_i - \underline{\varepsilon}_{i'}) := \kappa, \ \mathrm{mult}(\underline{\delta}_j - \underline{\delta}_{j'}) := \kappa^{-1}, \text{ and } \mathrm{mult}(\underline{\varepsilon}_i - \underline{\delta}_j) := 1. \tag{3}$$

For convenience, from now on we assume that $J_{\overline{\mathbf{0}}} \neq \{0\}$. We remark that our techniques and results can easily be adapted to ordinary Jordan algebras, and the reason for excluding them is that they have been dealt with in [23].

Remark 1.1. If $J \cong JP(0, n)$, then $\mathscr{P}^2(V)$ is not completely reducible. Therefore without loss of generality, from now on we exclude the Jordan superalgebras $JP(0, n)$.

Our next goal is to associate a set of restricted roots Σ to J. The Lie superalgebra $\mathfrak{g}$ that is associated to J is isomorphic to one of the types $\mathfrak{gl}$, $\mathfrak{gl} \times \mathfrak{gl}$, $\mathfrak{osp}$, or $\mathfrak{q} \times \mathfrak{q}$. Throughout the paper, we will use a standard matrix realization of $\mathfrak{g}$ that is given in Section 2. In this realization, there is a natural Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{h}_{\overline{\mathbf{0}}}$ is equal to the subspace of diagonal matrices and $\Theta(\mathfrak{h}) = \mathfrak{h}$. Note that $\mathfrak{h}_{\overline{\mathbf{1}}} = \{0\}$ except when $J \cong q(n)_+$ for $n \geq 2$.

Since $\Theta(\mathfrak{h}_{\overline{\mathbf{0}}}) = \mathfrak{h}_{\overline{\mathbf{0}}}$, we have a direct sum decomposition

$$\mathfrak{h}_{\overline{\mathbf{0}}} = \mathfrak{t}_{\overline{\mathbf{0}}} \oplus \mathfrak{a}_{\overline{\mathbf{0}}},$$

where $\mathfrak{t}_{\overline{\mathbf{0}}}$ and $\mathfrak{a}_{\overline{\mathbf{0}}}$ are the $+1$ and -1 eigenspaces of $\Theta\big|_{\mathfrak{h}_{\overline{\mathbf{0}}}}$, respectively. Let Δ denote the root system of $\mathfrak{g}$ corresponding to $\mathfrak{h}_{\overline{\mathbf{0}}}$, and set

$$\Sigma := \left\{\alpha\big|_{\mathfrak{a}_{\overline{\mathbf{0}}}} : \alpha \in \Delta\right\} \setminus \{0\}. \tag{4}$$

Assume that $\Sigma \neq \varnothing$ (see Remark 1.3). Then according to the structure of Σ, the Jordan superalgebras J can be divided into two classes (type A and type Q) defined below.

Jordan superalgebras of type A. Assume that J is one of the Jordan superalgebras that appear in Table 1. Then Σ is a root system of type $\mathsf{A}(r-1, s-1)$, where $r := \mathsf{r}_{J,+}$ and $s := \mathsf{r}_{J,-}$ are given in Table 1. We represent this root system as in (2). Furthermore, in these cases $\mathfrak{g}^\flat$ always has an invariant non-degenerate supersymmetric even bilinear form (see Table 4). Fix such a bilinear form $\langle \cdot, \cdot \rangle_\flat$ on $\mathfrak{g}^\flat$ (the choice of the bilinear form will not matter in what follows). Then the restriction $\langle \cdot, \cdot \rangle_\flat\big|_{\mathfrak{a}_{\overline{\mathbf{0}}} \times \mathfrak{a}_{\overline{\mathbf{0}}}}$ is also non-degenerate, and therefore it induces an isomorphism $\mathfrak{a}_{\overline{\mathbf{0}}} \cong \mathfrak{a}_{\overline{\mathbf{0}}}^*$. Via the latter isomorphism, $\langle \cdot, \cdot \rangle_\flat\big|_{\mathfrak{a}_{\overline{\mathbf{0}}} \times \mathfrak{a}_{\overline{\mathbf{0}}}}$ induces a bilinear form

$$\langle \cdot, \cdot \rangle_J : \mathfrak{a}_{\overline{\mathbf{0}}}^* \times \mathfrak{a}_{\overline{\mathbf{0}}}^* \to \mathbb{C}.$$

For $\alpha \in \Sigma$, we denote the corresponding restricted root space of $\mathfrak{g}$ by $\mathfrak{g}_\alpha$. We define the multiplicity of each $\alpha \in \Sigma$ to be

$$\mathrm{mult}(\alpha) := -\frac{1}{2}\mathrm{sdim}(\mathfrak{g}_\alpha),$$

where for any $\mathbb{Z}/2$-graded vector space $E := E_{\overline{\mathbf{0}}} \oplus E_{\overline{\mathbf{1}}}$ we define $\mathrm{sdim} E := \dim E_{\overline{\mathbf{0}}} - \dim E_{\overline{\mathbf{1}}}$. The last 3 columns of Table 1 give the graded dimensions of the restricted root spaces.

One can now verify directly that Σ, considered as a subset of the quadratic space $\left(\mathfrak{a}_{\overline{\mathbf{0}}}^*, \langle \cdot, \cdot \rangle_J\right)$ and equipped with the multiplicities defined above, is the deformed root system $\mathsf{A}_\kappa(r-1, s-1)$. Set

$$\theta_J := -\kappa.$$

Thus, the value of θ_J can be obtained from either of the two equalities

$$\theta_J = -\frac{\langle \underline{\delta}_1, \underline{\delta}_1 \rangle_J}{\langle \underline{\varepsilon}_1, \underline{\varepsilon}_1 \rangle_J} \text{ and } \theta_J = \frac{1}{2}\mathrm{sdim}(\underline{\varepsilon}_i - \underline{\varepsilon}_j),$$

and indeed in the cases that both of the quantities $-\frac{\langle \underline{\delta}_1, \underline{\delta}_1 \rangle_J}{\langle \underline{\varepsilon}_1, \underline{\varepsilon}_1 \rangle_J}$ and $\frac{1}{2}\mathrm{sdim}(\underline{\varepsilon}_i - \underline{\varepsilon}_j)$ are well-defined, they are equal. The values of θ_J are given in Table 1. The details of the computations that yield the values of the parameters $\mathsf{r}_{J,+}$, $\mathsf{r}_{J,-}$, and θ_J are postponed until Section 2.

Remark 1.2. In Case IV of Table 1, we assume that $t \in \mathbb{C}\backslash\{0, -1\}$ because D_0 is not simple and $D_{-1} \cong gl(1,1)_+$.

	J	Remarks	$\mathsf{r}_{J,+}$	$\mathsf{r}_{J,-}$	θ_J	$\pm(\underline{\varepsilon}_i - \underline{\varepsilon}_j)$	$\pm(\underline{\varepsilon}_i - \underline{\delta}_j)$	$\pm(\underline{\delta}_i - \underline{\delta}_j)$
I	$gl(m,n)_+$	$m,n \geq 1$	m	n	1	$2\|0$	$0\|2$	$2\|0$
II	$osp(n,2m)_+$	$m,n \geq 1$	m	n	$\frac{1}{2}$	$1\|0$	$0\|2$	$4\|0$
III	$(m,2n)_+$	$m,n \geq 1$	2	0	$\frac{m-1}{2} - n$	$m-1\|2n$	$-$	$-$
IV	D_t	$t \neq 0, -1$	1	1	$-\frac{1}{t}$	$-$	$0\|2$	$-$
V	F		2	1	$\frac{3}{2}$	$3\|0$	$0\|2$	$-$

TABLE 1. Σ of Type A.

Jordan superalgebras of type Q. Next assume that J is one of the Jordan superalgebras that appear in Table 2. Then Σ is a root system of type $\mathsf{Q}(r)$, where $r := r_J$ is given in Table 2. The graded dimension of all of the restricted root spaces is $(2|2)$.

	J	Remarks	r_J
VI	$p(n)_+$	$n \geq 2$	n
VII	$q(n)_+$	$n \geq 2$	n

TABLE 2. Σ of Type Q.

In the following remark, $(m,2n)_+$ denotes the Jordan superalgebra with underlying space $\mathbb{C}1 \oplus E$ and with product $a \circ b := (a,b)_E 1$, where E is an $(m|2n)$-dimensional vector superspace equipped with a nondegenerate even supersymmetric bilinear form $(\cdot,\cdot)_E$.

Remark 1.3. The only cases for which $\Sigma = \varnothing$ are the Jordan superalgebras of type $(0, 2n)_+$. Indeed it appears that the situation for these Jordan superalgebras differs substantially from the other cases that are considered in this paper, for the following reasons. First, the Zariski closure of the set of highest weights that occur in $\mathscr{P}(V)$ is not an affine subspace (see Definition 1.12 and Remark 1.15), and therefore it does not seem to be natural to consider the eigenvalues of the D_μ as a polynomial function on this Zariski closure (see Theorem 1.13). Second, even though $\mathscr{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module (see [7, Sec. 5.3]), the highest weights that occur in $\mathscr{P}(V)$ look quite different from those that occur in the cases $J \cong (m, 2n)_+$ for $m > 0$. In particular, the number of irreducible $\mathfrak{g}$-submodules occurring in the subspace $\mathscr{P}^k(V)$ of homogeneous elements of degree k in $\mathscr{P}(V)$ stabilizes for $k \geq 2n$. Therefore unlike the case $J \cong (m, 2n)_+$ for $m > 0$, one cannot expect a parametrization of irreducible summands of $\mathscr{P}(V)$ by hook partitions (see (5) below). We hope to investigate these interesting cases in the future. In the rest of this paper, we assume that $J \ncong (0, 2n)_+$.

In order to state our first theorem (Theorem 1.4), we need the parametrization of the irreducible summands of $\mathscr{P}(V)$ by partitions. For this parametrization, we choose a Borel subalgebra

$$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$$

of $\mathfrak{g}$ satisfying $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$. For the precise definition of $\mathfrak{b}$ and the embedding of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$, see Section 2. The quintuples $(\mathfrak{g}^\flat, \mathfrak{g}, \mathfrak{k}, \mathfrak{b}, V)$ that are associated to the Jordan superalgebras J are also listed in Table 4.

Let $\mathcal{P}$ denote the set of partitions. We represent elements of $\mathcal{P}$ by sequences of integers $\lambda := (\lambda_i)_{i=1}^\infty$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$, and $\lambda_i = 0$ for all sufficiently large $i \in \mathbb{N}$. As usual, the *weight* of any $\lambda \in \mathcal{P}$ is defined by $|\lambda| := \sum_{i=1}^\infty \lambda_i$. A partition $\lambda := (\lambda_i)_{i=1}^\infty \in \mathcal{P}$ is called *strict* if $\lambda_i > \lambda_{i+1}$ for all $i \leq \ell(\lambda)$, where $\ell(\lambda) := \max\{i \,:\, \lambda_i > 0\}$ denotes the length of λ. For $n \geq 0$ let $\mathcal{DP}(n)$ be the set of strict partitions λ such that $\ell(\lambda) \leq n$. For $m, n \geq 0$, let $\mathcal{H}(m, n)$ be the set of (m, n)-*hook* partitions, defined by

$$\mathcal{H}(m, n) := \{\lambda \in \mathcal{P} \,:\, \lambda_{m+1} \leq n\}. \tag{5}$$

For $d \geq 0$ set

$$\mathcal{H}_d(m, n) := \{\lambda \in \mathcal{H}(m, n) \,:\, |\lambda| = d\} \text{ and } \mathcal{DP}_d(n) := \{\lambda \in \mathcal{DP}(n) \,:\, |\lambda| = d\}.$$

Also, set

$$\mathcal{S}(m, n) := \begin{cases} \left\{-\frac{a}{b} \,:\, a, b \in \mathbb{Z},\ a \geq 1,\ \text{and } 1 \leq b \leq m-1\right\} & \text{if } n = 0, \\ \left\{-\frac{a}{b} \,:\, a, b \in \mathbb{Z},\ 0 \leq a \leq n,\ \text{and } b \geq 1\right\} & \text{if } m = 0, \\ \mathbb{Q}^{\leq 0} & \text{otherwise.} \end{cases}$$

Indeed $\mathcal{S}(m, n)$ is the set of admissible parameter values of the Sergeev-Veselov polynomials (see Theorem 1.8). The first main result of this paper is the following.

Theorem 1.4. *Let J be a finite dimensional unital complex simple Jordan superalgebra such that $J_{\overline{1}} \neq \{0\}$. Further, assume that J is not isomorphic to one of the Jordan superalgebras of types $(0, 2n)_+$ and $JP(0, n)$. Let $\mathfrak{g}$, $\mathfrak{b}$, and V be associated to J as above. Then the following assertions hold.*

(i) *When J is of type A, the $\mathfrak{g}$-module $\mathscr{P}(V)$ is completely reducible and multiplicity-free if and only if $\theta_J \notin \mathcal{S}(\mathsf{r}_{J,+}, \mathsf{r}_{J,-})$.*

(ii) *When J is of type Q, the $\mathfrak{g}$-module $\mathscr{P}(V)$ is completely reducible and multiplicity-free.*

Furthermore, whenever $\mathscr{P}(V)$ is completely reducible and multiplicity-free, for every $d \geq 0$ we have

$$\mathscr{P}^d(V) \cong \bigoplus_{\lambda \in \Omega_d} V_\lambda, \tag{6}$$

where V_λ is the irreducible $\mathfrak{g}$-module with the $\mathfrak{b}$-highest weight $\underline{\lambda}$ given in Table 3, and

$$\Omega_d := \begin{cases} \mathcal{H}_d(\mathsf{r}_{J,+}, \mathsf{r}_{J,-}) & \text{if } J \text{ is of type } \mathsf{A}, \\ \mathcal{DP}_d(\mathsf{r}_J) & \text{if } J \text{ is of type } \mathsf{Q}. \end{cases}$$

Remark 1.5. In Table 3, we represent the $\mathfrak{b}$-highest weight $\underline{\lambda}$ as a linear combination of the standard characters of $\mathfrak{h}_{\overline{0}}$, when $\mathfrak{g}$ is realized as in Section 2. The standard character of the Lie superalgebras of types $\mathfrak{gl}$, $\mathfrak{osp}$, and $\mathfrak{q}$ are given in Appendices B.1, B.2, and B.6 respectively. The highest weights $\lambda^{\mathrm{st}}_{m|n}$ and λ^{st}_n that appear in Cases I and VII of Table 3 are defined in Appendices B.1. and B.6 respectively.

Remark 1.6. We remark that several of the cases of Theorem 1.4 are already known. For J corresponding to Cases I–III and VII of Table 1 and Table 2, Theorem 1.4 can be found in [2], [4], [5], and [7]. Thus, the new cases of Theorem 1.4 are Cases IV–VI, for which the assertion is proved in Section 3.

In the rest of this section we assume that the $\mathfrak{g}$-module $\mathscr{P}(V)$ is completely reducible and multiplicity-free. Set

$$\Omega := \bigcup_{d \geq 0} \Omega_d.$$

Then from (6) it follows that

$$\mathscr{PD}(V)^{\mathfrak{g}} \cong (\mathscr{P}(V) \otimes \mathcal{S}(V))^{\mathfrak{g}} \cong \bigoplus_{\lambda,\mu \in \Omega} \left(V_\lambda \otimes V_\mu^*\right)^{\mathfrak{g}} \cong \bigoplus_{\lambda,\mu \in \Omega} \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\lambda). \tag{7}$$

For $\lambda \in \Omega$, let D_λ be the element of $\mathscr{PD}(V)^{\mathfrak{g}}$ that corresponds to $\mathrm{id}_{V_\lambda} \in \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda)$ via the isomorphism (7).

Remark 1.7. The family $(D_\lambda)_{\lambda \in \Omega}$ forms a basis of $\mathscr{PD}(V)^{\mathfrak{g}}$. This is because V_λ is of type M in the sense of [6, Sec. 3.1.2], that is, V_λ is irreducible as an ungraded module. In particular, there is no odd $\mathfrak{g}$-intertwining map $V_\lambda \to V_\lambda$. For Cases I–VI, this property of V_λ is an immediate consequence of highest weight theory for Lie superalgebras of types $\mathfrak{gl}$ and $\mathfrak{osp}$, and for Case VII, it is verified in [1, Sec. 3.1].

Our second main result (Theorem 1.13) yields an explicit formula for the eigenvalue $c_\mu(\lambda)$ of D_μ on V_λ. Before we state Theorem 1.13, we need to recall the definitions of the shifted super Jack polynomials of Sergeev and Veselov [29], and the factorial Schur Q-polynomials of Okounkov and Ivanov [14].

For $m, n \geq 0$ let $\mathscr{P}_{m,n}$ denote the $\mathbb{C}$-algebra of polynomials in $m+n$ variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$. Fix $\theta \in \mathbb{C}$ (if $n > 0$, we assume $\theta \neq 0$). Let $\Lambda^\natural_{m,n,\theta} \subseteq \mathscr{P}_{m,n}$ be the subalgebra of polynomials $f(x,y)$ with complex coefficients which are separately symmetric in $x := (x_1, \ldots, x_m)$ and in $y := (y_1, \ldots, y_n)$, and which satisfy the relation

$$f\left(x + \tfrac{1}{2}\mathsf{e}_i, y - \tfrac{1}{2}\mathsf{e}_j\right) = f\left(x - \tfrac{1}{2}\mathsf{e}_i, y + \tfrac{1}{2}\mathsf{e}_j\right)$$

on every hyperplane $x_i + \theta y_j = 0$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. Given any $\lambda \in \mathcal{H}(m,n)$, as in [29, Sec. 6] for $1 \leq i \leq m$ and $1 \leq j \leq n$ we define

$$\mathsf{p}_i(\lambda) := \lambda_i - \theta\left(i - \tfrac{1}{2}\right) - \tfrac{1}{2}(n - \theta m) \quad \text{and} \quad \mathsf{q}_j(\lambda) := \langle \lambda'_j - m \rangle - \theta^{-1}\left(j - \tfrac{1}{2}\right) + \tfrac{1}{2}\left(\theta^{-1}n + m\right), \tag{8}$$

where λ' denotes the transpose of λ, and

$$\langle x \rangle := \max\{x, 0\} \text{ for } x \in \mathbb{R}.$$

The $(m+n)$-tuple $(\mathsf{p}(\lambda), \mathsf{q}(\lambda))$, where $\mathsf{p}(\lambda) := (\mathsf{p}_1(\lambda), \dots, \mathsf{p}_m(\lambda))$ and $\mathsf{q}(\lambda) := (\mathsf{q}_1(\lambda), \dots, \mathsf{q}_n(\lambda))$, is called the *Frobenius coordinates* of λ. The following theorem characterizes shifted super Jack polynomials by their degree, symmetry, and vanishing properties.

Theorem 1.8. (Sergeev–Veselov [29, Thm 3], Knop–Sahi [17, Sec. 2]) *Let $m, n \geq 0$ be integers and let θ be a complex number such that $\theta \notin \mathcal{S}(m,n)$. Then for each $\lambda \in \mathcal{H}(m,n)$, there exists a unique polynomial $SP^*_\lambda \in \Lambda^\natural_{m,n,\theta}$ that satisfies the following properties.*

(i) $\deg(SP^*_\lambda) \leq |\lambda|$, *where* $\deg(SP^*_\lambda)$ *denotes the total degree of* SP^*_λ *in* x *and* y.
(ii) $SP^*_\lambda(\mathsf{p}(\mu), \mathsf{q}(\mu), \theta) = 0$ *for all* $\mu \in \mathcal{H}(m,n)$ *such that* $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.
(iii) $SP^*_\lambda(\mathsf{p}(\lambda), \mathsf{q}(\lambda), \theta) = H_\theta(\lambda)$, *where*

$$H_\theta(\lambda) := \prod_{1 \leq i \leq \ell(\lambda)} \prod_{1 \leq j \leq \lambda_i} (\lambda_i - j + \theta(\lambda'_j - i) + 1).$$

Furthermore, the family of polynomials $\big(SP^*_\lambda(x,y,\theta)\big)_{\lambda \in \mathcal{H}(m,n)}$ is a basis of $\Lambda^\natural_{m,n,\theta}$.

Remark 1.9. For $m, n > 0$, Theorem 1.8 is proved in [29]. If either $m = 0$ or $n = 0$, then up to scaling the SP^*_λ are the same as the interpolation polynomials P^ρ_λ defined by Knop and Sahi in [17]. Given $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, set $\rho_{k,\alpha} := (\rho_1, \dots, \rho_k)$ where $\rho_i := \frac{\alpha}{2}(k - 2i + 1)$ for $1 \leq i \leq k$. If $n = 0$ then $SP^*_\lambda(x, \theta) = P^\rho_\lambda$ for $\rho := \rho_{m,\theta}$, and if $m = 0$ then $SP^*_\lambda(y, \theta) = \frac{H_\theta(\lambda)}{H_{\theta^{-1}}(\lambda')} P^\rho_{\lambda'}$ for $\rho := \rho_{n,\theta^{-1}}$.

Next we state the characterization of factorial Schur Q-polynomials by their degree, symmetry and vanishing properties. For $n \in \mathbb{N}$, let $\mathscr{P}_n$ denote the $\mathbb{C}$-algebra of polynomials in n variables $x_1, \dots, x_n$. Further, let $\Gamma_n \subseteq \mathscr{P}_n$ be the subalgebra of symmetric polynomials $f(x_1, \dots, x_n)$ such that $f(t, -t, x_3, \dots, x_n)$ is independent of t (for $n = 1$ the latter condition is vacuous). In Theorem 1.10, for $\lambda \in \mathcal{DP}(n)$ we define $\lambda! := \prod_{1 \leq i \leq \ell(\lambda)} \lambda_i!$ and identify λ with the n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

Theorem 1.10. (Ivanov [14, Sec. 1]) *For every $\lambda \in \mathcal{DP}(n)$, there exists a unique polynomial $Q^*_\lambda \in \Gamma_n$ which satisfies the following properties.*

(i) $\deg(Q^*_\lambda) \leq |\lambda|$.
(ii) $Q^*_\lambda(\mu) = 0$ *for all* $\mu \in \mathcal{DP}(n)$ *such that* $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.
(iii) $Q^*_\lambda(\lambda) = H(\lambda)$, *where* $H(\lambda) := \lambda! \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}$.

Furthermore, the family of polynomials $\big(Q^*_\lambda\big)_{\lambda \in \mathcal{DP}(n)}$ is a basis of Γ_n.

Set

$$\Lambda_J := \begin{cases} \Lambda^\natural_{\mathsf{r}_{J,+}, \mathsf{r}_{J,-}, \theta_J} & \text{if } J \text{ is of type } \mathsf{A}, \\ \Gamma_{\mathsf{r}_J} & \text{if } J \text{ is of type } \mathsf{Q}. \end{cases} \quad \text{and} \quad \mathscr{P}_J := \begin{cases} \mathscr{P}_{\mathsf{r}_{J,+}, \mathsf{r}_{J,-}} & \text{if } J \text{ is of type } \mathsf{A}, \\ \mathscr{P}_{\mathsf{r}_J} & \text{if } J \text{ is of type } \mathsf{Q}. \end{cases} \tag{9}$$

There is a natural embedding of Λ_J as a subalgebra of $\mathscr{P}_J$.

Definition 1.11. For $\lambda \in \Omega$, we define $P_{J,\lambda} \in \Lambda_J$ as follows. When J is of type A we set

$$P_{J,\lambda} := \frac{|\lambda|!}{H_{\theta_J}(\lambda)} SP^*_\lambda(x, y, \theta_J),$$

where $x := \left(x_1, \dots, x_{\mathsf{r}_{J,+}}\right)$ and $y := \left(y_1, \dots, y_{\mathsf{r}_{J,-}}\right)$. When J is of type Q we set

$$P_{J,\lambda} := \frac{|\lambda|!}{H(\lambda)} Q^*_\lambda(x),$$

where $x := (x_1, \dots, x_{r_J})$.

Recall that for $\lambda \in \Omega$, we denote the $\mathfrak{b}$-highest weight of the irreducible $\mathfrak{g}$-module V_λ by $\underline{\lambda}$.

Definition 1.12. We define $\mathfrak{a}^*_\Omega$ to be the Zariski closure of the set $\left\{\underline{\lambda} : \lambda \in \Omega\right\}$ in $\mathfrak{h}^*_{\overline{\mathbf{0}}}$.

By a straightforward calculation using the explicit description of the $\mathfrak{b}$-highest weights given in Table 3, one can verify that $\mathfrak{a}^*_\Omega$ is a linear subspace of $\mathfrak{h}^*_{\overline{\mathbf{0}}}$ (see the proof of Proposition 4.2). Set

$$n_J := \begin{cases} \mathsf{r}_{J,+} + \mathsf{r}_{J,-} & \text{if } J \text{ is of type } \mathsf{A}, \\ \mathsf{r}_J & \text{if } J \text{ is of type } \mathsf{Q}. \end{cases}$$

Let

$$\tau_J : \mathfrak{a}^*_\Omega \to \mathbb{C}^{n_J} \tag{10}$$

be the affine linear map given in Table 3, where the elements of $\mathfrak{a}^*_\Omega$ are given in Cases I–VII by (30), (31), (32), (33), (38), (35), and (37) respectively, and the standard basis of $\mathbb{C}^{n_J}$ is denoted by $\mathsf{e}_1, \dots, \mathsf{e}_{n_J}$. We identify $\mathscr{P}_J$ with the algebra of polynomials on $\mathbb{C}^{n_J}$ in the natural way. Namely, for $v := (v_1, \dots, v_{n_J}) \in \mathbb{C}^{n_J}$, we identify the $x_i \in \mathscr{P}_J$ with the maps $v \mapsto v_i$ and the y_j (if they exist) with the maps $v \mapsto v_{j+r_{J,+}}$. The second main result of this paper is the following.

Theorem 1.13. *Let J, $\mathfrak{g}$, $\mathfrak{b}$, and V be as in Theorem 1.4. Assume that $\mathscr{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module. Then for every $\lambda, \mu \in \Omega$, the operator D_μ acts on V_λ by the scalar $P_{J,\mu}\left(\tau_J\left(\underline{\lambda}\right)\right)$, where $P_{J,\mu}$ is as in Definition 1.11, and $\underline{\lambda}$ is the $\mathfrak{b}$-highest weight of V_λ, given in Table 3.*

Remark 1.14. Theorem 1.13 is proved in [24] for Cases I–II, and in [1] for Case VII. We give a uniform proof for Cases I–IV and and VI in Sections 4 and 5 (see Proposition 5.2(iii)). With minor modifications, this strategy also works for Case VII. However, this uniform proof strategy does not work in Case V. In the latter case we prove Theorem 1.13 in Section 6 by a different method.

Remark 1.15. If $J \cong (0, 2n)_+$, then $\mathscr{P}(V)$ is completely reducible and multiplicity-free, but the Zariski closure of the set of highest weights is a union of $n + 1$ lines. Therefore it is not possible to give a natural formulation of Theorem 1.13.

Remark 1.16. In Case VII, the operators D_λ are closely related to certain operators I_λ that are constructed by Nazarov [21, Eq. (4.7)] using characters of the Sergeev algebra. Nazarov also defined certain explicit "Capelli" elements in the centre of the enveloping algebra of $\mathfrak{q}(n)$, and proved [21, Cor. 4.6] that their images under the left action of $\mathfrak{q}(n)$ on V are the I_λ. The precise connection between the D_λ and Nazarov's operators is determined in [1, Prop. 3.6].

The reason why the uniform proof of Theorem 1.13 fails for Case V is a property of the image of the Harish-Chandra homomorphism which is of independent interest. We denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathbf{U}(\mathfrak{g})$. The $\mathfrak{g}$-action on V induces a homomorphism of associative superalgebras

$$\mathsf{j} : \mathbf{U}(\mathfrak{g}) \to \mathscr{P}\mathscr{D}(V). \tag{11}$$

	$\underline{\lambda}$	τ_J
I	$(-\lambda^{\mathrm{st}}_{m\|n}) \oplus \lambda^{\mathrm{st}}_{m\|n}$	$\mu_{a,b} \mapsto \sum_{i=1}^{m} \left(a_i + \frac{m-2i+1-n}{2}\right) \mathsf{e}_i + \sum_{j=1}^{n} \left(b_j + \frac{m-2j+1+n}{2}\right) \mathsf{e}_{m+j}$
II	$-\sum_{i=1}^{m} 2\lambda_i \varepsilon_i - \sum_{j=1}^{n} \langle \lambda'_j - m \rangle (\delta_{2j-1} + \delta_{2j})$	$\mu_{a,b} \mapsto \sum_{i=1}^{m} \left(-\frac{1}{2} a_i + \frac{m+1-2n-2i}{4}\right) \mathsf{e}_i + \sum_{j=1}^{n} \left(-b_j + \frac{m+2+2n-4j}{2}\right) \mathsf{e}_{m+j}$
III	$(\lambda_1 - \lambda_2)\varepsilon_1 + (\lambda_1 + \lambda_2)\zeta$	$\mu_{a,b} \mapsto \left(\frac{1}{2}(a+b) + \frac{1}{4}(m-2n-1)\right) \mathsf{e}_1 + \left(\frac{1}{2}(b-a) - \frac{1}{4}(m-2n-1)\right) \mathsf{e}_2$
IV	$\left(\left(\frac{3+t}{1+t}\right)\|\lambda\| - 2\lambda_1\right)\varepsilon_1 + \left(\lambda_1 - \left(\frac{2+t}{1+t}\right)\|\lambda\|\right)(\delta_1 + \delta_2)$	$\mu_{a,b} \mapsto \left(-\frac{2+t}{1+t} a - \frac{3+t}{1+t} b - \frac{1}{2}\right) \mathsf{e}_1 + \left(\frac{1}{1+t} x - \frac{3+t}{1+t} \mathsf{y} + \frac{5+t}{1+t}\right) \mathsf{e}_2$
V	$(3\|\lambda\| - 2\lambda_1 - 2\lambda_2)\,\varepsilon_1 + (\lambda_1 - \lambda_2)\,(\delta_1 + \delta_2) + \|\lambda\|\zeta$	$\mu_{a,b,c} \mapsto \left(\frac{-a+2b+3c+1}{4}\right) \mathsf{e}_1 + \left(\frac{-a-2b+3c-5}{4}\right) \mathsf{e}_2 + \left(\frac{a-c+2}{2}\right) \mathsf{e}_3$
VI	$-\sum_{i=1}^{n} \lambda_i \varepsilon_i - \sum_{j=1}^{n} \lambda_j \delta_j$	$\mu_a \mapsto -\sum_{i=1}^{n} a_i \mathsf{e}_i$
VII	$(-\lambda^{\mathrm{st}}_n) \oplus \lambda^{\mathrm{st}}_n$	$\mu_a \mapsto \sum_{i=1}^{n} a_i \mathsf{e}_i$

TABLE 3. The $\mathfrak{b}$-highest weights $\underline{\lambda}$ and the affine maps τ_J

Let $\left(\mathbf{U}^{(i)}(\mathfrak{g})\right)_{i\geq 0}$ denote the standard filtration of $\mathbf{U}(\mathfrak{g})$. Let $\mathbf{Z}(\mathfrak{g}) \subseteq \mathbf{U}(\mathfrak{g})$ be the centre of $\mathbf{U}(\mathfrak{g})$, and set $\mathbf{Z}^{(i)}(\mathfrak{g}) := \mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^{(i)}(\mathfrak{g})$ for $i \geq 0$. Let

$$\mathrm{HC} : \mathbf{U}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h}) \cong \mathscr{P}(\mathfrak{h}^*) \tag{12}$$

be the Harish-Chandra projection corresponding to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}^-$ is the nilpotent subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{n}$. Thus for $D \in \mathbf{U}(\mathfrak{g})$ we define

$$\mathrm{HC}(D) := D_{\mathfrak{h}},$$

where $D = D_{\mathfrak{h}} + D'$ is the unique way of expressing D as a sum of two elements $D_{\mathfrak{h}} \in \mathbf{U}(\mathfrak{h}) \cong \mathcal{S}(\mathfrak{h})$ and $D' \in (\mathbf{U}(\mathfrak{g})\mathfrak{n} + \mathfrak{n}^-\mathbf{U}(\mathfrak{g}))$. Let

$$\mathsf{res} : \mathscr{P}(\mathfrak{h}^*) \to \mathscr{P}(\mathfrak{a}^*_\Omega) \tag{13}$$

denote the canonical restriction map, and let

$$\tau_J^* : \mathscr{P}_J \to \mathscr{P}(\mathfrak{a}^*_\Omega) \tag{14}$$

be the pullback of the map τ_J defined in (10), that is, $\tau_J^*(p) := p \circ \tau_J$ for $p \in \mathscr{P}_J$. We denote the degree filtration of the algebra $\mathscr{P}_J$ defined in (9) by $\left(\mathscr{P}_J^{(i)}\right)_{i\geq 0}$. In the following theorem, which will be proved in Section 4, we denote the exceptional (6|4)-dimensional Jordan superalgebra by F.

Theorem 1.17. *Let J be as in Theorem 1.4. Assume that $\mathscr{P}(V)$ is completely reducible and multiplicity-free. If $J \ncong F$, then* $\mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}^{(i)}(\mathfrak{g})\right)\right) = \tau_J^*\left(\Lambda_J^{(i)}\right)$ *for $i \geq 0$, where $\Lambda_J^{(i)} := \Lambda_J \cap \mathscr{P}_J^{(i)}$. If $J \cong F$, then* $\mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}(\mathfrak{g})\right)\right) \subsetneq \tau_J^*\left(\Lambda_J\right)$.

Since the map $\mathsf{j} : \mathbf{U}(\mathfrak{g}) \to \mathscr{PD}(V)$ is $\mathfrak{g}$-equivariant, we have $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) \subseteq \mathscr{PD}(V)^{\mathfrak{g}}$.

Corollary 1.18. *Let J be as in Theorem 1.4. Assume that $\mathscr{P}(V)$ is completely reducible and multiplicity-free. If $J \ncong F$, then* $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) = \mathscr{PD}(V)^{\mathfrak{g}}$. *If $J \cong F$, then* $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) \subsetneq \mathscr{PD}(V)^{\mathfrak{g}}$.

This phenomenon already occurs in the non-super case [12], [10]. Corollary 1.18 follows from Proposition 5.2(ii) and Proposition 5.3.

Remark 1.19. Weingart [30] computes the eigenvalues of a basis of invariant operators corresponding to the action of $\mathfrak{gl}(n)$ on $\Lambda(\mathcal{S}^2(\mathbb{C}^n))$ and $\Lambda(\Lambda^2(\mathbb{C}^n))$. These multiplicity-free representations arise naturally from the action of the even part of the Lie superalgebra $\mathfrak{p}(n)$ on its odd part. We will study similar actions in a forthcoming paper.

2. Realizations of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$, V, and Σ

In this section we describe explicit embeddings of $\mathfrak{b}$ and $\mathfrak{k}$ in $\mathfrak{g}$. We have the following three possibilities for $\mathfrak{g}$.

(i) $\mathfrak{g} \cong \mathfrak{gl}(r|s)$ or $\mathfrak{g} \cong \mathfrak{gl}(r|s) \oplus \mathfrak{gl}(r|s)$ for some $r, s \in \mathbb{N}$.
(ii) $\mathfrak{g} \cong \mathfrak{go}\mathfrak{sp}(r|2s)$ for some $r, s \in \mathbb{N}$.
(iii) $\mathfrak{g} \cong \mathfrak{q}(r) \oplus \mathfrak{q}(r)$ for some $r \geq 2$.

In each of the cases (i)–(iii) above, we consider the standard matrix realization of $\mathfrak{g}$ (or its direct summands) as given in Appendices B.1, B.2, and B.6, respectively. The embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}^\flat$ is determined uniquely by the semisimple element h given below. We identify $\mathfrak{k}$ and $\mathfrak{b}$ as subalgebras of this realization of $\mathfrak{g}$. We also give an explicit description of Σ. In what follows, $\mathrm{diag}(X_1, \ldots, X_n)$ denotes the block diagonal matrix formed by $X_1, \ldots, X_n$.

Case I. The matrix realization of $\mathfrak{g}^\flat \cong \mathfrak{gl}(2m|2n)$ is as in Appendix B.1. We set

$$h := \mathrm{diag}(-I_{m\times m}, I_{m\times m}, -I_{n\times n}, I_{n\times n}).$$

The matrix realization of $\mathfrak{g} \cong \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ is as in Appendix B.1, and the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}^\flat$ is given by

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}\right) \mapsto \begin{bmatrix} A & 0_{m\times m} & B & 0_{m\times n} \\ 0_{m\times m} & A' & 0_{m\times n} & B' \\ C & 0_{n\times m} & D & 0_{n\times n} \\ 0_{n\times m} & C' & 0_{n\times n} & D' \end{bmatrix}.$$

Invariant supersymmetric even bilinear forms on $\mathfrak{g}^\flat$ are of the form $\langle x, y\rangle := \alpha_1 \mathrm{str}(xy) + \alpha_2 \mathrm{str}(x)\mathrm{str}(y)$ for $\alpha_1, \alpha_2 \in \mathbb{C}$. The realization of $\mathfrak{a}_{\overline{\mathbf{0}}}$ as a subalgebra of $\mathfrak{g}^\flat$ is

$$\mathfrak{a}_{\overline{\mathbf{0}}} := \{\mathrm{diag}(\mathbf{d}_1, -\mathbf{d}_1, \mathbf{d}_2, -\mathbf{d}_2) \,:\, \mathbf{d}_1 \in \mathbb{C}^m \text{ and } \mathbf{d}_2 \in \mathbb{C}^n\}.$$

Note that the bilinear map $(x, y) \mapsto \mathrm{str}(x)\mathrm{str}(y)$ vanishes on $\mathfrak{a}_{\overline{\mathbf{0}}}$, and therefore without loss of generality we can choose the invariant form $\langle x, y\rangle_\flat$ to be $\langle x, y\rangle_\flat := \mathrm{str}(xy)$. Then Σ is a root system of type $\mathsf{A}(m-1, n-1)$ for the choice of $\underline{\varepsilon}_i := \varepsilon_i\big|_{\mathfrak{a}_{\overline{\mathbf{0}}}}$ and $\underline{\delta}_j := \delta_j\big|_{\mathfrak{a}_{\overline{\mathbf{0}}}}$, where $\{\varepsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n$ are the standard characters of the Cartan subalgebra $\mathfrak{h}_{m|n}$ of the left $\mathfrak{gl}(m|n)$ summand of $\mathfrak{g}$, defined in (57). By a direct calculation we obtain $\langle \underline{\varepsilon}_i, \underline{\varepsilon}_j\rangle_J = \frac{1}{2}\delta_{i,j}$ and $\langle \underline{\delta}_i, \underline{\delta}_j\rangle_J = -\frac{1}{2}\delta_{i,j}$, so that $\theta_J = 1$.

The embedding of $\mathfrak{k} \cong \mathfrak{gl}(m|n)$ in $\mathfrak{g}$ is the diagonal map $x \mapsto x \oplus x$. We set $\mathfrak{b} := \mathfrak{b}^{\mathrm{op}}_{m|n} \oplus \mathfrak{b}^{\mathrm{st}}_{m|n} \subseteq \mathfrak{g}$, where $\mathfrak{b}^{\mathrm{st}}_{m|n}$ and $\mathfrak{b}^{\mathrm{op}}_{m|n}$ are defined in Appendix B.1.

Case II. The matrix realization of $\mathfrak{g}^\flat \cong \mathfrak{osp}(4n|2m)$ is as in Appendix B.2. We set

$$h := \mathrm{diag}\left(-I_{2n\times 2n}, I_{2n\times 2n}, -I_{m\times m}, I_{m\times m}\right).$$

The matrix realization of $\mathfrak{g} \cong \mathfrak{gl}(m|2n)$ is as in Appendix B.1, and the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}^\flat$ is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} D & 0_{2n\times 2n} & C & 0_{2n\times m} \\ 0_{2n\times 2n} & -D^T & 0_{2n\times m} & B^T \\ B & 0_{m\times 2n} & A & 0_{m\times m} \\ 0_{m\times 2n} & -C^T & 0_{m\times m} & -A^T \end{bmatrix}.$$

We set $\langle x, y\rangle_\flat := \mathrm{str}(xy)$. The realization of $\mathfrak{a}_{\overline{0}}$ as a subalgebra of $\mathfrak{g}^\flat$ is

$$\mathfrak{a}_{\overline{0}} = \left\{\mathrm{diag}\left(\mathbf{d}, -\mathbf{d}, \mathbf{a}, -\mathbf{a}\right) \,:\, \mathbf{d} \in \mathbb{C}^m,\ \mathbf{a} := (a_1, \dots, a_{2n}) \in \mathbb{C}^{2n}, a_{2i-1} = a_{2i} \text{ for } 1 \le i \le n\right\},$$

and thus Σ is a root system of type $\mathsf{A}(m-1, n-1)$ with $\underline{\varepsilon}_i := \varepsilon_i\big|_{\mathfrak{a}_{\overline{0}}}$, $1 \le i \le m$, and $\underline{\delta}_j := \delta_{2j-1}\big|_{\mathfrak{a}_{\overline{0}}}$, $1 \le j \le n$, where $\{\varepsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^{2n}$ are the standard characters of the Cartan subalgebra $\mathfrak{h}_{m|2n} \subseteq \mathfrak{gl}(m|2n)$, defined in (57). By a direct calculation we obtain $\langle \underline{\varepsilon}_i, \underline{\varepsilon}_j\rangle_J = -\frac{1}{2}\delta_{i,j}$ and $\langle \underline{\delta}_i, \underline{\delta}_j\rangle_J = \frac{1}{4}\delta_{i,j}$, so that $\theta_J = \frac{1}{2}$.

The embedding of $\mathfrak{k} \cong \mathfrak{osp}(m|2n)$ in $\mathfrak{g}$ is as in Appendix B.3. We set $\mathfrak{b} := \mathfrak{b}^{\mathrm{op}}_{m|2n}$, where $\mathfrak{b}^{\mathrm{op}}_{m|2n}$ is defined in Appendix B.1.

Case III. Set $k := \lfloor \frac{m+1}{2} \rfloor$. The realization of $\mathfrak{g}^\flat \cong \mathfrak{osp}(m+3|2n)$ is as in Appendix B.2. We set

$$h := \begin{cases} \mathrm{diag}(-2, 0_{k\times k}, 2, 0_{k\times k}, 0_{2n\times 2n}) & \text{if } \ m+1 = 2k, \\ \mathrm{diag}(0, -2, 0_{k\times k}, 2, 0_{k\times k}, 0_{2n\times 2n}) & \text{if } \ m+1 = 2k+1. \end{cases}$$

The realization of $\mathfrak{g} \cong \mathfrak{gosp}(m+1|2n)$ is as in Appendix B.2. Let $\{\varepsilon_i\}_{i=1}^k \cup \{\delta_j\}_{j=1}^n \cup \{\zeta\}$ be the standard characters of the Cartan subalgebra $\tilde{\mathfrak{h}}_{m+1,n}$ of $\mathfrak{g}$. Then $\mathfrak{a}_{\overline{0}} = \bigcap_{i=2}^k \ker(\varepsilon_i) \cap \bigcap_{j=1}^n \ker(\delta_j)$, and thus $\Sigma = \left\{\varepsilon_1\big|_{\mathfrak{a}_{\overline{0}}}\right\}$. We consider Σ as a root system of type $\mathsf{A}(1,-1)$, where $\underline{\varepsilon}_1 - \underline{\varepsilon}_2 := \varepsilon_1\big|_{\mathfrak{a}_{\overline{0}}}$ (the choice of $\underline{\varepsilon}_1$ and $\underline{\varepsilon}_2$ does not matter). Since there are no $\underline{\delta}_j$'s, the value of θ_J is obtained only from the superdimension of $\underline{\varepsilon}_1 - \underline{\varepsilon}_2$.

Similar to Appendix B.2, let $\{\mathsf{e}_i\}_{i=1}^{m+1} \cup \{\mathsf{e}'_j\}_{j=1}^{2n}$ denote the natural homogeneous basis of the standard $\mathfrak{g}$-module $\mathbb{C}^{m+1|n}$. Then $\mathfrak{k} \cong \mathfrak{osp}(m|2n)$ is the subalgebra of $\mathfrak{g} \cong \mathfrak{gosp}(m+1|2n)$ given by

$$\mathfrak{k} := \begin{cases} \mathrm{Stab}_{\mathfrak{g}}(\mathsf{e}_1 - \mathsf{e}_{k+1}) & \text{if } m+1 = 2k, \\ \mathrm{Stab}_{\mathfrak{g}}(\mathsf{e}_2 - \mathsf{e}_{k+2}) & \text{if } m+1 = 2k+1. \end{cases}$$

The Borel subalgebra $\mathfrak{b} := \mathfrak{b}_{m+1|2n}$ is defined in (59).

Case IV. The Lie superalgebra $\mathfrak{g}^\flat$ is isomorphic to Scheunert's Lie superalgebra $\Gamma(-t, -1, 1+t)$ (see [25, Example I.1.5]). The realization of $\mathfrak{g} \cong \mathfrak{gl}(1|2)$ is as in Appendix B.1, and the embedding of $\mathfrak{k} \cong \mathfrak{osp}(1|2)$ in $\mathfrak{g} \cong \mathfrak{gl}(1|2)$ is as in Appendix B.3. The Borel subalgebra $\mathfrak{b} := \mathfrak{b}^{\mathrm{op}}_{1|2}$ is defined in Appendix B.1.

To identify Σ and compute the value of θ_J, we need the explicit realization of the root system of $\mathfrak{g}^\flat$. To distinguish the root systems of $\mathfrak{g}^\flat$ and $\mathfrak{g}$, we denote the root system of $\mathfrak{g}^\flat$ by $\Delta^\flat := \Delta^\flat_{\bar{0}} \sqcup \Delta^\flat_{\bar{1}}$ where

$$\Delta^\flat_{\bar{0}} := \{\pm 2\tilde{\varepsilon}_1, \pm 2\tilde{\varepsilon}_2, \pm 2\tilde{\varepsilon}_3\} \text{ and } \Delta^\flat_{\bar{1}} := \{\pm\tilde{\varepsilon}_1 \pm \tilde{\varepsilon}_2 \pm \tilde{\varepsilon}_3\}.$$

Let $(\cdot,\cdot)'_\flat$ denote the bilinear form induced on $\mathrm{Span}_{\mathbb{C}}\{\tilde{\varepsilon}_i : 1 \le i \le 3\}$ by the invariant form $(\cdot,\cdot)_\flat$. As usual, we choose $(\cdot,\cdot)_\flat$ such that the $\tilde{\varepsilon}_i$'s are orthogonal with respect to $(\cdot,\cdot)'_\flat$ and we have $(\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)'_\flat = -t$, $(\tilde{\varepsilon}_2,\tilde{\varepsilon}_2)'_\flat = -1$, and $(\tilde{\varepsilon}_3,\tilde{\varepsilon}_3)'_\flat = 1+t$. Let $\{h_i\}_{i=1}^3$ be a basis for the Cartan subalgebra of $\mathfrak{g}^\flat$ that is dual to the $\tilde{\varepsilon}_i$'s, i.e., $\tilde{\varepsilon}_i(h_j) = \delta_{i,j}$. Then $h := h_1 + h_2$ and therefore the fundamental roots of Δ are $\varepsilon_1 - \delta_1 := \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3\big|_{\mathfrak{h}}$ and $\delta_1 - \delta_2 := 2\tilde{\varepsilon}_3\big|_{\mathfrak{h}}$, where $\mathfrak{h}$ denotes the diagonal Cartan subalgebra of $\mathfrak{g}$. It follows that $\mathfrak{a}_{\bar{0}} = \mathrm{Span}_{\mathbb{C}}\{h_1, h_2\}$, and therefore Σ has only one odd root, hence it is of type $\mathsf{A}(0,0)$. If we choose $\underline{\varepsilon}_1 := \tilde{\varepsilon}_1\big|_{\mathfrak{a}_{\bar{0}}}$ then it follows that we should have $\underline{\delta}_1 = \tilde{\varepsilon}_2\big|_{\mathfrak{a}_{\bar{0}}}$, and by a straightforward calculation we obtain $\langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J = -t$ and $\langle\underline{\delta}_1,\underline{\delta}_1\rangle_J = -1$, so that $\theta_J = -\frac{1}{t}$.

Case V. The embedding of $\mathfrak{k} := \mathfrak{k}^{\mathrm{ex}} \cong \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ in $\mathfrak{g} \cong \mathfrak{gosp}(2|4)$ is defined in Appendix B.4. The Borel subalgebra $\mathfrak{b} := \mathfrak{b}^{\mathrm{ex}}_{2|4}$ is defined in (60).

As in Case IV, we let $(\cdot,\cdot)'_\flat$ be the bilinear form induced on the dual of the Cartan subalgebra of $\mathfrak{g}^\flat$ by the invariant form $(\cdot,\cdot)_\flat$ of $\mathfrak{g}^\flat$. The root system $\Delta^\flat := \Delta^\flat_{\bar{0}} \sqcup \Delta^\flat_{\bar{1}}$ of $\mathfrak{g}^\flat$ is

$$\Delta^\flat_{\bar{0}} := \{\pm\tilde{\varepsilon}_i \pm \tilde{\varepsilon}_j\}_{1\le i<j\le 3} \cup \{\pm\tilde{\varepsilon}_i\}_{i=1}^3 \cup \{\tilde{\delta}\} \text{ and } \Delta^\flat_{\bar{1}} := \left\{\tfrac{1}{2}\big(\pm\tilde{\varepsilon}_1 \pm \tilde{\varepsilon}_2 \pm \tilde{\varepsilon}_3 \pm \tilde{\delta}\big)\right\},$$

such that $(\tilde{\varepsilon}_i,\tilde{\varepsilon}_j)'_\flat = \delta_{i,j}$, $(\tilde{\delta},\tilde{\delta})'_\flat = -3$, and $(\tilde{\varepsilon}_i,\tilde{\delta})'_\flat = 0$. Let $\{h_{\tilde{\varepsilon}_i}\}_{i=1}^3 \cup \{h_{\tilde{\delta}}\}$ be a basis dual to $\{\tilde{\varepsilon}_i\}_{i=1}^3 \cup \{\tilde{\delta}\}$ for the Cartan subalgebra of $\mathfrak{g}^\flat$. We set $h := h_{\tilde{\varepsilon}_1} + h_{\tilde{\delta}}$. Then the fundamental roots of Δ are

$$\varepsilon_1 - \delta_1 := \tfrac{1}{2}(\tilde{\delta} - \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)\big|_{\mathfrak{h}},\ \delta_1 - \delta_2 := \tilde{\varepsilon}_3\big|_{\mathfrak{h}},\ \text{and } 2\delta_2 := (\tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)\big|_{\mathfrak{h}}.$$

From the description of $\mathfrak{k}$ it follows that $\delta_1 - \delta_2$ is a root of $\mathfrak{k}$, hence $\delta_1 - \delta_2\big|_{\mathfrak{a}} = 0$. Consequently,

$$\mathfrak{a}_{\bar{0}} = \mathrm{Span}_{\mathbb{C}}\left\{h_{\tilde{\delta}}, h_{\tilde{\varepsilon}_1}, h_{\tilde{\varepsilon}_2}\right\}.$$

One can now verify that Σ is a root system of type $\mathsf{A}(1,0)$, with fundamental roots

$$\underline{\varepsilon}_1 - \underline{\varepsilon}_2 := \tilde{\varepsilon}_2\big|_{\mathfrak{a}_{\bar{0}}} \text{ and } \underline{\varepsilon}_2 - \underline{\delta}_1 := \tfrac{1}{2}(\tilde{\delta} - \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)\big|_{\mathfrak{a}_{\bar{0}}}.$$

We can determine the value of θ_J without making a choice for the $\underline{\varepsilon}_i$ and the $\underline{\delta}_j$, as follows. First note that $\underline{\varepsilon}_1 - \underline{\varepsilon}_2 = \tilde{\varepsilon}_2\big|_{\mathfrak{a}}$, so that

$$\langle\underline{\varepsilon}_1 - \underline{\varepsilon}_2, \underline{\varepsilon}_1 - \underline{\varepsilon}_2\rangle_J = (\tilde{\varepsilon}_2,\tilde{\varepsilon}_2)_\flat = 1. \tag{15}$$

Since Σ is assumed to be a Sergeev-Veselov deformed root system, in particular we should have $\langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J = \langle\underline{\varepsilon}_2,\underline{\varepsilon}_2\rangle_J$ and $\langle\underline{\varepsilon}_1,\underline{\varepsilon}_2\rangle_J = 0$. Thus from (15) it follows that $\langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J = \frac{1}{2}$. Similarly, $\underline{\varepsilon}_1 - \underline{\delta}_1 = \frac{1}{2}(\tilde{\delta} - \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)\big|_{\mathfrak{a}_{\bar{0}}}$, so that

$$\langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J + \langle\underline{\delta}_1,\underline{\delta}_1\rangle_J = \langle\underline{\varepsilon}_1 - \underline{\delta}_1, \underline{\varepsilon}_1 - \underline{\delta}_1\rangle_J = \tfrac{1}{4}(\tilde{\delta} - \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2, \tilde{\delta} - \tilde{\varepsilon}_1 + \tilde{\varepsilon}_2)_\flat = -\tfrac{1}{4}.$$

Consequently, $\langle\underline{\delta}_1,\underline{\delta}_1\rangle_J = -\frac{1}{4} - \langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J = -\frac{3}{4}$. From the values of $\langle\underline{\varepsilon}_1,\underline{\varepsilon}_1\rangle_J$ and $\langle\underline{\delta}_1,\underline{\delta}_1\rangle_J$ we obtain $\theta_J = \frac{3}{2}$.

Case VI. The realization of $\mathfrak{g}^\flat \cong \mathfrak{p}(2n)$ is as in Appendix B.5. We set

$$h := \mathrm{diag}(-I_{n\times n}, I_{n\times n}, I_{n\times n} - I_{n\times n}).$$

The realization of $\mathfrak{g} \cong \mathfrak{gl}(n|n)$ is as in Appendix B.1, and the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}^\flat$ is given by the map

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A & 0_{n\times n} & 0_{n\times n} & B \\ 0_{n\times n} & -D^T & B^T & 0_{n\times n} \\ 0_{n\times n} & -C^T & -A^T & 0_{n\times n} \\ C & 0_{n\times n} & 0_{n\times n} & D \end{bmatrix}.$$

The realization of $\mathfrak{a}_{\overline{\mathbf{0}}}$ as a subalgebra of $\mathfrak{g}^\flat$ is

$$\mathfrak{a}_{\overline{\mathbf{0}}} := \{\mathrm{diag}(\mathbf{a}, -\mathbf{d}, -\mathbf{a}, \mathbf{d}) \,:\, \mathbf{a}, \mathbf{d} \in \mathbb{C}^n\},$$

and Σ is a root system of type $\mathsf{Q}(n)$. The embedding of $\mathfrak{k} \cong \mathfrak{p}(n)$ in $\mathfrak{g} \cong \mathfrak{gl}(n|n)$ is given in Appendix B.5. The Borel subalgebra $\mathfrak{b} := \mathfrak{b}^{\mathrm{mx}}_{n|n}$ is defined in Appendix B.1.

Case VII. The matrix realization of $\mathfrak{g}^\flat \cong \mathfrak{q}(2n)$ is as in Appendix B.6. The embedding of $\mathfrak{g} \cong \mathfrak{q}(n)\oplus\mathfrak{q}(n)$ in $\mathfrak{g}^\flat$ is the restriction of the one given in Case I. The subalgebra $\mathfrak{a}_{\overline{\mathbf{0}}}$ is the intersection with $\mathfrak{g}$ of the one given in Case I. The embedding of $\mathfrak{k} \cong \mathfrak{q}(n)$ in $\mathfrak{g} \cong \mathfrak{q}(n)\oplus\mathfrak{q}(n)$ is also the restriction of the diagonal map $x \mapsto x \oplus x$. We set $\mathfrak{b} := \mathfrak{b}^{\mathrm{op}}_n \oplus \mathfrak{b}^{\mathrm{st}}_n$, where $\mathfrak{b}^{\mathrm{st}}_n$ and $\mathfrak{b}^{\mathrm{op}}_n$ are defined in Appendix B.6.

We summarize the descriptions of $\mathfrak{g}^\flat$, $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{b}$ in Table 4. In addition, in the last column of Table 4 we give an explicit realization of V for Cases I–III and VI–VII, and the $\mathfrak{b}$-highest weight of V for Cases IV and V. The symbol Π in Cases VI and VII of Table 4 is the parity reversal functor, so that

$$((\mathbb{C}^{n|n})^* \otimes \mathbb{C}^{n|n})^{\Pi\otimes\Pi} := \left\{ v \in ((\mathbb{C}^{n|n})^* \otimes \mathbb{C}^{n|n}) \,:\, (\Pi\otimes\Pi)(v) = v \right\}.$$

	$\mathfrak{g}^\flat$	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{b}$	V
I	$\mathfrak{gl}(2m\|2n)$	$\mathfrak{gl}(m\|n) \oplus \mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m\|n)$	$\mathfrak{b}^{\mathrm{op}}_{m|n} \oplus \mathfrak{b}^{\mathrm{st}}_{m|n}$	$\mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*$
II	$\mathfrak{osp}(4n\|2m)$	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathfrak{b}^{\mathrm{op}}_{m|2n}$	$\mathcal{S}^2(\mathbb{C}^{m|2n})$
III	$\mathfrak{osp}(m+3\|2n)$	$\mathfrak{gosp}(m+1\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathfrak{b}_{m+1|2n}$	$(\mathbb{C}^{m+1|2n})^*$
IV	$D(2\|1,t)$	$\mathfrak{gl}(1\|2)$	$\mathfrak{osp}(1\|2)$	$\mathfrak{b}^{\mathrm{op}}_{1|2}$	$\left(-\frac{3+t}{1+t}\right)\varepsilon_1 + \left(\frac{2+t}{1+t}\right)(\delta_1+\delta_2)$
V	$F(3\|1)$	$\mathfrak{gosp}(2\|4)$	$\mathfrak{osp}(1\|2) \oplus \mathfrak{osp}(1\|2)$	$\mathfrak{b}^{\mathrm{ex}}_{2|4}$	$-3\varepsilon_1 - \zeta$
VI	$\mathfrak{p}(2n)$	$\mathfrak{gl}(n\|n)$	$\mathfrak{p}(n)$	$\mathfrak{b}^{\mathrm{mx}}_{n|n}$	$\Pi(\Lambda^2(\mathbb{C}^{n|n}))$
VII	$\mathfrak{q}(2n)$	$\mathfrak{q}(n) \oplus \mathfrak{q}(n)$	$\mathfrak{q}(n)$	$\mathfrak{b}^{\mathrm{op}}_n \oplus \mathfrak{b}^{\mathrm{st}}_n$	$(\mathbb{C}^{n|n} \otimes (\mathbb{C}^{n|n})^*)^{\Pi\otimes\Pi}$

TABLE 4. The quintuples $(\mathfrak{g}^\flat, \mathfrak{g}, \mathfrak{k}, \mathfrak{b}, V)$.

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4. Recall that by Remark 1.6, we will only need to consider Cases IV–VI. We address each case separately.

3.1. **Case IV.** Recall that in this case $\mathfrak{g} \cong \mathfrak{gl}(1|2)$. Let $\mathfrak{b}^{\mathrm{st}}_{1|2}$ be the Borel subgroup of $\mathfrak{g}$ defined as in Appendix B.1. The next proposition implies Theorem 1.4 in Case IV.

Proposition 3.1. *Assume that $J \cong D_t$ for $t \neq 0, -1$. Then $\mathscr{P}(V)$ is multiplicity-free if and only if $-\frac{1}{t} \notin \mathbb{Q}^{\leq 0}$. If the latter condition holds, then for every $d \geq 1$ we have*

$$\mathscr{P}^d(V) \cong \bigoplus_{k=1}^{d} V_{\eta_k},$$

where V_{η_k} is the irreducible $\mathfrak{g}$-module with $\mathfrak{b}$-highest weight

$$\eta_k := \left(d\left(\frac{3+t}{1+t}\right) - 2k\right)\varepsilon_1 + \left(-d\left(\frac{2+t}{1+t}\right) + k\right)(\delta_1 + \delta_2).$$

Proof. By the $\mathfrak{g}$-module isomorphism $\mathscr{P}^d(V) \cong \mathcal{S}^d(V^*)$ it is enough to prove the analogous statement for $\mathcal{S}^d(V^*)$. The $\mathfrak{b}^{\mathrm{st}}_{1|2}$-highest weight of V^* is

$$\eta := \left(\frac{3+t}{1+t}\right)\varepsilon_1 - \left(\frac{2+t}{1+t}\right)(\delta_1 + \delta_2).$$

For every $\mu := x_1\varepsilon_1 + y_1\delta_1 + y_2\delta_2 \in \mathfrak{h}^*_{\overline{\mathbf{0}}}$ such that $y_1 - y_2 \in \mathbb{Z}^{\geq 0}$, we denote the irreducible finite dimensional $\mathfrak{g}_{\overline{\mathbf{0}}}$-module with $(\mathfrak{b}^{\mathrm{st}}_{1|2} \cap \mathfrak{g}_{\overline{\mathbf{0}}})$-highest weight μ by M_μ. Let $K(\mu) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{b}^{\mathrm{st}}_{1|2}} M_\mu$ be the corresponding Kac module. As a $\mathfrak{g}_{\overline{\mathbf{0}}}$-module, $K(\mu)_{\overline{\mathbf{1}}}$ is isomorphic to $M_\mu \otimes M_{-\varepsilon_1+\delta_1}$. Therefore

$$K(\mu)_{\overline{\mathbf{1}}} \cong \begin{cases} M_{\mu-\varepsilon_1+\delta_1} & \text{if } y_1 = y_2, \\ M_{\mu-\varepsilon_1+\delta_1} \oplus M_{\mu-\varepsilon_1+\delta_2} & \text{if } y_1 > y_2. \end{cases} \tag{16}$$

Furthermore, μ is a typical $\mathfrak{b}^{\mathrm{st}}_{1|2}$-highest weight if and only if $x_1 + y_1 \neq 0$ and $x_1 + y_2 \neq 1$.

Since η is typical, we have $V^* \cong K(\eta)$. Let $\mathcal{F}$ denote the category of finite dimensional $\mathfrak{h}$-weight modules of $\mathfrak{g}$. Typicality of η implies that V^* is projective in $\mathcal{F}$. Consequently, the tensor product of V^* and any object of $\mathcal{F}$ is also projective (the proof of the latter statement is similar to [13, Prop. 3.8(b)]). It follows that $\mathcal{S}^d(V^*)$, which is a submodule of $(V^*)^{\otimes d}$, is also projective, and therefore it has a filtration by Kac modules (see [31, Prop. 2.5]).

Set $\gamma := \eta - 2\varepsilon_1 + \delta_1 + \delta_2$ and $\gamma_k := d\eta - (2k+1)\varepsilon_1 + (k+1)\delta_1 + k\delta_2$ for $0 \leq k \leq d-1$. Then we have an isomorphism of $\mathfrak{g}_{\overline{\mathbf{0}}}$-modules

$$\begin{aligned} \mathcal{S}^d(V^*)_{\overline{\mathbf{1}}} &\cong \mathcal{S}^{d-1}(V^*_{\overline{\mathbf{0}}}) \otimes V^*_{\overline{\mathbf{1}}} \\ &\cong \mathcal{S}^{d-1}\left(M_\gamma \oplus M_\eta\right) \otimes M_{\eta-\varepsilon_1+\delta_1} \cong \bigoplus_{k=0}^{d-1} M_{k\gamma+(d-k-1)\eta} \otimes M_{\eta-\varepsilon_1+\delta_1} \cong \bigoplus_{k=0}^{d-1} M_{\gamma_k}. \end{aligned} \tag{17}$$

By comparing (17) with (16), it follows that the Kac-module filtration of $\mathcal{S}^d(V^*)$ consists of exactly one copy of each of the modules $K(\gamma_k + \varepsilon_1 - \delta_1)$. If $\gamma_k + \varepsilon_1 - \delta_1$ is atypical for some k, then $\mathcal{S}^d(V^*)$ cannot be completely reducible because the subquotient $K(\gamma_k + \varepsilon_1 - \delta_1)$ is reducible but indecomposable. Thus, a necessary condition for complete reducibility of $\mathcal{S}^d(V^*)$ is that $\gamma_k + \varepsilon_1 - \delta_1$ is typical for every $0 \leq k \leq d-1$. But the latter necessary condition is also sufficient because typical modules always split off as direct summands.

Next we determine when all of the $\gamma_k + \varepsilon_1 - \delta_1$ are typical. Note that

$$\gamma_k + \varepsilon_1 - \delta_1 = \left(d\left(\frac{3+t}{1+t}\right) - 2k\right)\varepsilon_1 + \left(-d\left(\frac{2+t}{1+t}\right) + k\right)(\delta_1 + \delta_2) \quad \text{for} \quad 0 \leq k \leq d-1.$$

Therefore $\gamma_k+\varepsilon_1-\delta_1$ is typical for all $0\le k\le d-1$ if and only if $\frac{d}{1+t}\notin\{0,\dots,d\}$. It follows that $\mathcal{S}(V^*)$ is completely reducible if and only if $\frac{1}{1+t}\notin\{x\in\mathbb{Q}\,:\,0\le x\le 1\}$. Since $t\neq 0$, the latter condition can be expressed as $-\frac{1}{t}\notin\mathbb{Q}^{\le 0}$.

Finally, using the fact that $\mathfrak{b}$ is obtained from $\mathfrak{b}^{\mathrm{st}}_{1|2}$ by the composition $s_{\delta_1-\delta_2}\circ s_{\varepsilon_1-\delta_2}\circ s_{\varepsilon_1-\delta_1}$ of even and odd reflections, it is straightforward to verify that the $\mathfrak{b}$-highest weight of $K(\gamma_k+\varepsilon_1-\delta_1)$ is η_{k+1} (see [6, Lem 1.40]). □

3.2. **Case V.** For $d\ge 0$ we have a $\mathfrak{g}$-module isomorphism $\mathscr{P}^d(V)\cong\mathcal{S}^d(V^*)$. The weights of $V^*_{\overline{\mathbf{0}}}$ are $\{\varepsilon_1+\zeta,\varepsilon_1\pm\delta_1\pm\delta_2+\zeta,3\varepsilon_1+\zeta\}$ and the weights of $V^*_{\overline{\mathbf{1}}}$ are $\{2\varepsilon_1\pm\delta_1+\zeta,2\varepsilon_1\pm\delta_2+\zeta\}$.

Let $\mathfrak{b}:=\mathfrak{b}^{\mathrm{ex}}_{2|4}$ and $\mathfrak{b}_{2|4}$ be the Borel subalgebras that are chosen in Appendix B.2. Then $\mathfrak{b}$ can be obtained from $\mathfrak{b}_{2|4}$ by applying the sequence of odd reflections

$$r_{\varepsilon_1+\delta_1}\circ r_{\varepsilon_1+\delta_2}\circ r_{\varepsilon_1-\delta_2}\circ r_{\varepsilon_1-\delta_1}. \tag{18}$$

Let u be a $\mathfrak{b}$-highest weight vector of V^*. Then u has weight $\varepsilon_1+\delta_1+\delta_2+\zeta$. Also, let w be a $\mathfrak{b}_{2|4}$-highest weight vector of V^*. Then the weight of w is $3\varepsilon_1$, hence $w\in V^*_{\overline{\mathbf{0}}}$ and therefore w^k for $k\ge 2$ is a $\mathfrak{b}_{2|4}$-highest weight vector in $\mathcal{S}^k(V^*)$ of weight $3k\varepsilon_1$. For $s\ge 2$, we set

$$w_s:=e_{-\varepsilon_1-\delta_1}(e_{-\varepsilon_1-\delta_2}(e_{-\varepsilon_1+\delta_2}(e_{-\varepsilon_1+\delta_1}(w^s)))),$$

where the $e_{-\varepsilon_1\pm\delta_i}$ denote root vectors of $\mathfrak{g}$.

Lemma 3.2. *For $s\ge 2$, the vector w_s is a typical $\mathfrak{b}$-highest weight, whose weight is $(3s-4)\varepsilon_1+s\zeta$. In addition, $w_2\in\mathcal{S}^2(V^*_{\overline{\mathbf{0}}})$.*

Proof. It is straightforward to verify that $3s\varepsilon_1+s\zeta$ is a typical $\mathfrak{b}_{2|4}$-highest weight. From the relation between $\mathfrak{b}$ and $\mathfrak{b}_{2|4}$ via odd reflections given in (18), it follows that the $\mathfrak{b}$-highest weight vector of the irreducible summand of $\mathcal{S}^s(V^*)$ generated by w^s is w_s (see [6, Lem. 1.40]). Since the $(2\varepsilon_1+2\zeta)$-weight space of $\mathcal{S}^2(V^*)$ is indeed a subspace of $\mathcal{S}^2(V^*_{\overline{\mathbf{0}}})$, we obtain $w_2\in\mathcal{S}^2(V^*_{\overline{\mathbf{0}}})$. □

Definition 3.3. Let $\mathcal{W}_d$ denote the set of $\mathfrak{b}$-highest weight vectors given in (i) and (ii) below.

(i) Vectors of the form $u^q(w_2)^r w_s$ for integers q,r,s that satisfy $q,r\ge 0$, $s\ge 2$, and $d+2r+s=d$. Note that $u^q(w_2)^r w_s\neq 0$ since $u,w_2\in\mathcal{S}(V^*_{\overline{\mathbf{0}}})$. The weight of $u^q(w_2)^r w_s$ is equal to

$$(d+2s-4)\varepsilon_1+(d-2r-s)\,(\delta_1+\delta_2)+d\zeta. \tag{19}$$

(ii) The vector u^d, whose weight is $d\varepsilon_1+d\delta_1+d\delta_2+d\zeta$.

We denote the set of weights of the vectors in $\mathcal{W}_d$ by $\mathcal{E}_d$.

Let $\mathfrak{m}$ be the image of the unique embedding $\iota:\mathfrak{sp}(4)\to\mathfrak{g}_{\overline{0}}$. For every $\mu:=y_1\delta_1+y_2\delta_2$ such that $y_1\ge y_2$, let $M(\mu)$ denote the irreducible $\mathfrak{m}$-module with $\mathfrak{b}\cap\mathfrak{m}$-highest weight μ.

Lemma 3.4. *Let W be the irreducible $\mathfrak{g}$-module with $\mathfrak{b}$-highest weight $d\varepsilon_1+d\delta_1+d\delta_2+d\zeta$, for $d\ge 1$. Then both $M\big(d\delta_1+d\delta_2\big)$ and $M\big((d-1)\delta_1+(d-1)\delta_2\big)$ occur as $\mathfrak{m}$-submodules of $W_{\overline{\mathbf{0}}}$.*

Proof. The $\mathfrak{m}$-submodule of W that is generated by the $\mathfrak{b}$-highest weight of W is isomorphic to $M\big(d\delta_1+d\delta_2\big)$. From [6, Lem. 1.40] and the relation between $\mathfrak{b}$ and $\mathfrak{b}_{2|4}$ via odd reflections given in (18) it follows that the $\mathfrak{b}_{2|4}$-highest weight of W is $(d+2)\varepsilon_1+(d-1)\delta_1+(d-1)\delta_2+d\zeta$. Since $\mathfrak{b}_{2|4}\cap\mathfrak{m}=\mathfrak{b}\cap\mathfrak{m}$, the $\mathfrak{m}$-module generated by the $\mathfrak{b}_{2|4}$-highest weight of W is isomorphic to $M\big((d-1)\delta_1+(d-1)\delta_2\big)$. □

Lemma 3.5. *For every $k \geq 0$ there is an isomorphism of $\mathfrak{sp}(4)$-modules*

$$\mathcal{S}^k\big(M(\delta_1+\delta_2)\big) \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} M\big((k-2i)\delta_1 + (k-2i)\delta_2\big).$$

Proof. Note that $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$ and $M(\delta_1+\delta_2)$ is the standard 5-dimensional representation of $\mathfrak{so}(5)$. The statement now follows from the classical theory of spherical harmonics (for example see [9, Thm 5.6.11]). □

The next proposition proves Theorem 1.4 in Case V.

Proposition 3.6. *Assume that $J \cong F$. Then $\mathscr{P}^d(V)$ is a multiplicity-free direct sum of irreducible $\mathfrak{g}$-modules with $\mathfrak{b}$-highest weights in $\mathcal{E}_d$, where $\mathcal{E}_d$ is as in Definition 3.3.*

Proof. For every $\gamma \in \mathcal{E}_d$, we denote the irreducible $\mathfrak{g}$-module with $\mathfrak{b}$-highest weight γ by W_γ. If $\gamma \neq d\varepsilon_1 + d\delta_1 + d\delta_2 + d\zeta$, then by setting $a := s-2$ and $b := d-2r-s$ in (19) we can express γ as

$$\gamma := (d+2a)\varepsilon_1 + b\delta_1 + b\delta_2 + d\zeta, \tag{20}$$

where $0 \leq a \leq d-2$, $0 \leq b \leq d-a-2$ and $b \equiv d-a \pmod 2$. It is straightforward to verify that every γ of the form (20) is typical. Since typical submodules split off as direct summands, in order to prove the assertion of the proposition it suffices to show that the only irreducible subquotients of $\mathscr{P}^d(V)$ are those whose $\mathfrak{b}$-highest weight vectors are in $\mathcal{W}_d$.

For γ of the form (20), typicality of γ implies that W_γ is a Kac module, and therefore $\dim(W_\gamma)_{\overline{\mathbf{0}}} = 8 \dim M(b\delta_1 + b\delta_2)$. Set $N_k := \dim M(k\delta_1 + k\delta_2)$. By the Weyl character formula for $\mathfrak{sp}(4)$,

$$N_k = \frac{(2k+3)(k+2)(k+1)}{6} \text{ for } k \geq 0. \tag{21}$$

From (21), Lemma 3.4, and Lemma 3.5 it follows that

$$\begin{aligned}
\sum_{\gamma \in \mathcal{E}_d} \dim(W_\gamma)_{\overline{\mathbf{0}}} &= N_d + N_{d-1} + 8 \sum_{a=0}^{d-2} \sum_{\substack{0 \leq b \leq d-a-2 \\ b \equiv d-a \pmod 2}} \dim M(b\delta_1 + b\delta_2) \\
&= \frac{(d+1)(2d^2+4d+3)}{3} + 8 \sum_{a=0}^{d-2} \dim \mathcal{S}^{d-a-2}(\mathbb{C}^5) \\
&= \frac{(d+1)(2d^2+4d+3)}{3} + 8 \dim \mathcal{S}^{d-2}(\mathbb{C}^6) \\
&= \frac{(d+1)(2d^2+4d+3)}{3} + \frac{(d+3)(d+2)(d+1)d(d-1)}{15} = \dim \mathcal{S}^d(V^*)_{\overline{\mathbf{0}}}.
\end{aligned}$$

The relation $\sum_{\gamma \in \mathcal{E}_d} \dim(W_\gamma)_{\overline{\mathbf{0}}} = \dim \mathcal{S}^d(V^*)_{\overline{\mathbf{0}}}$ implies that if $\mathcal{S}^d(V^*)$ has an irreducible subquotient W such that $W \ncong W_\gamma$ for all $\gamma \in \mathcal{E}_d$, then $W_{\overline{\mathbf{0}}} = \{0\}$. Since $[\mathfrak{g},\mathfrak{g}]$ is generated by its odd elements, W must be a trivial $[\mathfrak{g},\mathfrak{g}]$-module. However, the $\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}]$-weights of V^* are $-\varepsilon_1 \pm \delta_1 \pm \delta_2$, $-\varepsilon_1$, $-3\varepsilon_1$, $-2\varepsilon_1 \pm \delta_1$, and $-2\varepsilon_1 \pm \delta_2$, and it is straightforward to verify that all of the $\mathfrak{h} \cap [\mathfrak{g},\mathfrak{g}]$-weights of $\mathcal{S}^d(V^*)$ are nonzero, which is a contradiction. Consequently, $\mathcal{S}^d(V^*) \cong \bigoplus_{\gamma \in \mathcal{E}_d} W_\gamma$. □

3.3. **Case VI.** For $d \geq 0$ we have an isomorphism of $\mathfrak{g}$-modules

$$\mathscr{P}^d(V) \cong \mathcal{S}^d(\Pi(\Lambda^2(\mathbb{C}^{n|n}))^*) \cong \Big(\Pi^d \Lambda^d(\Lambda^2(\mathbb{C}^{n|n}))\Big)^*. \tag{22}$$

Therefore in Case VI, Theorem 1.4 follows from the following proposition.

Proposition 3.7. *For every $d \geq 0$ we have*

$$\left(\Pi^d \Lambda^d(\Lambda^2(\mathbb{C}^{n|n}))\right)^* \cong \bigoplus_{\mu \in \mathcal{DP}_d(n)} V_\mu,$$

where V_μ denotes the $\mathfrak{gl}(n|n)$-module with $\mathfrak{b}_{n|n}^{\mathrm{mx}}$-highest weight $-\sum_{i=1}^n \mu_i(\varepsilon_i + \delta_i)$.

Proof. By Schur–Weyl–Sergeev duality,

$$(\mathbb{C}^{n|n})^{\otimes 2d} \cong \bigoplus_{\mu \in \mathcal{H}_{2d}(n,n)} E_{\mu_{n|n}^{\mathrm{st}}} \otimes F_\mu, \tag{23}$$

where $E_{\mu_{n|n}^{\mathrm{st}}}$ is the $\mathfrak{gl}(n|n)$-module with $\mathfrak{b}_{n|n}^{\mathrm{st}}$-highest weight $\mu_{n|n}^{\mathrm{st}}$ defined as in (58) and F_μ is the S_{2d}-module associated to the partition μ in the standard way.

For every $\lambda \in \mathcal{DP}(n)$ we define $\breve{\lambda} \in \mathcal{H}(n,n)$ to be the partition whose Young diagram is constructed by nesting the $(1,1)$-hooks with λ_i boxes in the first row and $\lambda_i + 1$ boxes in the first column, where $1 \leq i \leq \ell(\lambda)$. For example, if $\lambda = (4,2,1,0,\ldots)$ then $\breve{\lambda} = (4,3,3,1,0,\ldots)$. From (23) and a superized variation of the proof of [11, Thm 4.4.4] we obtain

$$\Lambda^d(\mathcal{S}^2(\mathbb{C}^{n|n})) \cong \bigoplus_{\mu \in \mathcal{DP}(n)_d} E_{(\breve{\mu})_{n|n}^{\mathrm{st}}}. \tag{24}$$

By comparing (22) and (24), we obtain that

$$\mathscr{P}^d(V) \cong \bigoplus_{\mu \in \mathcal{DP}(n)_d} W_\mu,$$

where W_μ is the $\mathfrak{gl}(n|n)$-module with $\mathfrak{b}_{n|n}^{\mathrm{op}}$-highest weight $-(\breve{\mu})_{n|n}^{\mathrm{st}}$.

Set $\lambda := \breve{\mu}$, and recall that $\langle x \rangle := \max\{x, 0\}$ for $x \in \mathbb{R}$. A straightforward calculation based on the method of [6, Sec. 2.4.1] shows that the $\mathfrak{b}_{n|n}^{\mathrm{mx}}$-highest weight of W_μ is

$$-\sum_{i=1}^n \langle \lambda_i - i + 1 \rangle \varepsilon_i - \sum_{i=1}^n \langle \lambda_i' - i \rangle \delta_i = -\sum_{i=1}^n \mu_i(\varepsilon_i + \delta_i). \qquad \square$$

4. Surjectivity of the Harish-Chandra map

This section is devoted to the proof of Theorem 1.17. The proof is divided into two parts: the case $J \not\cong F$, given in Proposition 4.2, and the case $J \cong F$, given in Proposition 4.3.

The associative superalgebra $\mathscr{PD}(V)$ has a natural filtration $\left(\mathscr{PD}^{(i)}(V)\right)_{i \geq 0}$ by order of operators. We denote the natural degree filtrations of $\mathscr{P}(\mathfrak{h}^*)$ and $\mathscr{P}(\mathfrak{a}_\Omega^*)$ by $\left(\mathscr{P}^{(i)}(\mathfrak{h}^*)\right)_{i \geq 0}$ and $\left(\mathscr{P}^{(i)}(\mathfrak{a}_\Omega^*)\right)_{i \geq 0}$, respectively.

In the following lemma, $B_t(z) := \sum_{i=0}^t \frac{1}{i+1} \sum_{j=0}^i (-1)^j \binom{i}{j} (z+j)^t$ denotes the Bernoulli polynomial of degree t.

Lemma 4.1. *Let $k, k' \in \mathbb{N}$, and let θ be a complex number such that $\theta \notin \mathbb{Q}^{\leq 0}$. For $t \in \mathbb{N}$, let $h_t(x,y,\theta) \in \Lambda_{k,k',\theta}^\natural$ be defined by*

$$h_t(x,y,\theta) := \sum_{i=1}^k B_t\left(x_i + \tfrac{1}{2}\right) + (-\theta)^{t-1} \sum_{j=1}^{k'} B_t\left(y_j + \tfrac{1}{2}\right).$$

Finally, let $\left(\Lambda^{\natural,(i)}_{k,k',\theta}\right)_{i\geq 0}$ *denote the degree filtration of* $\Lambda^{\natural}_{k,k',\theta}$*. Then for every* $d \geq 0$ *we have*

$$\Lambda^{\natural,(d)}_{k,k',\theta} = \mathrm{Span}_{\mathbb{C}} \left\{ h_1^{m_1} \cdots h_d^{m_d} \ : \ m_1, \ldots, m_d \in \mathbb{Z}^{\geq 0} \text{ and } \sum_{j=1}^{d} j m_j \leq d \right\}. \tag{25}$$

Proof. Clearly $h_t \in \Lambda^{\natural,(t)}_{k,k',\theta}$ for $t \in \mathbb{N}$. Therefore it remains to prove that in (25), the left hand side is a subset of the right hand side. For $N \in \mathbb{N}$, let $\Lambda_{N,\theta}$ denote the $\mathbb{C}$-algebra of polynomials $f(z_1, \ldots, z_N)$ which are symmetric in $z_i + \theta(1-i)$, and let $\Lambda_\theta := \varprojlim \Lambda_{N,\theta}$ be the algebra of θ-shifted symmetric functions. Let $\left(\Lambda^{(i)}_\theta\right)_{i\geq 0}$ be the degree filtration of Λ_θ. For $t \in \mathbb{N}$, let $b_t(z,\theta) \in \Lambda^{(t)}_\theta$ be defined by

$$b_t(z,\theta) := \sum_{i=1}^{\infty} \left[B_t\left(z_i + \frac{1}{2} + \theta\left(\frac{1}{2} - i\right)\right) - B_k\left(\frac{1}{2} + \theta\left(\frac{1}{2} - i\right)\right)\right].$$

In [29, Sec. 6], Sergeev and Veselov construct an epimorphism $\varphi^{\natural} : \Lambda_\theta \to \Lambda^{\natural}_{k,k',\theta}$ such that

$$\varphi^{\natural}\left(\Lambda^{(t)}_\theta\right) = \Lambda^{\natural,(t)}_{k,k',\theta} \text{ and } \deg\left(\varphi^{\natural}(b_t) - h_t\right) < t \text{ for } t \in \mathbb{N}. \tag{26}$$

Since $\mathrm{gr}(\Lambda_\theta)$ is isomorphic to the algebra of symmetric functions, the formal series $b^{\natural}_t$ constitute algebraically independent generators of Λ_θ. Consequently,

$$\Lambda^{(d)}_\theta \subseteq \mathrm{Span}_{\mathbb{C}} \left\{ b_1^{m_1} \cdots b_d^{m_d} \ : \ m_j \in \mathbb{Z}^{\geq 0} \text{ for } j \geq 1 \text{ and } \sum_{j=1}^{d} j m_j \leq d \right\}. \tag{27}$$

From (27) and (26) it follows that $\Lambda^{\natural,(d)}_{k,k',\theta}$ is a subset of the right hand side of (25). □

Proposition 4.2. *Assume that* $J \not\cong F$*. Then*

$$\mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}^{(i)}(\mathfrak{g})\right)\right) = \tau_J^*\left(\Lambda^{(i)}_J\right) \text{ for } i \geq 0. \tag{28}$$

Proof. The idea of the proof for all of the cases is similar and uses the explicit Harish-Chandra homomorphism (see for example [6, Prop. 2.25]). However, the explicit calculations in each case is different. Therefore our analysis is case by case.

Case I. In this case $\mathfrak{g} \cong \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and therefore $\mathbf{Z}(\mathfrak{g}) \cong \mathbf{Z}(\mathfrak{gl}(m|n)) \otimes \mathbf{Z}(\mathfrak{gl}(m|n))$. For $i \geq 0$ set $\mathcal{A}_i := \mathsf{j}\left(\mathbf{Z}^{(i)}(\mathfrak{gl}(m|n)) \otimes 1\right)$ and $\mathcal{B}_i := \mathsf{j}\left(1 \otimes \mathbf{Z}^{(i)}(\mathfrak{gl}(m|n))\right)$, where $\mathsf{j} : \mathbf{U}(\mathfrak{g}) \to \mathscr{PD}(V)$ is as in (11). From[24, Thm B.1] we obtain

$$\mathsf{j}(\mathbf{Z}^{(i)}(\mathfrak{g})) = \mathcal{A}_i = \mathcal{B}_i = \mathscr{PD}^{(i)}(V)^{\mathfrak{g}}. \tag{29}$$

For $a := (a_1, \ldots, a_m) \in \mathbb{C}^m$ and $b := (b_1, \ldots, b_n) \in \mathbb{C}^n$, set

$$\mu_{a,b} := \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j. \tag{30}$$

Then we have

$$\mathfrak{a}^*_\Omega = \left\{(-\mu_{a,b}, \mu_{a,b}) \ : \ a \in \mathbb{C}^m, \ b \in \mathbb{C}^n\right\}.$$

From (29) it follows that

$$\mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}^{(i)}(\mathfrak{g})\right)\right) = \mathsf{res}\left(\mathrm{HC}\left(1 \otimes \mathbf{Z}^{(i)}(\mathfrak{gl}(m|n))\right)\right).$$

Now for $k \in \mathbb{N}$ we define $f_k \in \mathscr{P}^{(k)}(\mathfrak{a}_\Omega^*)$ by

$$f_k(\mu_{a,b}) := \sum_{i=1}^{m} \left(a_i + \frac{m+1}{2} - \frac{n}{2} - i \right)^k + (-1)^{k-1} \sum_{j=1}^{n} \left(b_j + \frac{m+1}{2} + \frac{n}{2} - j \right)^k.$$

From the explicit description of the image of the Harish-Chandra homomorphism of $\mathfrak{gl}(m|n)$, given for example in [6], it follows that $\mathsf{res}\left(\mathrm{HC}\left(1 \otimes \mathbf{Z}(\mathfrak{gl}(m|n))\right)\right)$ is the $\mathbb{C}$-algebra generated by the f_k for $k \geq 1$. Furthermore, by graded-surjectivity of the Harish-Chandra homomorphism of $\mathfrak{gl}(m|n)$ (see the proof of [6, Thm 2.26]), for every $i \geq 0$ we have

$$\mathrm{HC}\big(1 \otimes \mathbf{Z}^{(i)}(\mathfrak{gl}(m|n))\big) = \mathrm{HC}\big(1 \otimes \mathbf{Z}(\mathfrak{gl}(m|n))\big) \cap \mathscr{P}^{(i)}(\mathfrak{h}^*).$$

Consequently, $f_1, \ldots, f_k \in \mathsf{res}\left(\mathrm{HC}\left(1 \otimes \mathbf{Z}^{(k)}(\mathfrak{gl}(m|n))\right)\right) = \mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}^{(k)}(\mathfrak{g})\right)\right)$ for $k \in \mathbb{N}$. Next observe that for $k \in \mathbb{N}$,

$$\left(\tau_J^*\right)^{-1}(f_k) = \sum_{i=1}^{m} x_i^k + (-1)^{k-1} \sum_{j=1}^{n} y_j^k \in \Lambda_J^{(k)}.$$

Since $\deg\left(h_k(x,y,1) - \left(\tau_J^*\right)^{-1}(f_k)\right) < k$, equality in (28) follows from Lemma 4.1.

Case II. For $a := (a_1, \ldots, a_m) \in \mathbb{C}^m$ and $b := (b_1, \ldots, b_n) \in \mathbb{C}^n$ set

$$\mu_{a,b} := \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j (\delta_{2j-1} + \delta_{2j}). \tag{31}$$

Then

$$\mathfrak{a}_\Omega^* = \{\mu_{a,b} \,:\, a \in \mathbb{C}^m,\ b \in \mathbb{C}^n\},$$

and from the explicit description of the Harish-Chandra homomorphism for $\mathfrak{gl}(m|2n)$, given for example in [6], it follows that $\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))$ is the $\mathbb{C}$-algebra generated by the polynomials $f_k \in \mathscr{P}(\mathfrak{a}_\Omega^*)$ for $k \in \mathbb{N}$, defined as

$$\begin{aligned} f_k(\mu_{a,b}) :&= \sum_{i=1}^{m} \left(a_i - \frac{m+1}{2} + n + i \right)^k \\ &\quad + (-1)^{k-1} \sum_{j=1}^{n} \left(b_j - \frac{m+1}{2} - n + 2j - 1 \right)^k + \left(b_j - \frac{m+1}{2} - n + 2j \right)^k. \end{aligned}$$

Furthermore, by graded-surjectivity of the Harish-Chandra homomorphism of $\mathfrak{g}$ (see the proof of [6, Thm 2.26]) we have $f_1, \ldots, f_k \in \mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}^{(k)}(\mathfrak{g})\right)\right)$ for $k \in \mathbb{N}$. Next observe that

$$\left(\tau_J^*\right)^{-1}(f_k) = (-2)^k \left[\sum_{i=1}^{m} x_i^k + \frac{(-1)^{k-1}}{2^k} \sum_{j=1}^{n} \left(y_j + \frac{1}{2} \right)^k + \left(y_j - \frac{1}{2} \right)^k \right] \in \Lambda_J^{(k)}.$$

Furthermore $\deg\left(\left(-\frac{1}{2}\right)^k \left(\tau_J^*\right)^{-1}(f_k) - h_k(x,y,\frac{1}{2})\right) < k$, so that Lemma 4.1 implies equality in (28).

Case III. For $a, b \in \mathbb{C}$ set

$$\mu_{a,b} := a\varepsilon_1 + b\zeta. \tag{32}$$

Then

$$\mathfrak{a}_\Omega^* := \{\mu_{a,b} \,:\, a, b \in \mathbb{C}\},$$

and from the explicit description of the Harish-Chandra homomorphism for $\mathfrak{osp}(m+1|2n)$ (see [6, Thm 2.26]) it follows that $\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g})))$ is generated by $f_1, f_2 \in \mathscr{P}(\mathfrak{a}_\Omega^*)$, defined as

$$f_1(\mu_{a,b}) := b \text{ and } f_2(\mu_{a,b}) := \left(a + \frac{m+1}{2} - n - 1\right)^2.$$

Furthermore, $f_k \in \mathsf{res}(\mathrm{HC}(\mathbf{Z}^{(k)}(\mathfrak{g})))$ for $k = 1, 2$ (f_1 lies in the image of the center of $\mathfrak{gosp}(m+1|2n)$, and f_2 lies in the image of the Casimir operator of $\mathfrak{osp}(m+1|2n)$). Next observe that

$$(\tau_J^*)^{-1}(f_1) = x_1 + x_2 \text{ and } (\tau_J^*)^{-1}\left(f_2 - f_1^2\right) = -4x_1x_2.$$

The statement now follows from the fact that $x_1 + x_2$ and x_1x_2 are algebraically independent generators of the algebra of symmetric polynomials in two variables (see [19, Sec. I.2]).

Case IV. For $a, b \in \mathbb{C}$, set

$$\mu_{a,b} := a\varepsilon_1 + b(\delta_1 + \delta_2). \tag{33}$$

Then

$$\mathfrak{a}_\Omega^* = \{\mu_{a,b} \,:\, a, b \in \mathbb{C}\},$$

and from the explicit description of the Harish-Chandra homomorphism of $\mathfrak{gl}(1|2)$ it follows that the $\mathbb{C}$-algebra $\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g})))$ is generated by $f_k \in \mathscr{P}^{(k)}(\mathfrak{a}_\Omega^*)$ for $k \in \mathbb{N}$, where

$$f_k(\mu_{a,b}) := (a+1) + (-1)^{k-1}((b-1)^k + b^k).$$

Setting $\tilde{a} := \frac{1}{2}(a+1)$ and $\tilde{b} := b - \frac{1}{2}$, we obtain that in the new coordinates $(\tilde{a}, \tilde{b})$, the polynomial f_k is equal to

$$\tilde{f}_k(\tilde{a}, \tilde{b}) := 2^k \left[\tilde{a}^k + \frac{(-1)^{k-1}}{2^k}\left(\left(\tilde{b} - \frac{1}{2}\right)^k + \left(\tilde{b} + \frac{1}{2}\right)^k\right)\right].$$

Let $W_k \subseteq \mathscr{P}^{(k)}(\mathfrak{a}_\Omega^*)$ be defined as

$$W_k := \mathrm{Span}_{\mathbb{C}}\left\{f_1^{m_1} \cdots f_k^{m_k} \,:\, m_j \in \mathbb{Z}^{\geq 0} \text{ for } j \geq 1 \text{ and } \sum_{j=1}^{k} j m_j \leq k\right\}.$$

For $k \in \mathbb{N}$, let $h_k(x, y, \frac{1}{2}) \in \Lambda^{\natural}_{1,1,\frac{1}{2}}$ be defined as in Lemma 4.1. Since

$$\deg\left(\tilde{f}_k(\tilde{a}, \tilde{b}) - h_k(\tilde{a}, \tilde{b}, \tfrac{1}{2})\right) < k \;\text{ for } k \in \mathbb{N},$$

Lemma 4.1 implies that $\dim W_k = |\mathcal{H}_k(1,1)|$. It is straightforward to verify that $(\tau_J^*)^{-1}(f_k) \in \Lambda^{\natural,(k)}_{1,1,-\frac{1}{t}}$ for $k \in \mathbb{N}$, so that $(\tau_J^*)^{-1}(W_k) \subseteq \Lambda^{\natural,(k)}_{1,1,-\frac{1}{t}}$. Since $\dim W_k = |\mathcal{H}_k(1,1)| = \dim \Lambda^{\natural,(k)}_{1,1,-\frac{1}{t}}$, we obtain $(\tau_J^*)^{-1}(W_k) = \Lambda^{\natural,(k)}_{1,1,-\frac{1}{t}}$. This completes the proof of (28).

Case VI. For $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $b := (b_1, \ldots, b_n) \in \mathbb{C}^n$ set $\tilde{\mu}_{a,b} := \sum_{i=1}^n (a_i\varepsilon_i + b_i\delta_i)$. From the explicit description of the image of the Harish-Chandra homomorphism of $\mathfrak{gl}(n|n)$ corresponding to $\mathfrak{b}^{\mathrm{mx}}_{n|n}$ it follows that $\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))$ is generated by the polynomials $\tilde{f}_k \in \mathscr{P}(\mathfrak{h}^*)$ for $k \geq 1$, where $\tilde{f}_k(\tilde{\mu}_{a,b}) := \sum_{i=1}^n (x_i + \frac{1}{2})^r + (-1)^{r-1}\sum_{i=1}^n (y_i + \frac{1}{2})^r$. For $k \in \mathbb{N}$ let $h_k(x, y, 1) \in \Lambda^{\natural}_{n,n,1}$ be defined as

in Lemma 4.1. Since $\deg(h_k - \tilde{f}_k) < k$, graded surjectivity of the Harish-Chandra homomorphism of $\mathfrak{gl}(n|n)$ (see the proof of [6, Thm 2.26]) and Lemma 4.1 imply that

$$\mathrm{HC}\left(\mathbf{Z}^{(k)}(\mathfrak{g})\right) = \mathrm{Span}_{\mathbb{C}}\left\{\tilde{f}_1^{m_1}\cdots\tilde{f}_k^{m_k} \,:\, m_j \in \mathbb{Z}^{\geq 0} \text{ for } j \geq 1 \text{ and } \textstyle\sum_{j=1}^{k} j m_j \leq k\right\}. \tag{34}$$

Next for $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ set

$$\mu_a := \sum_{i=1}^{n} a_i(\varepsilon_i + \delta_i). \tag{35}$$

Then

$$\mathfrak{a}_{\Omega}^{*} = \{\mu_a \,:\, a \in \mathbb{C}^n\}.$$

Set $f_k := \tilde{f}_k\big|_{\mathfrak{a}_{\Omega}^{*}}$, so that

$$f_k(\mu_a) := \sum_{i=1}^{n}\left(a_i + \frac{1}{2}\right)^k + (-1)^{k-1}\sum_{j=1}^{n}\left(a_i - \frac{1}{2}\right)^k.$$

Also, set $p_r(\mu_a) := \sum_{i=1}^{n} a_i^r$ for $r \geq 0$. Then f_k is a linear combination of the p_{2i-1} for $1 \leq i \leq \lfloor \frac{k+1}{2} \rfloor$. Therefore from (34) it follows that

$$\mathsf{res}(\mathrm{HC}(\mathbf{Z}^{(k)}(\mathfrak{g}))) = \mathrm{Span}_{\mathbb{C}}\left\{\prod_{i\in\mathbb{N}} p_{2i-1}^{m_i} \,:\, m_j \in \mathbb{Z}^{\geq 0} \text{ for } j \geq 1 \text{ and } \sum_{j\in\mathbb{N}}(2j-1)m_j \leq k\right\}. \tag{36}$$

The statement of the proposition is now a consequence of the fact that the right hand side of (36) is equal to $\tau_J^*\left(\Lambda_J^{(k)}\right)$ (this follows for example from [22, Thm 2.11] and [22, Rem. 2.6]).

Case VII. The argument is similar to Case I, based on the explicit description of the Harish-Chandra homomorphism of $\mathfrak{q}(n)$ (see [6, Thm 2.46]). In this case $\mathfrak{a}_{\Omega}^{*} := \{(-\mu_a, \mu_a) \,:\, a \in \mathbb{C}^n\}$, where for $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ we define

$$\mu_a := \sum_{i=1}^{n} a_i\varepsilon_i. \tag{37}$$

□

Proposition 4.3. *Assume that $J \cong F$. Then*

$$\mathsf{res}\left(\mathrm{HC}\left(\mathbf{Z}(\mathfrak{g})\right)\right) \subsetneq \tau_J^*\left(\Lambda_J\right).$$

Proof. Recall that $\mathfrak{g} \cong \mathfrak{gosp}(2|4)$. For $a, b, c \in \mathbb{C}$ set

$$\mu_{a,b,c} := a\varepsilon_1 + b\delta_1 + b\delta_2 + c\zeta. \tag{38}$$

Then

$$\mathfrak{a}_{\Omega}^{*} := \{\mu_{a,b,c} \,:\, a, b, c \in \mathbb{C}\}, \tag{39}$$

and from the explicit description of the Harish-Chandra homomorphism of $\mathfrak{osp}(2|4)$ (see for example [6, Thm 2.26]) it follows that $\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g})))$ is the $\mathbb{C}$-algebra generated by the following three families of polynomials.

(i) The polynomials f_k for $k \in \mathbb{N}$, where $f_k(\mu_{a,b,c}) := (a+2)^{2k} - (b+2)^{2k} - (b+1)^{2k}$.

(ii) The polynomials

$$F_g(\mu_{a,b,c}) := (a+2)\left((b+2)^2 - (a+2)^2\right)\left((b+1)^2 - (a+2)^2\right)\left[g\left(a+2, (b+2)^2, (b+1)^2\right)\right],$$

where $g(\mathsf{s}, \mathsf{t}_1, \mathsf{t}_2)$ is a polynomial in the variables $\mathsf{s}, \mathsf{t}_1, \mathsf{t}_2$ which is symmetric in $\mathsf{t}_1, \mathsf{t}_2$.

(iii) The polynomial $Q(\mu_{a,b,c}) := c$.

By a straightforward calculation based on the above generators we can verify that

$$(\tau_J^*)^{-1}\left(\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))\right) \subseteq \Lambda^{\natural}_{2,1,\frac{3}{2}}.$$

To complete the proof of the proposition, it suffices to show that

$$h_3(x_1,x_2,y_1,\tfrac{3}{2}) \notin (\tau_J^*)^{-1}\left(\mathsf{res}(\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))\right), \tag{40}$$

where $h_3(x_1,x_2,y_1,\frac{3}{2}) \in \Lambda^{\natural}_{2,1,\frac{3}{2}}$ is defined as in Lemma 4.1. Consider the change of coordinates

$$\begin{cases} \tilde{a} := a+b+\frac{7}{2} = 2x_1+3y_1 \\ \tilde{b} := a-b+\frac{1}{2} = 2x_2+3y_1 \\ \tilde{c} := c = x_1+x_2+y_1. \end{cases}$$

In the $(\tilde{a},\tilde{b},\tilde{c})$ coordinates, the generators f_k, F_g, and Q turn into $\tilde{f}_k$, $\tilde{F}_g$, and $\tilde{Q}$, defined below.

(i) $\tilde{f}_k(\tilde{a},\tilde{b},\tilde{c}) := \frac{1}{2^{2k}}\left[(\tilde{a}+\tilde{b})^{2k} - (\tilde{a}-\tilde{b}+1)^{2k} - (\tilde{a}-\tilde{b}-1)^{2k}\right]$.

(ii) $\tilde{F}_g(\tilde{a},\tilde{b},\tilde{c}) := \frac{1}{32}\left[(\tilde{a}+\tilde{b})(4\tilde{a}^2-1)(4\tilde{b}^2-1)\right] g\left(\frac{\tilde{a}+\tilde{b}}{2}, (\frac{\tilde{a}-\tilde{b}+1}{2})^2, (\frac{\tilde{a}-\tilde{b}-1}{2})^2\right)$.

(iii) $\tilde{Q}(\tilde{a},\tilde{b},\tilde{c}) := \tilde{c}$.

Note that the $\tilde{f}_k$ are independent of c. Also, $\tau_J^*\left(h_3(x_1,x_2,y_1,\frac{3}{2})\right)$ can be expressed in $(\tilde{a},\tilde{b},\tilde{c})$-coordinates as

$$\begin{aligned} \frac{81}{64}\tilde{c}^3 - \frac{135}{128}(\tilde{a}+\tilde{b})\tilde{c}^2 + \left(\frac{171}{256}(\tilde{a}^2+\tilde{b}^2) + \frac{27}{128}\tilde{a}\tilde{b} - \frac{51}{64}\right)\tilde{c} \\ + \left(-\frac{53}{512}(\tilde{a}^3+\tilde{b}^3) - \frac{63}{512}(\tilde{a}^2\tilde{b}+\tilde{a}\tilde{b}^2) + \frac{35}{128}(\tilde{a}+\tilde{b})\right). \end{aligned} \tag{41}$$

Now assume that (40) is not true. It follows that the polynomial (41) belongs to the $\mathbb{C}$-algebra generated by the $\tilde{f}_k$'s, the $\tilde{F}_g$'s, and $\tilde{Q}$. Since the variables $\tilde{a},\tilde{b},\tilde{c}$ are algebraically independent, the coefficient of $\tilde{c}^2$ in (41) should belong to the $\mathbb{C}$-algebra generated by the $\tilde{f}_k$ and the $\tilde{F}_g$'s. It follows that there exist polynomials $\phi(u_1,\ldots,u_N) \in \mathbb{C}[u_1,\ldots,u_N]$ for some $N \in \mathbb{N}$ and $g_\circ(\tilde{a},\tilde{b}) \in \mathbb{C}[\tilde{a},\tilde{b}]$ such that

$$(\tilde{a}+\tilde{b}) = (\tilde{a}+\tilde{b})(4\tilde{a}^2-1)(4\tilde{b}^2-1)g_\circ(\tilde{a},\tilde{b}) + \phi\left(\tilde{f}_1(\tilde{a},\tilde{b}),\ldots,\tilde{f}_N(\tilde{a},\tilde{b})\right). \tag{42}$$

Setting $\tilde{a} = \frac{1}{2}$ and $\tilde{b} = 0$ in (42), we obtain

$$\tfrac{1}{2} = \phi\left(-(\tfrac{3}{2})^2,\ldots,-(\tfrac{3}{2})^{2N}\right), \tag{43}$$

and setting $\tilde{a} = -\frac{1}{2}$ and $\tilde{b} = 0$ in (42), we obtain

$$-\tfrac{1}{2} = \phi\left(-(\tfrac{3}{2})^2,\ldots,-(\tfrac{3}{2})^{2N}\right). \tag{44}$$

Clearly (43) and (44) cannot be true simulteneously. This contradiction implies that (40) must be true. □

5. Proof of Theorem 1.13 when $J \ncong F$.

The proof of Theorem 1.13 differs in the two cases $J \ncong F$ and $J \cong F$. Indeed in the case $J \ncong F$ Proposition 4.2 allows us to give a uniform proof. Thus, we first prove Theorem 1.13 in the case $J \ncong F$, and then we give a separate argument for the case $J \cong F$. We begin with the following lemma, whose proof is similar to that of [24, Lem. 5.4]. Recall that $c_\mu(\lambda)$ for $\lambda, \mu \in \Omega$ denotes the scalar by which D_μ acts on V_λ.

Lemma 5.1. *Let $\mu \in \Omega_d$ where $d \geq 0$. Then $c_\mu(\mu) = d!$, and $c_\mu(\lambda) = 0$ for all $\lambda \in \bigcup_{k=0}^{d} \Omega_k \backslash \{\mu\}$.*

Proof. If $\lambda \in \Omega_k$ where $k < d$, then $V_\lambda \subseteq \mathscr{P}^k(V)$, and thus $c_\mu(\lambda) = 0$ because $D_\mu \mathscr{P}^k(V) = \{0\}$. Next assume $\lambda \in \Omega_d$. The map $\mathsf{m} : \mathscr{P}\mathscr{D}(V) \otimes \mathscr{P}(V) \to \mathscr{P}(V)$ defined by $D \otimes f \mapsto Df$ and the canonical isomorphism $\mathsf{m} : \mathscr{P}(V) \otimes \mathcal{S}(V) \to \mathscr{P}\mathscr{D}(V)$ are $\mathfrak{g}$-equivariant. Since $D_\mu \in \mathsf{m}(V_\mu \otimes V_\mu^*)$, the restriction of D_μ to V_λ is a $\mathfrak{g}$-equivariant map $V_\lambda \to V_\mu$, so that $c_\mu(\lambda) = 0$ unless $\lambda = \mu$.

Finally, assume $\lambda = \mu$. Let $\langle \cdot, \cdot \rangle$ be the duality pairing between $\mathscr{P}^d(V) \cong \mathcal{S}^d(V)^*$ and $\mathcal{S}^d(V)$, and let $\beta : \mathscr{P}^d(V) \times \mathcal{S}^d(V) \to \mathbb{C}$ be the bilinear map $\beta(a, b) := \partial_b(a)$, where

$$\partial : \mathcal{S}(V) \to \mathscr{D}(V)$$

is the canonical isomorhpism between the symmetric algebra $\mathcal{S}(V)$ and the algebra of constant coefficient differential operators $\mathscr{D}(V)$. A direct calculation shows that $\beta(\cdot, \cdot) = d!\langle \cdot, \cdot \rangle$. Now choose a basis $v_1, \ldots, v_t$ for V_μ and a dual basis $v_1^*, \ldots, v_t^*$ for V_μ^*. Then

$$D_\mu v_k = \mathsf{m}(\textstyle\sum_{i=1}^t v_i \otimes v_t^*) v_k = \sum_{i=1}^t v_i \partial_{v_i^*} v_k = d! v_k. \qquad \square$$

Recall the maps

$$\mathsf{j} : \mathbf{U}(\mathfrak{g}) \to \mathscr{P}\mathscr{D}(V), \quad \mathrm{HC} : \mathbf{U}(\mathfrak{g}) \to \mathscr{P}(\mathfrak{h}^*), \text{ and } \mathsf{res} : \mathscr{P}(\mathfrak{h}^*) \to \mathscr{P}(\mathfrak{a}_\Omega^*),$$

defined in (11), (12), and (13). For $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ and $\lambda \in \Omega$ let $\overline{\mathrm{HC}}(D)(\underline{\lambda})$ denote the scalar by which D acts on the irreducible $\mathfrak{g}$-module $V_\lambda \subseteq \mathscr{P}(V)$ whose $\mathfrak{b}$-highest weight is $\underline{\lambda}$. Then we have

$$\overline{\mathrm{HC}}(\mathsf{j}(z)) = \mathsf{res}(\mathrm{HC}(z)) \text{ for } z \in \mathbf{Z}(\mathfrak{g}). \tag{45}$$

Recall that by $\left(\mathscr{P}_J^{(i)}\right)_{i \geq 0}$ we denote the standard degree filtration of the polynomial algebra $\mathscr{P}_J$ defined in (9). Let $\tau_J : \mathfrak{a}_\Omega^* \to \mathbb{C}^{n_J}$ and $\tau_J^* : \mathscr{P}_J \to \mathscr{P}(\mathfrak{a}_\Omega^*)$ be defined as in (10) and (14), respectively. Since τ_J is a bijection, τ_J^* is an isomorphism of $\mathbb{C}$-algebras.

If $J \ncong F$, then by Proposition 4.2, for every $\mu \in \Omega_d$ there exists an element $z_\mu \in \mathbf{Z}^{(d)}(\mathfrak{g})$ such that

$$\mathsf{res}(\mathrm{HC}(z_\mu)) = \tau_J^*\left(P_{J,\mu}\right),$$

where $P_{J,\mu}$ is as in Definition 1.11. Theorem 1.13 follows from Proposition 5.2(iii).

Proposition 5.2. *Assume that $J \ncong F$. Then the following assertions hold.*

(i) $\mathsf{j}(z_\mu) = D_\mu$ *for all* $\mu \in \Omega$.
(ii) $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) = \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$.
(iii) $\overline{\mathrm{HC}}(D_\mu)(\underline{\lambda}) = P_{J,\mu}(\tau_J(\underline{\lambda}))$ *for all* $\lambda, \mu \in \Omega$.

Proof. (i) By a direct computation, from Theorem 1.8 and Theorem 1.10 it follows that $P_{J,\mu}\left(\tau_J(\underline{\mu})\right) = d!$ and

$$P_{J,\mu}\left(\tau_J(\underline{\lambda})\right) = 0 \text{ for all } \lambda \in \bigcup_{k=0}^{d} \Omega_k \backslash \{\mu\}.$$

Set $D'_\mu := \mathsf{j}(z_\mu)$. Then $D'_\mu \in \mathscr{P}\mathscr{D}^{(d)}(V)^{\mathfrak{g}}$ because the map j preserves the filtrations. From (45) it follows that for $\lambda \in \bigcup_{k=0}^{d} \Omega_k$, the operator D'_μ acts on V_λ by the scalar $P_J(\tau_J(\underline{\lambda}))$. Since elements of $\mathscr{P}\mathscr{D}^{(d)}(V)$ are uniquely determined by their restrictions to $\mathscr{P}^d(V)$, Lemma 5.1 implies that $D_\mu = D'_\mu$.

(ii) Since the family $(D_\mu)_{\mu\in\Omega}$ is a basis of $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$, the above argument implies that $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) = \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$.

(iii) This follows immediately from the fact that $D_\mu = D'_\mu$. □

We remark that Proposition 5.2(ii) does not hold when $J \cong F$. This is proved in Proposition 5.3.

Proposition 5.3. *Assume that $J \cong F$. Then $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) \subsetneq \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$*

Proof. Suppose that $\mathsf{j}(\mathbf{Z}(\mathfrak{g})) = \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$. Then from (45) and the explicit description of the image of the Harish-Chandra homomorphism of $\mathfrak{osp}(2|4)$ it follows that

$$\overline{\mathrm{HC}}\left(\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}\right) = \overline{\mathrm{HC}}\left(\mathsf{j}\left(\mathbf{Z}(\mathfrak{g})\right)\right) = \mathsf{res}\left(\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))\right) \subseteq \tau_J^*(\Lambda_J). \tag{46}$$

Fix $d \geq 0$ and set $\Omega_{\leq d} := \bigcup_{k=0}^{d} \Omega_k$. By Proposition C.6 we have $\overline{\mathrm{HC}}(D_\lambda) \in \mathscr{P}^{(d)}(\mathfrak{a}_\Omega^*)$ for $\lambda \in \Omega_{\leq d}$, and therefore from (46) it follows that $\overline{\mathrm{HC}}(D_\lambda) \in \tau_J^*\left(\Lambda_J^{(d)}\right)$ for $\lambda \in \Omega_{\leq d}$. Recall that elements of $\mathscr{P}\mathscr{D}(V)$ are uniquely determined by their restrictions to $\mathscr{P}(V)$. Since the Capelli operators $(D_\lambda)_{\lambda\in\Omega_{\leq d}}$ are linearly independent, it follows that the polynomials $\left(\overline{\mathrm{HC}}(D_\lambda)\right)_{\lambda\in\Omega_{\leq d}}$ are also linearly independent. Since $|\Omega_{\leq d}| = \sum_{k=0}^{d} |\mathcal{H}_k(1,2)| = \dim \Lambda_J^{(d)}$, it follows that the family $\left(\overline{\mathrm{HC}}(D_\lambda)\right)_{\lambda\in\Omega_{\leq d}}$ also spans $\tau_J^*\left(\Lambda_J^{(d)}\right)$. Since d can be any non-negative integer, from (45) it follows that

$$\tau_J^*(\Lambda_J) = \bigcup_{d=0}^{\infty} \tau_J^*(\Lambda_J^{(d)}) \subseteq \mathsf{res}\left(\overline{\mathrm{HC}}(\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}})\right) = \mathsf{res}\left(\mathrm{HC}(\mathbf{Z}(\mathfrak{g}))\right),$$

which contradicts Proposition 4.3. □

6. Proof of Theorem 1.13 when $J \cong F$

Recall that in this case $\mathfrak{g} \cong \mathfrak{gosp}(2|4)$, and $\mathfrak{a}_\Omega^*$ is the subspace of $\mathfrak{h}^*$ that is given in (39). Let $\sigma : \mathbb{C}^3 \to \mathfrak{a}_\Omega^*$ be defined by $\sigma(a,b,c) := \mu_{a,b,c}$, where $\mu_{a,b,c}$ is defined in (38). We define the map $\sigma^* : \mathscr{P}(\mathfrak{a}_\Omega^*) \to \mathscr{P}(\mathbb{C}^3) \cong \mathbb{C}[a,b,c]$ by $\sigma^*(f) := f \circ \sigma_F$.

Lemma 6.1. *Assume that $J \cong F$. Then $\sigma^*(\tau_J^*(\Lambda_J))$ is the subalgebra of $\mathbb{C}[a,b,c]$ consisting of polynomials $f(a,b,c)$ which satisfy the following two properties.*

(i) $f(a,b,c) = f(a,-b-3,c)$.

(ii) $f(a+1,b+\frac{1}{2},c) = f(a-1,b-\frac{1}{2},c)$ *on the affine hyperplane* $a-b+\frac{1}{2}=0$.

Proof. This is straightforward from the explicit description of τ_J given in Table 3. □

Now for $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ we have $\sigma^*\left(\overline{\mathrm{HC}}(D)\right) \in \mathbb{C}[a,b,c]$. We have a direct sum decomposition $\mathfrak{g} \cong \mathfrak{g}' \oplus \mathfrak{z}$ where $\mathfrak{g}' := [\mathfrak{g},\mathfrak{g}] \cong \mathfrak{osp}(2|4)$ and $\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) \cong \mathbb{C}$ is the centre of $\mathfrak{g}$.

From now on, let $\mathcal{G}$ denote the simply connected complex Lie supergroup corresponding to $\mathfrak{g}$, and let $\mathcal{V}$ be the affine superspace corresponding to V. The $\mathfrak{g}$-action on V lifts to a $\mathcal{G}$-action on $\mathcal{V}$. Set $\mathcal{K} := \mathrm{stab}_{\mathcal{G}}(e)$ (see Appendix C). Since $\mathfrak{z}$ is $\mathcal{K}$-invariant,

$$\mathbf{U}(\mathfrak{g})^{\mathcal{K}} \cong \left(\mathbf{U}(\mathfrak{g}') \otimes \mathbf{U}(\mathfrak{z})\right)^{\mathcal{K}} \cong \mathbf{U}(\mathfrak{g}')^{\mathcal{K}} \otimes \mathbf{U}(\mathfrak{z}).$$

Remark C.5 implies that every $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ can be realized as an element of $\mathscr{D}(\mathcal{G}/\mathcal{K})$. Let $\Psi_{\mathcal{G},\mathcal{K}}$ be defined as in (61). Since there is a $\mathcal{K}$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$, Proposition C.1 implies that every $D \in \mathscr{D}(\mathcal{G}/\mathcal{K})$ lies in $\Psi_{\mathcal{G},\mathcal{K}}\left(\mathbf{U}(\mathfrak{g})^{\mathcal{K}}\right)$.

Lemma 6.2. *Assume that $J \cong F$. Let $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$.*

(i) *If $D\big|_{\mathcal{G}/\mathcal{K}} \in \Psi_{\mathcal{G},\mathcal{K}}\left(1 \otimes \mathbf{U}(\mathfrak{z})\right)$, then $\sigma^*\left(\overline{\mathrm{HC}}(D)\right) \in \mathbb{C}[c]$.*

(ii) *If $D\big|_{\mathcal{G}/\mathcal{K}} \in \Psi_{\mathcal{G},\mathcal{K}}\left(\mathbf{U}(\mathfrak{g}')^{\mathcal{K}} \otimes 1\right)$, then $\sigma^*\left(\overline{\mathrm{HC}}(D)\right) \in \mathbb{C}[a,b]$.*

Proof. The proof of (i) is straightforward. For (ii), let $\mathcal{G}'$ be the subsupergroup of $\mathcal{G}$ corresponding to $\mathfrak{g}' \subseteq \mathfrak{g}$. For $\tilde{X} \in \mathbf{U}(\mathfrak{g})^{\mathcal{K}}$, let $\mathrm{L}_{\tilde{X}}$ denote the left invariant differential operator induced by $\tilde{X}$ on $\mathcal{G}/\mathcal{K}$ (see Appendix C). Similarly, any $\tilde{X} \in \mathbf{U}(\mathfrak{g}')^{\mathcal{K}}$ induces a left invariant differential operator on $\mathcal{G}'/\mathcal{K}$, which we denote by $\mathrm{L}_{\tilde{X}}$ as well. For $\tilde{X} \in \mathbf{U}(\mathfrak{g}')^{\mathcal{K}} \subseteq \mathbf{U}(\mathfrak{g})^{\mathcal{K}}$, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{G}/\mathcal{K}}(G/K) & \xrightarrow{\phi \mapsto \phi|_{\mathcal{G}'/\mathcal{K}}} & \mathcal{O}_{\mathcal{G}'/\mathcal{K}}(G'/K) \\ {\scriptstyle \mathrm{L}_{\tilde{X}}}\downarrow & & \downarrow{\scriptstyle \mathrm{L}_{\tilde{X}}} \\ \mathcal{O}_{\mathcal{G}/\mathcal{K}}(G/K) & \xrightarrow[\phi \mapsto \phi|_{\mathcal{G}'/\mathcal{K}}]{} & \mathcal{O}_{\mathcal{G}'/\mathcal{K}}(G'/K) \end{array}$$

is commutative. According to Remark C.5, there is a $\mathcal{G}$-equivariant embedding

$$\mathfrak{p}_e^{\#}(V_{\overline{0}})\big|_{\mathscr{P}(V)} : \mathscr{P}(V) \hookrightarrow \mathcal{O}_{\mathcal{G}/\mathcal{K}}(G/K).$$

For $\lambda \in \Omega$, let $p_\lambda \in V_\lambda \subseteq \mathscr{P}(V)$ be a $\mathfrak{b}$-highest weight vector and set $\phi_\lambda := \mathfrak{p}_e^{\#}(V_{\overline{0}})(p_\lambda)$. Since $\phi_\lambda \neq 0$ and p_λ is a homogeneous element of $\mathscr{P}(V)$, we obtain $\phi_\lambda\big|_{\mathcal{G}'/\mathcal{K}} \neq 0$. Set $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{g}'$. From Proposition C.6 and the fact that $\mathfrak{g}' = \mathfrak{k} + (\mathfrak{b} \cap \mathfrak{g}')$ it follows that D acts on V_λ is by the scalar $\underline{\lambda}(\tilde{D}_{\mathfrak{h}'})$ for some $\tilde{D}_{\mathfrak{h}'} \in \mathbf{U}(\mathfrak{h}')$ that only depends on D. Since the map $\underline{\lambda} \mapsto \underline{\lambda}(\tilde{D}_{\mathfrak{h}'})$ only depends on $\underline{\lambda}\big|_{\mathfrak{h}'}$, it is indeed an element of $\mathscr{P}(\mathfrak{a}'^*)$, where $\mathfrak{a}'^* := \{\mu_{a,b,0} : a, b \in \mathbb{C}\}$, for $\mu_{a,b,c}$ defined as in (38). Finally, to complete the proof observe that $\sigma^*\left(\mathscr{P}(\mathfrak{a}'^*)\right) = \mathbb{C}[a,b]$. □

Proposition 6.3. *Let $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$. If $D\big|_{\mathcal{G}/\mathcal{K}} \in \Psi_{\mathcal{G},\mathcal{K}}(\mathbf{U}(\mathfrak{g}')^{\mathcal{K}} \otimes 1)$, then $h := \sigma^*\left(\overline{\mathrm{HC}}(D)\right)$ satisfies the relation $h(a,b) = h(a,-b-3)$.*

Proof. Let us denote the one-dimensional $\mathfrak{so}(2)$-module with weight $k\varepsilon_1$ by $M(k\varepsilon_1)$, and the irreducible $\mathfrak{sp}(4)$-module with $\mathfrak{b} \cap \mathfrak{sp}(4)$-highest weight $k_1\delta_1 + k_2\delta_2$ by $M'(k_1\delta_1 + k_2\delta_2)$. We have $V^* \cong V_{\overline{0}}^* \oplus V_{\overline{1}}^*$, where

$$V_{\overline{0}}^* \cong M(\varepsilon_1) \otimes M'(\delta_1 + \delta_2) \oplus M(3\varepsilon_1) \otimes M'(0) \quad \text{and} \quad V_{\overline{1}}^* \cong M(2\varepsilon) \otimes M'(\delta_1),$$

as $\mathfrak{so}(2) \oplus \mathfrak{sp}(4)$-modules. Set

$$U^* := M(\varepsilon_1) \otimes M'(\delta_1 + \delta_2) \quad \text{and} \quad W^* := M(3\varepsilon_1) \otimes M'(0) \oplus M(2\varepsilon) \otimes M'(\delta_1).$$

Then $V^* \cong U^* \oplus W^*$, from which we obtain a natural tensor product decomposition

$$\mathscr{P}(V) \cong \mathscr{P}(U) \otimes \mathscr{P}(W). \tag{47}$$

Using (47), we identify $\mathscr{P}(U)$ and $\mathscr{P}(W)$ with subalgebras $\mathscr{P}(U) \otimes 1$ and $1 \otimes \mathscr{P}(W)$ of $\mathscr{P}(V)$. By dualizing the relation $V^* \cong U^* \oplus W^*$ we obtain a direct sum decomposition $V \cong U \oplus W$ for subspaces U, W of V. The latter direct sum decomposition yields a tensor product decomposition

$$\mathscr{D}(V) \cong \mathscr{D}(U) \otimes \mathscr{D}(W), \tag{48}$$

which allows us to identify $\mathscr{D}(U)$ and $\mathscr{D}(W)$ with subalgebras of $\mathscr{D}(V)$.

It is straightforward to verify that $U^* = (V^*)^{\mathfrak{b}_{\overline{1}}}$. Thus every $\mathfrak{b}_{\overline{0}}$-invariant vector in $\mathscr{P}(U) \subset \mathscr{P}(V)$ is also a $\mathfrak{b}$-highest weight vector in $\mathscr{P}(V)$. By Lemma 3.5, we have an $\mathfrak{so}(2) \oplus \mathfrak{sp}(4)$-module isomorphism

$$\mathscr{P}^k(U) \cong \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} M(k\varepsilon_1) \otimes M'\big((k-2i)\delta_1 + (k-2i)\delta_2\big).$$

Recall that $\mu_{a,b,c}$ denotes the element of $\mathfrak{a}_\Omega^*$ defined in (38). The Zariski closure of the set

$$S := \left\{ k\varepsilon_1 + (k-2i)\delta_1 + (k-2i)\delta_2 \,:\, i,k \in \mathbb{Z}^{\geq 0} \text{ and } 0 \leq i \leq \lfloor \frac{k}{2} \rfloor \right\}$$

is equal to $\{\mu_{a,b,0} \,:\, a,b \in \mathbb{C}\}$.

Let $\mathscr{P}^+(W)$ and $\mathscr{D}^+(W)$ denote the augmentation ideals of $\mathscr{P}(W)$ and $\mathscr{D}(W)$, respectively. Using (47), we obtain a decomposition

$$\mathscr{P}\mathscr{D}(V) \cong \mathscr{P}\mathscr{D}(U) \oplus \mathscr{P}\mathscr{D}(U)^\perp, \tag{49}$$

where

$$\mathscr{P}\mathscr{D}(U)^\perp := \mathscr{P}\mathscr{D}(V)\mathscr{D}^+(W) + \mathscr{P}^+(W)\mathscr{P}\mathscr{D}(U).$$

Write $D := D_U \oplus D_U^\perp$, where $D_U \in \mathscr{P}\mathscr{D}(U)$ and $D_U^\perp \in \mathscr{P}\mathscr{D}(U)^\perp$. For $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$, choose a nonzero vector

$$v_{k,i} \in M(k\varepsilon_1) \otimes M'\big((k-2i)\delta_1 + (k-2i)\delta_2\big).$$

Then $Dv_{k,i} = c_{k,i}v_{k,i}$ for some scalar $c_{k,i} \in \mathbb{C}$, hence $D_U^\perp v_{k,i} = Dv_{k,i} - D_U v_{k,i} \in \mathscr{P}(U)$. Since $\mathscr{D}^+(W)v_{k,i} = 0$, we obtain $D_U^\perp v_{k,i} \in \mathscr{P}(U) \cap (\mathscr{P}^+(W)\mathscr{P}(V)) = \{0\}$, so that $Dv_{k,i} = D_U v_{k,i}$. Since the decomposition (49) is $\mathfrak{g}_{\overline{0}}$-invariant, we have $D_U \in \mathscr{P}\mathscr{D}(U)^{\mathfrak{g}_{\overline{0}}}$. Therefore from [12, Sec. 11.4] it follows that D_U lies in the algebra generated by the degree operator and the image of the Casimir operator of $\mathfrak{sp}(4)$. Let $f_1, f_2 \in \mathscr{P}(\mathfrak{a}_\Omega^*)$ denote the eigenvalues of the degree and Casimir operators, respectively, and set $h_i := \sigma^*(f_i)$ for $i = 1, 2$. The degree operator acts on $v_{k,i}$ by the scalar k. It follows that $h_1(a,b) = a$, and therefore $h_1(a,b) = h_1(a,-b-3)$. Similarly, the Casimir operator of $\mathfrak{sp}(4)$ acts on $v_{k,i}$ by the scalar $(k+2)^2+(k+1)^2$, so that $h_2(a,b) := (b+2)^2+(b+1)^2$, and therefore $h_2(a,b) = h_2(a,-b-3)$. The statement of the proposition follows from the fact that h belongs to the subalgebra of $\mathbb{C}[a,b]$ generated by h_1 and h_2. □

Proposition 6.4. *Let $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$. If $D\big|_{\mathcal{G}/\mathcal{K}} \in \Psi_{\mathcal{G},\mathcal{K}}(\mathbf{U}(\mathfrak{g}')^{\mathcal{K}} \otimes 1)$, then $h := \sigma^*\left(\overline{\mathrm{HC}}(D)\right)$ satisfies the relation $h(a+1, b+\frac{1}{2}) = h(a-1, b-\frac{1}{2})$ for all $a, b \in \mathbb{C}$ such that $a - b + \frac{1}{2} = 0$.*

Proof. Let $\mathfrak{b}_{2|4} \subseteq \mathfrak{g}$ be the Borel subalgebra defined as in (59). Note that $\mathfrak{b}_{2|4} \neq \mathfrak{b}$. For $\lambda \in \Omega$, let $\underline{\lambda}$ denote the $\mathfrak{b}_{2|4}$-highest weight of V_λ. The Borel subalgebra $\mathfrak{b}$ can be obtained from $\mathfrak{b}_{2|4}$ by the composition of the odd reflections given in (18). Thus from [6, Lem. 1.40] it follows that if $\underline{\lambda}$ is one of the typical highest weights of the form given in (19), then $\underline{\underline{\lambda}} := \underline{\lambda} + 4\varepsilon_1$, whereas if $\underline{\lambda} = d\varepsilon_1 + d\delta_1 + d\delta_2 + d\zeta$, then $\underline{\underline{\lambda}} = \underline{\lambda} + 2\varepsilon_1 - \delta_1 - \delta_2$. In particular, the $\mathfrak{b}_{2|4}$-highest weight $\underline{\underline{\lambda}}$ always lies in $\mathfrak{a}_\Omega^*$, where $\mathfrak{a}_\Omega^*$ is the subspace of $\mathfrak{h}^*$ given in (39).

Now for every $\mathfrak{b}_{2|4}$-highest weight $\underline{\underline{\lambda}} \in \mathfrak{a}_\Omega^*$, let $f(\underline{\underline{\lambda}})$ denote the scalar by which D acts on V_λ. We have $\mathfrak{g} = \mathfrak{b}_{2|4} + \mathfrak{k}^{\mathrm{ex}}$, and thus by Proposition C.6 the map $\underline{\underline{\lambda}} \mapsto f(\underline{\underline{\lambda}})$ can be extended to an element of $\mathscr{P}(\mathfrak{a}_\Omega^*)$. Set $h_1 := \sigma^*(f)$. From Lemma 6.2(ii) it follows that indeed $h_1 \in \mathbb{C}[a,b]$. Since the eigenvalue of D on V_λ is independent of the choice of the Borel subalgebra, from typical highest weights we obtain

$$h_1(a+4, b) = h(a,b) \text{ for } a, b \in \mathbb{C}, \tag{50}$$

and from the atypical highest weights, which correspond to the partitions $\lambda := (d, 0, \ldots)$, we obtain

$$h_1(a+2, a-1) = h(a, a) \text{ for } a \in \mathbb{C}. \tag{51}$$

Consequently, if $a - b + \frac{1}{2} = 0$ then $a + 1 = b + \frac{1}{2}$ and thus from (50) and (51) it follows that

$$h(a+1, b+\tfrac{1}{2}) = h_1(a+3, b-\tfrac{1}{2}) = h(a-1, b-\tfrac{1}{2}). \qquad \square$$

6.1. **Proof of Theorem 1.13 when $J \cong F$.** Fix $\mu \in \Omega_d$ for $d \geq 0$. Then $D_\mu \in \mathscr{P}\mathscr{D}^{(d)}(V)^{\mathfrak{g}}$ and therefore by Proposition C.6 there exists $h \in \mathscr{P}^d(\mathfrak{a}_\Omega^*)$ such that for every $\lambda \in \Omega$, the operator D_μ acts on V_λ by the scalar $h(\underline{\lambda})$. By Lemma 5.1 we have

$$h(\underline{\mu}) = d! \text{ and } h(\underline{\lambda}) = 0 \text{ for } \lambda \in \bigcup_{k=0}^{d} \Omega_k \backslash \{\lambda\}. \tag{52}$$

Let $f \in \mathscr{P}_J^{(d)}$ be defined by $f := (\tau_J^*)^{-1}(h)$. From Lemma 6.1, Lemma 6.2, Proposition 6.3, and Proposition 6.4 it follows that $f \in \Lambda_J^{(d)}$. Furthermore,

$$\tau_J(\underline{\lambda}) = \left(\lambda_1 + \tfrac{1}{4}, \lambda_2 - \tfrac{5}{4}, d - \lambda_1 - \lambda_2 + 1\right) = \left(\mathsf{p}_1(\lambda), \mathsf{p}_2(\lambda), \mathsf{q}_1(\lambda)\right), \tag{53}$$

where $\mathsf{p}_1(\lambda)$, $\mathsf{p}_2(\lambda)$, and $\mathsf{q}_1(\lambda)$ are defined as in (8) for $m = 2$, $n = 1$, and $\theta = \frac{3}{2}$. From (52) and (53) it follows that f and $P_{J,\mu}$ satisfy the same degree, symmetry, and vanishing properties, so that $f = P_{J,\mu}$ by Theorem 1.8.

Appendix A. The TKK construction

Recall that a vector superspace $J := J_{\overline{0}} \oplus J_{\overline{1}}$ is called a *Jordan superalgebra* if it is equipped with a supercommutative bilinear product $J \times J \to J$ which satisfies the Jordan identity

$$(-1)^{|x||z|}[L_x, L_{yz}] + (-1)^{|y||x|}[L_y, L_{zx}] + (-1)^{|z||y|}[L_z, L_{xy}] = 0 \text{ for homogeneous } x, y, z \in J,$$

where we define $L_a : J \to J$ for $a \in J$ to be the left multiplication map $x \mapsto ax$, and denote the parity of a homogeneous element $a \in J$ by $|a|$.

Following [3], by a *short grading* of a Lie superalgebra $\mathfrak{l}$ we mean a $\mathbb{Z}$-grading of $\mathfrak{l}$ of the form $\mathfrak{l} := \bigoplus_{t\in\mathbb{Z}} \mathfrak{l}(t)$, such that $\mathfrak{l}(t) = \{0\}$ for $t \notin \{0, \pm 1\}$. Using the Kantor functor, in [16] Kac associates to J a simple Lie superalgebra $\mathfrak{g}_J$ (the *TKK Lie superalgebra*) with a short grading

$$\mathfrak{g}_J := \mathfrak{g}_J(-1) \oplus \mathfrak{g}_J(0) \oplus \mathfrak{g}_J(1).$$

We recall the definition of $\mathfrak{g}_J$. Set $\mathfrak{g}_J(-1) := J$, $\mathfrak{g}_J(0) := \mathrm{Span}_{\mathbb{C}}\{L_a, [L_a, L_b] : a, b \in J\} \subseteq \mathrm{End}_{\mathbb{C}}(J)$, and $\mathfrak{g}_J(1) := \mathrm{Span}_{\mathbb{C}}\{P, [L_a, P] : a \in J\} \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}^2(J), J)$, where $P : \mathcal{S}^2(J) \to J$ is the map $P(x, y) := xy$, and $[L_a, P](x, y) := a(xy) - (ax)y - (-1)^{|x||y|}(ay)x$. The Lie superbracket of $\mathfrak{g}_J$ is defined by the following relations.

(i) $[A, a] := A(a)$ for $A \in \mathfrak{g}_J(0)$ and $a \in \mathfrak{g}_J(-1)$.
(ii) $[A, a](x) := A(a, x)$ for $A \in \mathfrak{g}_J(1)$, $a \in \mathfrak{g}_J(-1)$, and $x \in J$.
(iii) $[A, B](x, y) := A(B(x, y)) - (-1)^{|A||B|}B(A(x), y) - (-1)^{|A||B|+|x||y|}B(A(y), x)$ for $A \in \mathfrak{g}_J(0)$, $B \in \mathfrak{g}_J(1)$, and $x, y \in J$.

For the classification of finite dimensional complex simple Jordan superalgebras and their corresponding TKK Lie superalgebras, see the articles by Kac [16] and Cantarini–Kac [3].

If J has a unit $1_J \in J$, then the elements $e := 1_J$, $f := -2P$, and $h := 2L_{1_J}$ of $\mathfrak{g}_J$ satisfy (1). It follows that $\mathfrak{s} := \mathrm{Span}_{\mathbb{C}}\{e, f, h\}$ is a subalgebra of $\mathfrak{g}_J$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Indeed $\mathfrak{s}$ is a *short subalgebra* of $\mathfrak{g}_J$ (see [3]). We recall the definition of a short subalgebra.

Definition A.1. Let $\mathfrak{l}$ be a complex Lie superalgebra. A *short subalgebra* of $\mathfrak{l}$ is a Lie subalgebra $\mathfrak{a} \subseteq \mathfrak{l}_{\overline{0}}$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, with a basis e, f, h that satisfies the relations (1), such that the eigenspace decomposition of ad $\left(-\frac{1}{2}h\right)$ defines a short grading of $\mathfrak{l}$.

Remark A.2. Let $\mathfrak{l}$ be a complex Lie superalgebra and let $\mathfrak{a} \subseteq \mathfrak{l}_{\overline{0}}$ be a short subalgebra of $\mathfrak{l}$.

(a) Assume that $\mathfrak{l}$ is a subalgebra of another Lie superalgebra $\tilde{\mathfrak{l}}$ such that $\dim \tilde{\mathfrak{l}} = \dim \mathfrak{l} + 1$. Since every finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module is completely reducible, it follows that $\tilde{\mathfrak{l}} \cong \mathfrak{l} \oplus \mathbb{C}$ as $\mathfrak{a}$-modules, so that $\mathfrak{a}$ is a short subalgebra of $\tilde{\mathfrak{l}}$ as well.

(b) Every central extension $0 \to \mathbb{C} \to \hat{\mathfrak{l}} \to \mathfrak{l} \to 0$ splits on $\mathfrak{a}$. An argument similar to part (a) implies that the image of $\mathfrak{a}$ under the splitting section is a short subalgebra of $\hat{\mathfrak{l}}$.

When J is isomorphic to $gl(m,n)_+$, $p(n)_+$, or $q(n)_+$, it will be more convenient for us to replace $\mathfrak{g}_J$ by a non-simple Lie superalgebra which has a more natural matrix realization (see also Remark A.3). To this end, we define the Lie superalgebra $\mathfrak{g}^\flat$ by

$$\mathfrak{g}^\flat := \begin{cases} \mathfrak{gl}(2m|2n) & \text{if } J \cong gl(m,n)_+, \\ \mathfrak{p}(2n) & \text{if } J \cong p(n)_+, \\ \mathfrak{q}(2n) & \text{if } J \cong q(n)_+, \\ \mathfrak{g}_J & \text{otherwise.} \end{cases} \tag{54}$$

For a precise description of $\mathfrak{p}(2n)$ and $\mathfrak{q}(2n)$ see Appendix B. From Remark A.2 it follows that the short subalgebra $\mathfrak{s}$ of $\mathfrak{g}_J$ corresponds to a unique short subalgebra of $\mathfrak{g}^\flat$. We use the same symbols $\mathfrak{s}$, e, f, and h for denoting the short subalgebra of $\mathfrak{g}^\flat$ and its corresponding basis.

By restriction of the adjoint representation, $\mathfrak{g}^\flat$ is equipped with an $\mathfrak{s}$-module structure. This $\mathfrak{s}$-module is a direct sum of trivial and adjoint representations of $\mathfrak{s}$, hence it integrates to a representation of the adjoint group $\mathrm{PSL}_2(\mathbb{C})$. Furthermore,

$$w := \exp(\mathrm{ad}(f)) \exp(-\mathrm{ad}(e)) \exp(\mathrm{ad}(f)) \tag{55}$$

represents the nontrivial element of the Weyl group of $\mathrm{PSL}_2(\mathbb{C})$.

Set $\mathfrak{g}^\flat(t) := \{x \in \mathfrak{g}^\flat : [h,x] = -2tx\}$ for $t \in \{0, \pm 1\}$. The Lie superalgebra $\mathfrak{g}^\flat(0)$ naturally acts on $\mathfrak{g}^\flat(-1) \cong J$. Set

$$\mathfrak{g} := \mathfrak{g}^\flat(0) \text{ and } V := \mathfrak{g}^\flat(-1)^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}^{1|0}).$$

Thus the $\mathfrak{g}$-module V is the dual of the $\mathfrak{g}$-module J.

Remark A.3. The reason for replacing $\mathfrak{g}_J$ by $\mathfrak{g}^\flat$ is to obtain a convenient way of associating partitions to the irreducible $\mathfrak{g}$-modules that occur in $\mathscr{P}(V)$. For example, assume that $J \cong gl(m,n)_+$, where $gl(m,n)_+$ denotes the Jordan superalgebra of $(m+n) \times (m+n)$ matrices in (m,n)-block form. Then $\mathfrak{g}_J \cong \mathfrak{sl}(2m|2n)$ if $m \neq n$, and $\mathfrak{g}_J \cong \mathfrak{psl}(2m|2n)$ if $m = n$. In both cases, $\mathfrak{g}_J(0)$ is closely related to $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, but it is *not* isomorphic to it. However, $\mathfrak{g} := \mathfrak{g}^\flat(0) \cong \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, and the irreducible summands of $\mathscr{P}(V) \cong \mathscr{P}((\mathbb{C}^{m|n})^* \otimes \mathbb{C}^{m|n})$ are naturally parametrized by (m,n)-hook partitions.

Appendix B. Classical Lie superalgebras

In this Appendix we give explicit realizations of classical Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{gosp}(m|2n)$, $\mathfrak{p}(n)$, and $\mathfrak{q}(n)$. We describe root systems of $\mathfrak{gl}(m|n)$, $\mathfrak{gosp}(m|2n)$, and $\mathfrak{q}(n)$, and choose Borel subalgebras in these Lie superalgebras.

B.1. **The Lie superalgebra $\mathfrak{gl}(m|n)$.** Let $m, n \geq 1$ be integers. We use the usual realization of $\mathfrak{gl}(m|n)$ as $(m+n) \times (m+n)$ matrices in (m,n)-block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{56}$$

where A is $m \times m$ and D is $n \times n$. The diagonal Cartan subalgebra of $\mathfrak{gl}(m|n)$ is

$$\mathfrak{h}_{m|n} := \{\mathrm{diag}(\mathbf{s}, \mathbf{t}) \,:\, \mathbf{s} := (s_1, \ldots, s_m) \in \mathbb{C}^m \text{ and } \mathbf{t} := (t_1, \ldots, t_n) \in \mathbb{C}^n\}\,. \tag{57}$$

The standard characters $\varepsilon_i, \delta_j : \mathfrak{h}_{m|n} \to \mathbb{C}$ are defined by

$$\varepsilon_i(\mathrm{diag}(\mathbf{s}, \mathbf{t})) := s_i \text{ for } 1 \leq i \leq m \;\text{ and }\; \delta_j(\mathrm{diag}(\mathbf{s}, \mathbf{t})) := t_j \text{ for } 1 \leq j \leq n.$$

We define $\mathfrak{b}^{\mathrm{st}}_{m|n}$ (respectively, $\mathfrak{b}^{\mathrm{op}}_{m|n}$) to be the Borel subalgebras of $\mathfrak{gl}(m|n)$ corresponding to the fundamental systems Π^{st} (respectively, Π^{op}), where

$$\Pi^{\mathrm{st}} := \{\varepsilon_i - \varepsilon_{i+1}\}_{i=1}^{m-1} \cup \{\varepsilon_m - \delta_1\} \cup \{\delta_j - \delta_{j+1}\}_{j=1}^{n-1} \text{ and } \Pi^{\mathrm{op}} := -\Pi^{\mathrm{st}}$$

For every partition $\lambda \in \mathcal{H}(m,n)$, we set

$$\lambda^{\mathrm{st}}_{m|n} := \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \langle \lambda'_j - m \rangle \delta_j. \tag{58}$$

In the spacial case $m = n$, we define $\mathfrak{b}^{\mathrm{mx}}_{n|n}$ to be the Borel subalgebra corresponding to the fundamental system

$$\Pi^{\mathrm{mx}} := \{\delta_i - \varepsilon_i\}_{i=1}^{n} \cup \{\varepsilon_j - \delta_{j-1}\}_{j=2}^{n}\,.$$

B.2. **The Lie superalgebra $\mathfrak{gosp}(m|2n)$.** Let $m, n \geq 1$ be integers. We begin with an explicit realization of $\mathfrak{osp}(m|2n)$. Set $r := \lfloor \frac{m}{2} \rfloor$. Let J^+ be the $m \times m$ matrix defined by

$$J^+ := \begin{bmatrix} 1 & 0_{1\times r} & 0_{1\times r} \\ 0_{r\times 1} & 0_{r\times r} & I_{r\times r} \\ 0_{r\times 1} & I_{r\times r} & 0_{r\times r} \end{bmatrix} \text{ if } m = 2r+1, \text{ and } J^+ := \begin{bmatrix} 0_{r\times r} & I_{r\times r} \\ I_{r\times 1} & 0_{r\times r} \end{bmatrix} \text{ if } m = 2r,$$

Also, let J^- be the $2n \times 2n$ matrix defined by

$$J^- := \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

Let $\{\mathsf{e}_i\}_{i=1}^{m} \cup \{\mathsf{e}'_j\}_{j=1}^{2n}$ be the standard homogeneous basis of $\mathbb{C}^{m|2n}$, and let $\mathsf{B} : \mathbb{C}^{m|2n} \times \mathbb{C}^{m|2n} \to \mathbb{C}$ be the even supersymmetric bilinear form defined by

$$\mathsf{B}(\mathsf{e}_i, \mathsf{e}_j) = J^+_{i,j},\; \mathsf{B}(\mathsf{e}'_i, \mathsf{e}'_j) = J^-_{i,j},\; \text{and } \mathsf{B}(\mathsf{e}_i, \mathsf{e}'_j) = 0.$$

We realize the Lie superalgebra $\mathfrak{osp}(m|2n)$ as the subalgebra of $\mathfrak{gl}(m|2n)$ that leaves the bilinear form $\mathsf{B} : \mathbb{C}^{m|2n} \times \mathbb{C}^{m|2n} \to \mathbb{C}$ invariant. For $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{t} \in \mathbb{C}^n$, set

$$\mathsf{d}(\mathbf{s}, \mathbf{t}) := \begin{cases} \mathrm{diag}(\mathbf{s}, -\mathbf{s}, \mathbf{t}, -\mathbf{t}) & \text{if } m = 2r, \\ \mathrm{diag}(0, \mathbf{s}, -\mathbf{s}, \mathbf{t}, -\mathbf{t}) & \text{if } m = 2r+1. \end{cases}$$

Recall from Appendix B.1 that we denote the standard Cartan subalgebra of $\mathfrak{gl}(m|2n)$ by $\mathfrak{h}_{m|2n}$. Then $\overline{\mathfrak{h}}_{m|2n} := \mathfrak{h}_{m|2n} \cap \mathfrak{osp}(m|2n)$ is a Cartan subalgebra of $\mathfrak{osp}(m|2n)$. We have

$$\overline{\mathfrak{h}}_{m|2n} = \{\mathsf{d}(\mathbf{s}, \mathbf{t}) \,:\, \mathbf{s} \in \mathbb{C}^r \text{ and } \mathbf{t} \in \mathbb{C}^n\}\,,$$

and the standard characters of $\overline{\mathfrak{h}}_{m|2n}$ are given by

$$\varepsilon_i(\mathsf{d}(\mathbf{s},\mathbf{t})) := s_i \text{ for } 1 \le i \le r \text{ and } \delta_j(\mathsf{d}(\mathbf{s},\mathbf{t})) := t_j \text{ for } 1 \le j \le n.$$

Let $\breve{\mathfrak{b}}_{m|2n} \subseteq \mathfrak{osp}(m|2n)$ be the Borel subalgebra corresponding to the fundamental system $\mathit{\Pi}$, where

$$\mathit{\Pi} := \begin{cases} \{\varepsilon_i - \varepsilon_{i+1}\}_{i=1}^{r-1} \cup \{\varepsilon_r - \delta_1\} \cup \{\delta_j - \delta_{j+1}\}_{j=1}^{n-1} \cup \{\delta_n\} & \text{if } m = 2r+1, \\ \{\varepsilon_i - \varepsilon_{i+1}\}_{i=1}^{r-1} \cup \{\varepsilon_r - \delta_1\} \cup \{\delta_j - \delta_{j+1}\}_{j=1}^{n-1} \cup \{2\delta_n\} & \text{if } m = 2r. \end{cases}$$

Finally, we set $\mathfrak{gosp}(m|2n) := \mathfrak{osp}(m|2n) \oplus \mathbb{C}I \subseteq \mathfrak{gl}(m|2n)$, where $I := I_{(m+2n)\times(m+2n)}$. We also set

$$\mathfrak{b}_{m|2n} := \breve{\mathfrak{b}}_{m|2n} \oplus \mathbb{C}I. \tag{59}$$

We extend the standard characters ε_i, δ_j of $\overline{\mathfrak{h}}_{m|2n}$ to the subalgebra $\tilde{\mathfrak{h}}_{m|2n} := \mathfrak{h}_{m|2n} \cap \mathfrak{gosp}(m|2n)$ of diagonal matrices in $\mathfrak{gosp}(m|2n)$, by setting $\varepsilon_i(I) = \delta_j(I) = 0$. Let $\zeta : \tilde{\mathfrak{h}}_{m|2n} \to \mathbb{C}$ be the linear functional defined uniquely by

$$\zeta\big|_{\overline{\mathfrak{h}}_{m|2n}} = 0 \text{ and } \zeta(I) = 1.$$

The set $\{\varepsilon_i\}_{i=1}^r \cup \{\delta_j\}_{j=1}^n \cup \{\zeta\}$ is a basis for the dual of $\tilde{\mathfrak{h}}_{m|2n}$. We remark that when $\mathfrak{g}$ is of type $\mathfrak{gosp}$ (i.e., in Cases III and V), we have $\zeta(h) = 2$ where $h \in \mathfrak{g}^\flat$ is defined as in (1).

In Case V, where $\mathfrak{g} = \mathfrak{gosp}(2|4)$, we need to consider an exceptional Borel subalgebra

$$\mathfrak{b}^{\mathrm{ex}}_{2|4} := \hat{\mathfrak{b}}_{2|4} \oplus \mathbb{C}I, \tag{60}$$

where $\hat{\mathfrak{b}}_{2|4}$ is the Borel subalgebra of $\mathfrak{osp}(2|4)$ corresponding to the fundamental system

$$\mathit{\Pi}^{\mathrm{ex}} := \{-\varepsilon_1 - \delta_1, \delta_1 - \delta_2, 2\delta_2\}.$$

B.3. **The anisotropic embedding of $\mathfrak{osp}(m|2n)$ in $\mathfrak{gl}(m|2n)$.** We will need another realization of $\mathfrak{osp}(m|2n)$ inside $\mathfrak{gl}(m|2n)$ which will be used in the description of the spherical subalgebra $\mathfrak{k}$. Set $r := \lfloor \frac{m}{2} \rfloor$, and let $\tilde{J}^-$ be the $2n \times 2n$ matrix defined by

$$\tilde{J}^- := \mathrm{diag}(\underbrace{\tilde{J}, \ldots, \tilde{J}}_{n \text{ times}}) \text{ where } \tilde{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $\tilde{\mathsf{B}} : \mathbb{C}^{m|2n} \times \mathbb{C}^{m|2n} \to \mathbb{C}$ be the even supersymmetric bilinear form which is given in the standard basis $\{\mathsf{e}_i\}_{i=1}^m \cup \{\mathsf{e}_j\}_{j=1}^{2n}$ of $\mathbb{C}^{m|2n}$ by

$$\tilde{\mathsf{B}}(\mathsf{e}_i, \mathsf{e}_j) := \delta_{i,j}, \ \tilde{\mathsf{B}}(\mathsf{e}'_i, \mathsf{e}'_j) := \tilde{J}^-_{i,j}, \text{ and } \tilde{\mathsf{B}}(\mathsf{e}_i, \mathsf{e}'_j) := 0.$$

Thus, the matrix of $\tilde{\mathsf{B}}(\cdot,\cdot)$ in the standard basis of $\mathbb{C}^{m|2n}$ is

$$\begin{bmatrix} I_{m\times m} & 0_{m\times 2n} \\ 0_{2n\times m} & \tilde{J}^- \end{bmatrix}.$$

The subalgebra of $\mathfrak{gl}(m|2n)$ that leaves the bilinear form $\tilde{\mathsf{B}}$ invariant is isomorphic to $\mathfrak{osp}(m|2n)$.

B.4. **The exceptional embedding of $\mathfrak{osp}(1|2)\oplus\mathfrak{osp}(1|2)$ in $\mathfrak{gosp}(2|4)$.** We consider the realization of $\mathfrak{osp}(2|4)$ given in Appendix B.2. Set

$$g := \begin{bmatrix} 0 & \sqrt{-1} & 0 & 0 & 0 & 0 \\ -\sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} \\ 0 & 0 & 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 & 0 & 0 \end{bmatrix}.$$

We set $\mathfrak{k}^{\mathrm{ex}}$ to be the subalgebra of fixed points of the map $\mathfrak{osp}(2|4) \to \mathfrak{osp}(2|4)$ given by $x \mapsto \mathrm{Ad}_g(x)$. One can verify that $\mathfrak{k}^{\mathrm{ex}} \cong \mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$. We will consider $\mathfrak{k}^{\mathrm{ex}}$ as a subalgebra of $\mathfrak{gosp}(2|4)$.

B.5. **The Lie superalgebra $\mathfrak{p}(n)$.** Let $n \geq 1$ be an integer, and let $\check{\mathsf{B}} : \mathbb{C}^{n|n} \times \mathbb{C}^{n|n} \to \mathbb{C}$ be the odd supersymmetric bilinear form defined by

$$\check{\mathsf{B}}(\mathsf{e}_i, \mathsf{e}'_j) := \delta_{i,j},\ \check{\mathsf{B}}(\mathsf{e}_i, \mathsf{e}_j) := 0,\ \text{and}\ \check{\mathsf{B}}(\mathsf{e}'_i, \mathsf{e}'_j) := 0,$$

where $\{\mathsf{e}_i\}_{i=1}^n \cup \{\mathsf{e}'_i\}_{i=1}^n$ is the standard homogeneous basis of $\mathbb{C}^{n|n}$. The Lie superalgebra $\mathfrak{p}(n)$ is the subalgebra of $\mathfrak{gl}(n|n)$ that leaves $\check{\mathsf{B}}(\cdot,\cdot)$ invariant. It consists of matrices in (n,n)-block form

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix},\ \text{where } B = B^T \text{ and } C = -C^T.$$

In this paper we will not need a description of the root system and highest weight modules of $\mathfrak{p}(n)$.

B.6. **The Lie superalgebra $\mathfrak{q}(n)$.** Let $n \geq 1$ be an integer. The Lie superalgebra $\mathfrak{q}(n)$ is the subalgebra of $\mathfrak{gl}(n|n)$ that consists of matrices in (n,n)-block form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Let $\mathfrak{h}$ be the subalgebra of matrices of the latter form where A and B are diagonal. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{q}(n)$. The standard characters $\{\varepsilon_i\}_{i=1}^n$ of $\mathfrak{h}_{\overline{\mathbf{0}}}$ are the restrictions of the corresponding standard characters of $\mathfrak{gl}(n|n)$. Let $\mathfrak{b}_n^{\mathrm{st}}$ (respectively, $\mathfrak{b}_n^{\mathrm{op}}$) be the Borel subalgebra of $\mathfrak{q}(n)$ associated to the fundamental system $\mathit{\Pi}^{\mathrm{st}} := \{\varepsilon_i - \varepsilon_{i+1}\}_{i=1}^{n-1}$ (respectively, $\mathit{\Pi}^{\mathrm{op}} := \{\varepsilon_{i+1} - \varepsilon_i\}_{i=1}^{n-1}$). For every partition $\lambda \in \mathcal{DP}(n)$, we set $\lambda_n^{\mathrm{st}} := \sum_{i=1}^n \lambda_i \varepsilon_i$.

Appendix C. Facts from supergeometry

All of the supermanifolds that are considered in this appendix are complex analytic. We denote the underlying complex manifold of a supermanifold $\mathcal{X}$ by $|\mathcal{X}|$, and the sheaf of superfunctions on $\mathcal{X}$ by $\mathcal{O}_{\mathcal{X}}$. Morphisms of supermanfolds are expressed as $(\mathsf{f}, \mathsf{f}^{\#}) : \mathcal{X} \to \mathcal{Y}$, where $\mathsf{f} : |\mathcal{X}| \to |\mathcal{Y}|$ is the complex analytic map between the underlying spaces and $\mathsf{f}^{\#} : \mathcal{O}_{\mathcal{Y}} \to \mathsf{f}_*\mathcal{O}_{\mathcal{X}}$ is the associated morphism of sheaves of superalgebras.

Let $\mathcal{L}$ be a connected Lie supergroup and let $\mathcal{M}$ be a Lie subsupergroup of $\mathcal{L}$. Set $\mathfrak{l} := \mathrm{Lie}(\mathcal{L})$ and $\mathfrak{m} := \mathrm{Lie}(\mathcal{M})$. The right action of $\mathcal{L}$ on $\mathcal{L}$ induces a canonical isomorphism of superalgebras from $\mathbf{U}(\mathfrak{l})$ onto the algebra of left invariant holomorphic differential operators on $\mathcal{L}$. Under this isomorphism elements of $\mathbf{U}(\mathfrak{l})^{\mathcal{M}}$, the subalgebra of $\mathcal{M}$-invariants in $\mathbf{U}(\mathfrak{l})$, are mapped to holomorphic differential operators which are left $\mathcal{L}$-invariant and right $\mathcal{M}$-invariant. The latter differential operators induce

$\mathcal{L}$-invariant differential operators on the homogeneous space $\mathcal{L}/\mathcal{M}$. Consequently, we obtain a homomorphism of superalgebras

$$\Psi_{\mathcal{L},\mathcal{M}} : \mathbf{U}(\mathfrak{l})^{\mathcal{M}} \to \mathscr{D}(\mathcal{L}/\mathcal{M}), \tag{61}$$

where $\mathscr{D}(\mathcal{L}/\mathcal{M})$ denotes the algebra of $\mathcal{L}$-invariant differential operators on $\mathcal{L}/\mathcal{M}$. By a superization of the argument of [20, Prop. 9.1], we obtain the following statement.

Proposition C.1. *Let $\mathscr{D}^{(d)}(\mathcal{L}/\mathcal{M})$ denote the subspace of elements of $\mathscr{D}(\mathcal{L}/\mathcal{M})$ of order at most d. Assume that there exists an $\mathcal{M}$-invariant complement of $\mathfrak{m}$ in $\mathfrak{l}$. Then*

$$\Psi_{\mathcal{L},\mathcal{M}}\left(\mathbf{U}^{(d)}(\mathfrak{l})^{\mathcal{M}}\right) = \mathscr{D}^{(d)}(\mathcal{L}/\mathcal{M}) \text{ for every } d \geq 0.$$

In the rest of this appendix we will assume that J is a Jordan superalgebra of type A. Let $\mathfrak{g}$, $\mathfrak{k}$, and V be as in Section 1.

Lemma C.2. *There is a vector $v_{\mathfrak{k}} \in V_{\overline{\mathbf{0}}}$ such that $\mathfrak{k} = \mathrm{stab}_{\mathfrak{g}}(v_{\mathfrak{k}})$.*

Proof. In all of the cases where J is of type A we have $V \cong V^* \cong J$ as $\mathfrak{k}$-modules, hence we can set $v_{\mathfrak{k}}$ equal to the element of $V_{\overline{\mathbf{0}}}$ corresponding to $1_J \in J$. □

Lemma C.3. *The map $\mathfrak{g} \to V$, $x \mapsto x \cdot v_{\mathfrak{k}}$ is surjective.*

Proof. The kernel of the linear map $x \mapsto x \cdot v_{\mathfrak{k}}$ is $\mathfrak{k}$. The statement now follows in all of the cases by verifying that the graded dimension of the image of this map and of V are the same. □

Let $\tilde{\mathfrak{b}} := \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}$ be a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \tilde{\mathfrak{b}} + \mathfrak{k}$. Let $\mathcal{G}$ be a complex Lie supergroup such that $\mathrm{Lie}(\mathcal{G}) = \mathfrak{g}$, and let $\mathcal{V}$ be the complex affine superspace corresponding to V. We assume that $|\mathcal{G}|$ is a connected Lie group, and that the action of $\mathfrak{g}$ on V can be globalized to an action of $\mathcal{G}$ on $\mathcal{V}$. The stabilizer of $v_{\mathfrak{k}} \in |\mathcal{V}| = V_{\overline{\mathbf{0}}}$ is a complex Lie supergroup $(\mathcal{K}, \mathcal{O}_{\mathcal{K}})$ such that $\mathrm{Lie}(\mathcal{K}) = \mathfrak{k}$.

Proposition C.4. *The orbit map of $v_{\mathfrak{k}}$ factors through an embedding $(\mathsf{p}_{v_{\mathfrak{k}}}, \mathsf{p}^{\#}_{v_{\mathfrak{k}}}) : \mathcal{G}/\mathcal{K} \hookrightarrow \mathcal{V}$ whose image is an open subsupermanifold of $\mathcal{V}$.*

Proof. This follows from the fact that the differential of the orbit map $(\mathsf{p}_{v_{\mathfrak{k}}}, \mathsf{p}^{\#}_{v_{\mathfrak{k}}})$ is a bijection for all $g \in \mathcal{G}$, which is a consequence of Lemma C.3 and $\mathcal{G}$-equivariance of $(\mathsf{p}_{v_{\mathfrak{k}}}, \mathsf{p}^{\#}_{v_{\mathfrak{k}}})$. □

Remark C.5. Using the embedding $\mathcal{G}/\mathcal{K} \hookrightarrow \mathcal{V}$ of Proposition C.4 and the natural injection $\mathscr{P}(V) \hookrightarrow \mathcal{O}_{\mathcal{V}}(|\mathcal{V}|)$, we obtain a $\mathcal{G}$-equivariant embedding

$$\mathsf{p}^{\#}_{v_{\mathfrak{k}}}(|\mathcal{V}|)\big|_{\mathscr{P}(V)} : \mathscr{P}(V) \hookrightarrow \mathcal{O}_{\mathcal{G}/\mathcal{K}}\left(|\mathcal{G}/\mathcal{K}|\right).$$

Furthermore, connectedness of $|\mathcal{G}|$ implies $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}} = \mathscr{P}\mathscr{D}(V)^{\mathcal{G}}$. Therefore we can restrict every $D \in \mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$ to the open subsupermanifold $\mathcal{G}/\mathcal{K}$ of $\mathcal{V}$, and indeed $D\big|_{\mathcal{G}/\mathcal{K}} \in \mathscr{D}(\mathcal{G}/\mathcal{K})$.

For the next proposition, recall that every linear functional $\varphi : \tilde{\mathfrak{h}} \to \mathbb{C}$ induces a natural homomorphism of $\mathbb{C}$-algebras $\mathcal{S}\big(\tilde{\mathfrak{h}}\big) \to \mathbb{C}$ given by $x_1 \cdots x_k \mapsto \varphi(x_1) \cdots \varphi(x_k)$ for $x_1, \ldots, x_k \in \tilde{\mathfrak{h}}$. Set $\mathcal{S}^{(d)}(\tilde{\mathfrak{h}}) := \bigoplus_{i=0}^{d} \mathcal{S}^i\big(\tilde{\mathfrak{h}}\big)$.

Proposition C.6. *Assume that $\mathscr{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module, and let $D \in \mathscr{P}\mathscr{D}^{(d)}(V)^{\mathfrak{g}}$. Then there exists an element $x_D \in \mathcal{S}^{(d)}\big(\tilde{\mathfrak{h}}\big)$ such that for every irreducible $\mathfrak{g}$-module $W \subseteq \mathscr{P}(V)$, the action of D on W is by the scalar $\tilde{\lambda}(x_D)$, where $\tilde{\lambda}$ is the $\tilde{\mathfrak{b}}$-highest weight of W.*

Proof. Let $\mathcal{G}$, $\mathcal{V}$, and $\mathcal{K}$ be defined as above, and let $(\mathfrak{q}, \mathfrak{q}^{\#}) : \mathcal{G} \to \mathcal{G}/\mathcal{K}$ be the canonical quotient map. Then $\mathfrak{q}^{\#}(|\mathcal{G}/\mathcal{K}|) : \mathcal{O}_{\mathcal{G}/\mathcal{K}}(|\mathcal{G}/\mathcal{K}|) \to \mathcal{O}_{\mathcal{G}}(G)$ is an injection. For any $\tilde{X} \in \mathbf{U}(\mathfrak{g})$, let $\mathrm{L}_{\tilde{X}}$ (respectively, $\mathrm{R}_{\tilde{X}}$) denote the action of $\tilde{X}$ on $\mathcal{O}_{\mathcal{G}}(G)$ by left invariant (respectively, right invariant) differential operators. By Proposition C.1, there exists $\tilde{D} \in \mathbf{U}^{(d)}(\mathfrak{g})^{\mathcal{K}}$ such that $\mathfrak{q}^{\#}(|\mathcal{G}/\mathcal{K}|)(Df) = \mathrm{L}_{\tilde{D}}\mathfrak{q}^{\#}(|\mathcal{G}/\mathcal{K}|)(f)$ for every $f \in \mathcal{O}_{\mathcal{G}/\mathcal{K}}(|\mathcal{G}/\mathcal{K}|)$.

Now set $f := \mathfrak{p}^{\#}_{v_{\mathfrak{k}}}(|V|)(\phi_{\tilde{\lambda}})$ where $\phi_{\tilde{\lambda}} \in \mathscr{P}(V)$ is a highest weight vector of W, and let $\tilde{f} := \mathfrak{q}^{\#}(|\mathcal{G}/\mathcal{K}|)(f)$. Let $\mathcal{N}$ be the connected Lie subsupergroup of $\mathcal{G}$ such that $\mathrm{Lie}(\mathcal{N}) = \tilde{\mathfrak{n}}$. Then $\tilde{f}$ is left $\mathcal{N}$-invariant and right $\mathcal{K}$-invariant. We can express $\tilde{D}$ in the form $\tilde{D} = D_1 + D_2 + D_3$, where

$$D_1 \in \tilde{\mathfrak{n}}\mathbf{U}^{(d-1)}(\mathfrak{g}),\ D_2 \in \mathbf{U}^{(d)}\big(\tilde{\mathfrak{h}}\big),\ \text{and}\ D_3 \in \mathbf{U}^{(d-1)}(\mathfrak{g})\mathfrak{k}.$$

From $\mathcal{K}$-invariance of f it follows that $\mathrm{L}_{D_3}\tilde{f} = 0$. Furthermore, we can write D_1 as a sum of elements of the form XD' where $X \in \tilde{\mathfrak{n}}$ and $D' \in \mathbf{U}(\mathfrak{g})$. Let $\mathcal{H} := (H, \mathcal{O}_{\mathcal{H}})$ denote the connected Lie subsupergroup of $\mathcal{G}$ such that $\mathrm{Lie}(\mathcal{H}) = \tilde{\mathfrak{h}}$. For $h \in H$ we have

$$\mathrm{L}_{XD'}\tilde{f}(h) = \mathrm{L}_X\left(\mathrm{L}_{D'}\tilde{f}\right)(h) = \mathrm{R}_{-\mathrm{Ad}_h X}\left(\mathrm{L}_{D'}\tilde{f}\right)(h).$$

Since $\mathrm{L}_{D'}\tilde{f}$ is left $\mathcal{N}$-invariant and $\mathrm{Ad}_h(\tilde{\mathfrak{n}}) \subseteq \tilde{\mathfrak{n}}$, it follows that $\mathrm{L}_{XD'}\tilde{f}(h) = 0$. Consequently, we have shown that for $x_D := D_2$,

$$\mathrm{L}_{\tilde{D}}\tilde{f}(h) = \mathrm{L}_{D_2}\tilde{f}(h) = \tilde{\lambda}\big(D_2\big)\tilde{f}(h) = \tilde{\lambda}(x_D)\tilde{f}(h).$$

It remains to prove that $\tilde{f} \neq 0$. To this end, note that the canonical multiplication morphism

$$\mathcal{N} \times \mathcal{H} \times \mathcal{K} \to \mathcal{G}$$

is a local isomorphism at the identity element. Thus, from analyticity, left $\mathcal{N}$-invariance, and right $\mathcal{K}$-invariance of $\tilde{f}$, it follows that $\tilde{f}\big|_H$ is not identically zero. □

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