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THE CAPELLI EIGENVALUE PROBLEM FOR QUANTUM GROUPS

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Abstract. We introduce and study quantum Capelli operators inside newly constructed quantum Weyl algebras associated to three families of symmetric pairs ([19]). Both the center of a particular quantized enveloping algebra and the Capelli operators act semisimply on the polynomial part of these quantum Weyl algebras. We show how to transfer well-known properties of the center arising from the theory of quantum symmetric pairs to the Capelli operators. Using this information, we provide a natural realization of Knop-Sahi interpolation polynomials as functions that produce eigenvalues for quantum Capelli operators.

1. Introduction

In the mid 1980's, Macdonald introduced a new family of parametrized orthogonal polynomials, referred to as Macdonald polynomials. These polynomials can be viewed as generalizations of the orthogonal polynomials that appear as zonal spherical functions for real and p-adic symmetric spaces ([20]). About a decade later, Knop ([9]) and Sahi ([26]) defined parameterized versions of interpolation polynomials (in Type A) with close connections to Macdonald polynomials. For certain parameters, the Knop-Sahi interpolation polynomials can be viewed as q-analogs of polynomials that produce eigenvalues for Capelli operators ([11],[12],[27]). Knop-Sahi interpolation polynomials are referred to by a number of other names in the literature including interpolation Macdonald polynomials ([25]), quantum Capelli polynomials ([9]), and shifted Macdonald polynomials ([24]).

Drinfeld and Jimbo discovered quantized enveloping algebras, which are Hopf algebra deformations of universal enveloping algebras of Lie algebras, around the same time as Macdonald initiated the study of his new family of orthogonal polynomials. Shortly afterwards, the quest began for realizing Macdonald polynomials as zonal sperical functions on quantum symmetric spaces. This realization was ultimately carried out in a series of papers ([4], [15],[16], [21], [22], [23]). The theory relies on definitions of quantum symmetric pairs using special coideal subalgebras constructed first for classical Lie algebras using reflection equations ([20], [23]) and then in general via generators and relations ([13], [14], [8]).

In this paper, we complete another part of the quantum picture, namely finding a natural realization of the Knop-Sahi interpolation polynomials as functions that produce eigenvalues for quantum Capelli operators. The crucial piece in the story is identifying quantum Capelli operators, which are invariants with respect to a particular quantized enveloping algebra, inside three families of newly constructed quantum Weyl algebras $\mathscr{P}\mathscr{D}_\theta$ ([19]). In other words, we formulate and solve the Capelli eigenvalue problem in the quantum case, thus providing quantum analogs of results in [27]. Similar results for Capelli operators and their eigenvalues have been obtained in the Lie superalgebra setting ([1], [28], [29], [30]). We also show that these two quantum realizations, one for Macdonald polynomials and the other for Knop-Sahi interpolation polynomials, are closely connected. In particular, we use results from the Macdonald polyomial setting concerning central

elements of the quantized enveloping algebra in order to establish basic properties of quantum Capelli operators.

The quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$ is associated to one of the following three (infinitesimal) symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{k}$ is the Lie subalgebra fixed by the involution θ of $\mathfrak{g}$:

- $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_n, \mathfrak{so}_n)$
- $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$
- $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_n \oplus \mathfrak{gl}_n, \mathfrak{gl}_n)$

We refer to the first pair as Type AI, the second as Type AII, and the third as the Type A diagonal case, thus matching the language used in the classification of symmetric pairs via Satake diagrams. These correspond to the Jordan algebras of Hermitian matrices over $\mathbb{R}$, $\mathbb{H}$, and $\mathbb{C}$ in Type AI, Type AII, and the Type A diagonal case respectively. Indeed, in each case, the Riemannian symmetric space G/K can be identified naturally with the open subset of positive definite Hermitian matrices. We focus on these three families precisely because the Jordan algebra setting is central in Kostant and Sahi's study of Capelli identities, Capelli operators, and interpolation polynomials ([11], [12], [27]).

As explained in [19], the quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$ is a deformation of the twisted tensor product of two algebras, the polynomial part $\mathscr{P}_\theta$ and the constant coefficient differential part $\mathscr{D}_\theta$. The polynomial part $\mathscr{P}_\theta$ is the algebra of quantized functions on the space of $n \times n$ symmetric matrices for Type AI, $n \times n$ skew symmetric matrices for Type AII, and all $n \times n$ matrices in the Type A diagonal case. The algebra $\mathscr{D}_\theta$ is isomorphic to the opposite algebra of $\mathscr{P}_\theta$.

Let $\mathcal{B}_\theta$ denote the right coideal subalgebra of $U_q(\mathfrak{g})$ which is a quantum analog of $U(\mathfrak{k})$ inside of $U(\mathfrak{g})$ as defined in [14] (see also [8]). Recall that there is a second root system, called the restricted root system, associated to a symmetric pair $\mathfrak{g}, \mathfrak{k}$. For each of the symmetric pairs under consideration in this paper, the restricted root system Σ is of type A_{n-1}. Denote by Λ_Σ^+ the set of partitions of length n realized as weights associated to Σ. Note that these partitions defined by Σ form a subset of the partitions defined by the root system of $\mathfrak{g}$. Thus, with respect to this inclusion, elements of Λ_Σ^+ are also weights for $\mathfrak{g}$.

Let $M = \mathrm{Mat}_n$ in type AI, $M = \mathrm{Mat}_{2n}$ in type AII, and $M = \mathrm{Mat}_n \times \mathrm{Mat}_n$ in the diagonal case. The initial definition of $\mathscr{P}_\theta$ (see [19]) is a subalgebra of $\mathcal{B}_\theta$ invariants inside the quantized function algebra $\mathcal{O}_q(M)$ on M. Moreover, $\mathscr{P}_\theta$ inherits a left $U_q(\mathfrak{g})$-module structure from $\mathcal{O}_q(M)$. As a left $U_q(\mathfrak{g})$-module, $\mathscr{P}_\theta$ is multiplicity free and is isomorphic to a direct sum of simple highest weight modules $L(2\lambda)$ where $\lambda \in \Lambda_\Sigma^+$. The algebra $\mathscr{P}_\theta$ can be viewed as a subalgebra of the quantum homogeneous space $\mathcal{O}_q[G/K]$, which in turn is a subalgebra of $\mathcal{O}_q[G]$, where G, K is the symmetric pair of Lie groups associated to $\mathfrak{g}, \mathfrak{k}$. The algebra $\mathscr{D}_\theta$ also admits a decomposition into left $U_q(\mathfrak{g})$-modules that is isomorphic to a direct sum of the $L^*(2\lambda)$, for $\lambda \in \Lambda_\Sigma^+$ where $L^*(2\lambda)$ is a left $U_q(\mathfrak{g})$-module dual of $L(2\lambda)$.

The quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$ inherits the structure of a left $U_q(\mathfrak{g})$-module via its construction that is compatible with the module actions on the subalgebras $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$. Thus, $\mathscr{P}\mathscr{D}_\theta$ is isomorphic to the direct sum of modules of the form $L(2\lambda) \otimes L^*(2\mu)$ where λ, μ both run over partitions in Λ_Σ^+. Note that $L(2\lambda) \otimes L^*(2\lambda) \cong \mathrm{End}\, L(2\lambda)$. Let C_λ be the vector corresponding to the identity in $L(2\lambda) \otimes L^*(2\lambda)$ via this isomorphism. The space of left $U_q(\mathfrak{g})$ invariants of $L(2\lambda) \otimes L^*(2\mu)$ is zero if $\lambda \neq \mu$ and equal to the one-dimensioal space spanned by C_λ for $\lambda = \mu$. The quantum Capelli operators are the elements $C_\lambda, \lambda \in \Lambda_\Sigma^+$.

One can define an action of $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$ in a manner similar to the action of the classical Weyl algebra on its polynomial subalgebra. Since $\mathscr{P}_\theta$ is a $U_q(\mathfrak{g})$-module, elements of $U_q(\mathfrak{g})$ also act on $\mathscr{P}_\theta$. These actions lead to $U_q(\mathfrak{g})$-module maps from $\mathscr{P}\mathscr{D}_\theta$ and $U_q(\mathfrak{g})$ into $\mathrm{End}\, \mathscr{P}_\theta$ where the module

action for $\mathscr{P}_\theta$ is the left action and the module action for $U_q(\mathfrak{g})$ is the (left) adjoint action. Let $\mathcal{F}(U_q(\mathfrak{g}))$ denote the locally finite subalgebra of $U_q(\mathfrak{g})$ with respect to the adjoint action. We define a $U_q(\mathfrak{g})$-module map that takes certain elements of $\mathcal{F}(U_q(\mathfrak{g}))$, including most of the center of $U_q(\mathfrak{g})$, to the quantum Weyl algebra that is compatible with the maps into End $\mathscr{P}_\theta$. This enables us to establish the following connection between the center and the Capelli operators.

Theorem A. *There is an isomorphism between a polynomial subring Z of the center of $U_q(\mathfrak{g})$ and the algebra generated by the quantum Capelli operators so that the action of the two agree on $\mathscr{P}_\theta$.*

In [18], quantum Weyl algebras over $m \times n$ matrices where $m \neq n$ are studied. These quantum Weyl algebras come equipped with a left action of $U_q(\mathfrak{gl}_m)$ and a right action of $U_q(\mathfrak{gl}_n)$. A key tool in [18] is a mapping of most of the locally finite subalgebras of $U_q(\mathfrak{gl}_m)$ and of $U_q(\mathfrak{gl}_n)$ to subalgebras of the quantum Weyl algebra using very similar arguments to the ones found here.

By [17], there is a Harish-Chandra type map from the center to a subalgebra of the Cartan subalgebra of $U_q(\mathfrak{g})$ so that the image of the center $Z(U_q(\mathfrak{g}))$ consists of invariants with respect to a dotted action of the Weyl group for the restricted root system. Note that elements of $Z(U_q(\mathfrak{g}))$ act semisimply on $\mathscr{P}_\theta$ with eigenspaces corresponding to simple modules $L(2\lambda)$, $\lambda \in \Lambda^+_\Sigma$. The image of a central element inside of the Cartan subalgebra can be viewed as an eigenvalue function: the eigenvalue for a central element z on $L(2\lambda)$ is obtained by evaluating the image of z under this Harish-Chandra map at λ, or more precisely, at q^λ (see Section 10.1 for a definition of this type of evaluation).

It follows from Theorem A that the eigenvalue functions for quantum Capelli operators live inside the same subalgebra of the Cartan part and, moreover, inherit the desired Weyl group invariance property from the center. Write $\mathcal{E}_\lambda$ for the eigenvalue function associated to the Capelli operator C_λ. We shall see that there is an interpretation for this eigenvalue function as a polynomial in n variables. The explicit map from Z to the quantum Capelli operators of Theorem A ensures that the degree of $\mathcal{E}_\lambda$ is equal to $|\lambda|$.

Let $H_{2\lambda}$ denote the highest weight generating vector for the copy of $L(2\lambda)$ inside of $\mathscr{P}_\theta$. The algebra $\mathscr{P}_\theta$ has a natural degree function which turns it into a graded algebra. Using a careful analysis involving the relations of $\mathscr{P}\mathscr{D}_\theta$, we show that: $C_\lambda \cdot H_{2\mu} = 0$ for all $H_{2\mu}$, $\mu \neq \lambda$, of degree less than or equal to that of C_λ. Moreover, $C_\lambda \cdot H_{2\lambda} \neq 0$. These results are used to show that the polynomials $\mathcal{E}_\lambda$ satisfy the vanishing property, as well as a nondegeneracy condition, of the Knop-Sahi interpolation polynomials. This leads to our main result where we refer to the notation of [24] for the explicit definition of the polynomials $P^*_\lambda(x; a, g)$.

Theorem B. *For each $\lambda \in \Lambda^+_\Sigma$, $\mathcal{E}_\lambda$ is equal to the polynomial $P^*_\lambda(x; a, g)$ (up to a normalization scalar) where*

- $(a, g) = (q^4, q^2)$ *in Type AI*
- $(a, g) = (q^2, q^4)$ *in Type AII,*
- $(a, g) = (q^2, q^2)$ *in the Type A diagonal case.*

It should be noted that the parameters obtained in Theorem B are precisely the same parameters as those for the realization of Macdonald polynomials as zonal spherical functions. This is due both to the connection between Macdonald polynomials and Knop-Sahi interpolation polynomials as well as the relationship between eigenspaces of $\mathscr{P}_\theta$ and quantum zonal spherical functions. For more details on this connection, see Remark 10.5.

An original motivation for this paper was to understand and extend Bershtein's results on quantum Capelli operators in [3]. One can view the results in [3] as a quantum analog of the classical setup relying on a Hermitian symmetric pair of Type AIII. By [11], Section 1, the classical versions

of Bershtein's approach and the Type A diagonal case of this paper produce the same Capelli operator eigenvalues. This paper shows that the same happens in the quantum setting. Indeed, the eigenvalues for the quantum Capelli operators in the Type A diagonal case of Theorem B match those in [3]. Note that fixed Lie subalgebra $\mathfrak{k}$ in [3] contains the entire Cartan subalgebra and so the corresponding symmetric pair is maximally compact. An advantage of the approach here is that the symmetric pairs are explicitly defined so that they are in maximally split form. This allows us to use the theory of quantum symmetric pairs and symmetric spaces as developed in [13],[14],[15], [16] and [17].

The remainder of this paper is organized as follows. Section 2 sets notation for root systems, quantized enveloping algebras, the three types of symmetric pairs and their quantum versions. In Section 3, we give generators and relations for the quantized function algebras $\mathcal{O}_q(\mathrm{Mat}_N)$ on $N \times N$ matrices and for the three quantum homogeneous spaces $\mathscr{P}_\theta$. In Section 4, we obtain detailed descriptions of highest weight vectors inside $\mathscr{P}_\theta$ with respect to the action of $U_q(\mathfrak{g})$ and use them to write down explicit module decompositions of the quantum homogeneous spaces $\mathscr{P}_\theta$.

In Section 5, we describe the quantum Weyl algebras $\mathscr{P}\mathscr{D}_\theta$ of [19] in terms of generators and relations. We then define a $U_q(\mathfrak{g})$ equivariant action of $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$ that resembles the action for the classical Weyl algebra on the polynomial subalgebra. Using the relations for $\mathscr{P}\mathscr{D}_\theta$, we analyze the action of the quantum Weyl algebra on the highest weight vectors $H_{2\lambda}, \lambda \in \Lambda_\Sigma^+$, inside of $\mathscr{P}_\theta$. This information is used to determine the action of the Capelli operators C_λ on highest weight vectors $H_{2\mu}$ needed for the proof of Theorem B.

Section 6 shows how to identify certain elements u of the Cartan subalgebra of $U_q(\mathfrak{g})$ with elements a of $\mathscr{P}\mathscr{D}_\theta$ so that u and a agree with respect to their action on $\mathscr{P}_\theta$. The elements of the Cartan subalgebra that have this property are part of the locally finite subalgebra $\mathcal{F}(U_q(\mathfrak{g}))$ of $U_q(\mathfrak{g})$. The locally finite subalgebra is a well-studied object for quantized enveloping algebras of semisimple Lie algebras. In Section 7, we give a complete description of $\mathcal{F}(U_q(\mathfrak{g}))$ by translating the results for the locally finite part of $U_q(\mathfrak{sl}_N)$ to the $\mathfrak{gl}_N$ setting. This allows us to define a mapping Υ on almost all of $\mathcal{F}(U_q(\mathfrak{g}))$ into $\mathscr{P}\mathscr{D}_\theta$ which is compatible with the action on $\mathscr{P}_\theta$.

Section 8 is devoted to the center of $U_q(\mathfrak{g})$ and the image of a special subalgebra Z of the center (as in Theorem A) to $\mathscr{P}\mathscr{D}_\theta$ via Υ. Generators for the image of Z inside End $\mathscr{P}_\theta$ are identified and bounds on the degree of their image inside of $\mathscr{P}\mathscr{D}_\theta$ are determined. Using the standard Harish-Chandra map and a restricted version based on those in [17], we explain the dotted restricted Weyl group invariance for central elements. We further show that both Z and its image under the restricted Harish-Chandra map are isomorphic to a polynomial ring in n generators.

Quantum Capelli operators, are defined in Section 9. Comparing the degree of quantum Capelli operators with the degree of the image of generators of Z under Υ yields Theorem A (Theorem 9.3). Eigenvalue functions associated to quantum Capelli operators are introduced in Section 10.2 and their relation to the image of central elements of Z under the restricted Harish-Chandra map is explained. We show that these eigenvalue functions satisfy the defining properties of Knop-Sahi polynomials by combining facts about the center of $U_q(\mathfrak{g})$ with structural properties of $\mathscr{P}\mathscr{D}_\theta$. This establishes Theorem B (Theorem 10.4).

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2. Quantized enveloping algebras

2.1. **Roots and weights.** Let N be a positive integer and let $\epsilon_1, \dots, \epsilon_N$ denote a fixed orthonormal basis for $\mathbb{R}^N$ with respect to the standard inner product $(\cdot, \cdot)$. Let Φ_N denote the root system of Type A_{N-1} with positive simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, N-1$. Write $\omega_1, \dots, \omega_{N-1}$ for the fundamental weights associated to Φ_N. Set $P = \sum_{i=1}^{N-1} \mathbb{Z}\omega_i$ equal to the weight lattice and $Q = \sum_{i=1}^{N-1} \mathbb{Z}\alpha_i$ equal to the root lattice. Write $P^+ = \sum_{i=1}^{N-1} \mathbb{N}\omega_i$ for the subset of dominant integral weights. Let Q^+ denote the subset $\sum_{i-1}^{N} \mathbb{N}\alpha_i$ of the root lattice.

Let Λ_N denote the set of vectors $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_N\epsilon_N$ where each $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Set Λ_N^+ equal to the subset of Λ_N consisting of those weights $\lambda \in \Lambda_N$ satisfying $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$. In other words, elements in Λ_N^+ correspond to the set of partitions $(\lambda_1, \dots, \lambda_N)$. Recall that $\epsilon_1 + \cdots + \epsilon_N$ is orthogonal to each α_i with respect to the given inner product. Moreover, it is straightforward to check that the fundamental weight ω_i satisfies

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{N}(\epsilon_1 + \cdots + \epsilon_N) \tag{1}$$

for each $i = 1, \dots, N-1$. Set $\hat{\omega}_i = \epsilon_1 + \cdots + \epsilon_i$ for $i = 1, \dots, N$ and so the above equality becomes $\omega_i = \hat{\omega}_i - \frac{i}{N}\hat{\omega}_N$. We refer to the weights $\hat{\omega}_1, \dots, \hat{\omega}_N$ as the fundamental partitions associated to the root system Φ_N. Note that Λ_N^+ is the $\mathbb{N}$ linear combination of the fundamental partitions $\hat{\omega}_1, \dots, \hat{\omega}_N$.

Let w_0 denote the longest element of the Weyl group in type A_{N-1}. We have $w_0\alpha_i = -\alpha_{N-i}$ and $w_0\epsilon_j = \epsilon_{N+1-j}$ for $i = 1, \dots, N-1$ and $j = 1, \dots, N$. It follows that the set $w_0\Lambda_N^+$ is the $\mathbb{N}$ linear combination of the elements $w_0\hat{\omega}_i = \epsilon_{N-i+1} + \cdots + \epsilon_N$ for $i = 1, \dots, N$.

Let $\mathfrak{gl}_N$ denote the complex general linear Lie algebra consisting of $N \times N$ matrices and let $\mathfrak{sl}_N$ be the Lie subalgebra equal to the subspace of $N \times N$ matrices with trace 0. Recall that Φ_N is the root system for $\mathfrak{sl}_N$ and the finite-dimensional simple $\mathfrak{sl}_N$-modules are parameterized by their highest weights which are elements of P^+. For $\mathfrak{gl}_N$, one uses Λ_N to parameterize the finite-dimensional simple $\mathfrak{gl}_N$-modules. More generally, the relationship between ω_i and $\hat{\omega}_i$ in (1) will help us translate results from the $\mathfrak{sl}_N$ setting to that of $\mathfrak{gl}_N$.

The root system for $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$ is just the disjoint union $\Phi_N^{(1)} \cup \Phi_N^{(2)}$ of two copies of the root system for $\mathfrak{gl}_N$ and the corresponding Weyl group is just the direct product of two copies of the Weyl group for Φ_N . Write $\epsilon_1, \dots, \epsilon_N$ for the orthonormal basis for the first copy of $\mathfrak{gl}_N$ and $\epsilon_{N+1}, \dots, \epsilon_{2N}$ for the second copy. Set $\hat{\omega}_i = \epsilon_1 + \cdots + \epsilon_i$ and $\hat{\omega}_{i+N} = \epsilon_{1+N} + \cdots + \epsilon_{i+N}$ for $i = 1, \dots, N$. The longest Weyl group element is $w_0 \times w_0$ which simply acts as w_0 on each copy of Φ_N. Other notions are extended from $\mathfrak{gl}_N$ to $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$ in a similar fashion. Sometimes we denote weights for $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$ using a single symbol, say λ. In other instances, we use the sum $\gamma \oplus \gamma'$ to represent the weight $\gamma_1\epsilon_1 + \cdots + \gamma_N\epsilon_N + \gamma'_1\epsilon_{1+N} + \cdots + \gamma'_N\epsilon_{2N}$.

2.2. **The quantized enveloping algebra.** Write $e_1, \dots, e_{N-1}$, $f_1, \dots, f_{N-1}$, $h_{\epsilon_1}, \cdots, h_{\epsilon_N}$ for the standard Chevalley generators for $\mathfrak{gl}_N$. For the direct sum $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$, we write e_i, f_i, h_{ϵ_j} where $i \in \{1, \dots, N-1\} \cup \{N+1, \dots, 2N-1\}$ and $j \in \{1, \dots, 2N\}$ for the standard generators. Here, the e_i, f_i, h_{ϵ_j}, $i \leq N-1, j \leq N$ generate the first copy of $\mathfrak{gl}_N$ while e_i, f_i, h_{ϵ_j}, $i, j \geq N+1$ generate the second copy.

Let q be an indeterminate. The quantized enveloping algebra $U_q(\mathfrak{gl}_N)$ is an algebra over $\mathbb{C}(q)$ generated by $K_{\epsilon_1}^{\pm 1}, \dots, K_{\epsilon_N}^{\pm 1}$, $E_1, \dots, E_{N-1}$, $F_1, \dots, F_{N-1}$ subject to the algebra relations as stated in [21] (see also [10], Section 10). Note that $K_{\epsilon_1} \cdots K_{\epsilon_N}$ is a central element of $U_q(\mathfrak{gl}_N)$. Set

$K_i = K_{\epsilon_i} K_{\epsilon_{i+1}}^{-1}$ for $i = 1, \ldots, N-1$. The subalgebra of $U_q(\mathfrak{gl}_N)$ generated by

$$K_1^{\pm 1}, \ldots, K_{N-1}^{\pm 1}, E_1, \ldots, E_{N-1}, F_1, \ldots, F_{N-1}$$

is the quantized enveloping algebra $U_q(\mathfrak{sl}_N)$.

Given an integer linear combination $\beta = \sum_{i=1}^N \beta_j \epsilon_j$, write K_β for the product $K_{\epsilon_1}^{\beta_1} \cdots K_{\epsilon_N}^{\beta_N}$. The algebra $U_q(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct Δ, counit ϵ, and antipode S defined on generators by

- $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$, $\epsilon(E_i) = 0$ and $S(E_i) = -K_i^{-1} E_i$
- $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$, $\epsilon(F_i) = 0$ and $S(F_i) = -F_i K_i$
- $\Delta(K) = K \otimes K$, $\epsilon(K) = 1$ and $S(K) = K^{-1}$

for $i = 1, \ldots, N-1$ and for all $K = K_\beta, \beta \in \sum_j \mathbb{Z}\epsilon_j$. It follows from the defining relations for $U_q(\mathfrak{gl}_N)$ that

$$K_\beta E_i K_\beta^{-1} = q^{(\beta, \alpha_i)} E_i \quad \text{and} \quad K_\beta F_i K_\beta^{-1} = q^{-(\beta, \alpha_i)} F_i \tag{2}$$

for $i = 1, \ldots, N-1$ and all $\beta \in \sum_j \mathbb{Z}\epsilon_j$. The subalgebra $U_q(\mathfrak{sl}_N)$ is also a Hopf algebra using the same coproduct, couinit, and antipode. Given an arbitrary element u in one of these Hopf algebras, we express the coproduct of u by $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$.

We also consider the quantized enveloping algebra of $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$. Note that

$$U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N) \cong U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$$

as Hopf algebras. Write $E_i, F_i, K_{\epsilon_j}^{\pm 1}$, $1 \le i \le N-1, 1 \le j \le N$ for the generators of the first copy of $U_q(\mathfrak{gl}_N)$ and $E_i, F_i, K_{\epsilon_j}^{\pm 1}$, $N+1 \le i \le 2N-1, N+1 \le j \le 2N$ for the generators of the second copy. Formally, we may identify E_i with $E_i \otimes 1$ and E_{i+N} with $1 \otimes E_i$ for $1 \le i \le N-1$ with similar identifications for the F_i and $K_{\epsilon_j}^{\pm 1}$.

Let $U^0(\mathfrak{gl}_N)$ denote the subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $K_{\epsilon_1}^{\pm 1}, \ldots, K_{\epsilon_N}^{\pm 1}$. Similarly, we write $U^0(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ for the subalgebra of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ generated by $K_{\epsilon_1}^{\pm 1}, \ldots, K_{\epsilon_{2N}}^{\pm 1}$. We drop $\mathfrak{gl}_N$ or $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$ from the notation and simply write U^0 when the associated Lie algebra can be understood from context.

Let $U^+(\mathfrak{gl}_N)$ denote the subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $E_1, \ldots, E_{N-1}$. Similarly, let $U^+(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ denote the subalgebra of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ generated by E_i, for $i = 1, \ldots N-1$ and $i = N+1, \ldots, 2N-1$. Just as for U^0, we drop $\mathfrak{gl}_N$ or $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$ from the notation and simply write U^+ when the associated Lie algebra can be understood from context. Define the subalgebra U^- in the same way with each E_i replaced by F_i.

In the study of $U_q(\mathfrak{sl}_N)$, it is sometimes necessary to pass to the simply connected quantized enveloping algebra $\check{U}_q(\mathfrak{sl}_N)$. This Hopf algebra is an extension of $U_q(\mathfrak{sl}_N)$ obtained by enlarging U^0 to $\check{U}^0$ where $\check{U}^0$ is the group algebra generated by the weight lattice P (see for example [5], 3.2.10). In another words, $\check{U}_q(\mathfrak{sl}_N)$ is the Hopf algebra generated by $U_q(\mathfrak{sl}_N)$ and $\check{U}^0$ where the generators of $U_q(\mathfrak{sl}_N)$ and the elements $K_\beta, \beta \in P$ satisfy (2).

Recall that the augmentation ideal of a Hopf algebra H, denoted by H_+, is the kernel of the counit ϵ. Given a subset M of H, we write M_+ for the intersection of M with H_+. For example, we write $(U_q(\mathfrak{gl}_N))_+$ for the augmentation ideal of $U_q(\mathfrak{gl}_N)$. Similarly, we denote the intersection of U^+ (resp. U^-) with the augmentation ideal of $U_q(\mathfrak{gl}_N)$ (or $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ depending on context) by U_+^+ (resp. U_+^-).

We write $L(\lambda)$ for the simple $U_q(\mathfrak{gl}_N)$-module of highest weight λ. In other words, $L(\lambda)$ is generated by a highest weight vector v_λ such that $E_i v_\lambda = 0$ for all $i = 1, \ldots, N-1$ and $K_\beta v_\lambda =$

$q^{(\beta,\lambda)}v_\lambda$ for all weights β. Recall that $L(\lambda)$ is finite-dimensional viewed as a $U_q(\mathfrak{sl}_N)$-module if and only if $\lambda \in P^+$ and is finite-dimensional viewed as a $U_q(\mathfrak{gl}_N)$-module if and only if $\lambda \in \Lambda_N$. These notions extend to the setting of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ in a straightforward manner and we will similarly denote highest weight modules by $L(\lambda)$ where here λ is understood to be a weight for $\mathfrak{gl}_N \oplus \mathfrak{gl}_N$.

2.3. **The adjoint action.** Given a Hopf algebra H and a two-sided H-bimodule M, the bimodule M admits an H-module structure via the adjoint action defined by

$$(\text{ad } a)m = \sum a_{(1)} m S(a_{(2)})$$

for all $a \in H$ and $m \in M$. The locally finite part $\mathcal{F}_H(M)$ of M is the submodule consisting of those elements that generate a finite module with respect to the adjoint action of H. More precisely,

$$\mathcal{F}_H(M) = \{a \in M |\ \dim[(\text{ad } H)m] < \infty\}.$$

When M is also a Hopf algebra, the locally finite part $\mathcal{F}_H(M)$ is a subalgebra of M ([6]). We typically drop H from the notation $\mathcal{F}(M)$ when H can be understood from context.

For $H = U_q(\mathfrak{gl}_N)$, $U_q(\mathfrak{sl}_N)$ or $\check{U}_q(\mathfrak{sl}_N)$, the adjoint action is determined by the following formulas:

$$\begin{aligned}(\text{ad } E_i)\cdot m &= E_i m - K_i m K_i^{-1} E_i \\ (\text{ad } F_i)\cdot m &= F_i m K_i - m F_i K_i \\ (\text{ad } K)\cdot m &= K m K^{-1}\end{aligned}$$

for all $i = 1, \dots, N-1$, $K = K_\beta$ for all weights β with K_β in the specified quantized enveloping algebra, and $m \in M$. Note that these formulas carry over easily to $H = U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$.

2.4. **Three symmetric pair families.** We are interested in quantum homogeneous spaces associated to three families of symmetric pairs $\mathfrak{g}, \mathfrak{k}$ where $\mathfrak{k}$ is the Lie subalgebra fixed by the involution θ. Here, $\mathfrak{g} = \mathfrak{gl}_n$ for the first family, $\mathfrak{g} = \mathfrak{gl}_{2n}$ for the second family, and $\mathfrak{g} = \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ for the third family. Below, we describe the involution for each family and then define the generators for the associated quantum analog $\mathcal{B}_\theta$ of $U(\mathfrak{g}^\theta)$.

For each of the three families, we associate an R-matrix $R_\mathfrak{g}$ and a solution J to the reflection equation $R_\mathfrak{g} J_1 R_\mathfrak{g}^{t_1} J_2 = J_2 R_\mathfrak{g}^{t_1} J_1 R_\mathfrak{g}$ where J is an $N \times N$ matrix, $J_1 = J \otimes I$ and $J_2 = I \otimes J$ where I is the $N \times N$ identity matrix. These R-matrices are closely related to the following matrix R in $\text{Mat}_N \times \text{Mat}_N$ defined by

$$R = \sum_{1 \le i \le N} q e_{ii} \otimes e_{ii} + \sum_{1 \le i < j \le N} (e_{ii} \otimes e_{jj} + e_{jj} \otimes e_{ii}) + (q - q^{-1}) \sum_{1 \le j < i \le N} e_{ij} \otimes e_{ji} \tag{3}$$

where Mat_N is the space of $N \times N$ matrices and the e_{ij} are matrix units.

Type AI: $\mathfrak{g} = \mathfrak{gl}_n$ and θ is defined by $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$ and $\theta(h_{\epsilon_j}) = -h_{\epsilon_j}$ each $i = 1, \dots, n-1$ and $j = 1, \dots, n$. Hence $\mathfrak{k}$ is generated by $f_i - e_i$ for $i = 1, \dots, n-1$. Passing to the quantum case, $\mathcal{B}_\theta$ is generated by $F_i - E_i K_i^{-1}$, for $i = 1, \dots, n-1$. In this case, $R_\mathfrak{g} = R$ as defined in (3) with $N = n$. The associated solution to the reflection equation is $J = I_n$, the $n \times n$ identity matrix.

Type AII: $\mathfrak{g} = \mathfrak{gl}_{2n}$ and θ is defined by

- $\theta(e_i) = e_i, \theta(f_i) = f_i, \theta(h_i) = h_i$ for $i = 1, 3, \dots, 2n-1$.
- $\theta(f_i) = -[e_{i-1}, [e_{i+1}, e_i]]$ for $i = 2, 4, \dots, 2n-2$.
- $\theta(h_{\epsilon_{2i-1}}) = -h_{\epsilon_{2i}}$ for $i = 1, \dots, n$.

Hence $\mathfrak{k}$ is generated by e_i, f_i, h_i, for $i = 1, 3, \ldots, 2n-1$ and $f_i - [e_{i-1}, [e_{i+1}, e_i]]$ for $i = 2, 4, \ldots, 2n-2$. Passing to the quantum case, $\mathcal{B}_\theta$ is generated by

- $K_i^{\pm 1}, E_i, F_i$ for $i = 1, 3, \ldots, 2n-1$.
- $B_i = F_i - q^3((\text{ad } E_{i-1}E_{i+1}) \cdot E_i)K_i^{-1} = F_i - q^3[E_{i-1}, [E_{i+1}, E_i]_q]_q K_i^{-1}$ for $i = 2, 4, \ldots, 2n-2$ where $[a, b]_q$ denotes the q-commutator $ab - qba$ of a and b.

In this case, $R_\mathfrak{g} = R$ as defined in (3) with $N = 2n$. The associated solution to the reflection equation is $J = \sum_{k=1}^{n}(e_{2k-1,2k} - qe_{2k,2k-1})$.

Type A diagonal case: $\mathfrak{g} = \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ and θ is defined by $\theta(f_i) = -e_{n+i}$, $\theta(f_{n+i}) = -e_i$, and $\theta(h_{\epsilon_j}) = -h_{\epsilon_{n+j}}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$ and so $\mathfrak{k}$ is generated by $f_i - e_{n+i}$, $f_{n+i} - e_i$, and $h_{\epsilon_j} - h_{\epsilon_{n+j}}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. Passing to the quantum case, the corresponding quantum symmetric pair coideal subalgebra $\mathcal{B}_\theta$ is generated by

$$B_i = F_i - qE_{n+i}K_i^{-1}, \quad B_{n+i} = F_{n+i} - qE_iK_{n+i}^{-1}, \quad \text{and} \quad (K_{\epsilon_j}^{-1}K_{\epsilon_{n+j}})^{\pm 1}$$

for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. In this case, $R_\mathfrak{g}$ is the $4n^2 \times 4n^2$ block diagonal matrix with diagonal (R, I_{n^2}, I_{n^2}, R) were R is the matrix defined by (3) with $N = n$. The associated solution to the reflection equation is $J = \sum_{k=1}^{n}(e_{k,n+k} + e_{n+k,k})$. We often drop the phrase Type A and simply refer to this family as the diagonal case or diagonal type.

2.5. **Restricted root systems.** Write π for the set of simple roots for $\mathfrak{g}$ and Φ for the root system generated by π. By Section 2.1, for Types AI and AII, we have $\pi = \{\alpha_1, \ldots, \alpha_{N-1}\}$ and the root system $\Phi = \Phi_N$ is of type A_{N-1} where $N = n$ in Type AI and $N = 2n$ in Type AII. For the diagonal case, $\pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \cup \{\alpha_{n+1}, \ldots, \alpha_{2n-1}\}$ where $\{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\{\alpha_{n+1}, \ldots, \alpha_{2n-1}\}$ each separately generate a root system of Type A_n and together generate a root system $\Phi = \Phi_n^{(1)} \cup \Phi_n^{(2)}$.

Note that θ induces an involution, which we also call θ, on the root system Φ. More generally, θ can be extended to an involution on $\sum_{i=1}^{N} \mathbb{Z}\epsilon_i$ where $N = n$ in Type AI, and $N = 2n$ in Type AII and the diagonal case. Set $\tilde{\beta} = (\beta - \theta(\beta))/2$ for each $\beta \in \sum_{i=1}^{N} \mathbb{Z}\epsilon_i$. The set of all $\tilde{\beta}$ where β runs over elements in Φ, forms another root system, called the restricted root system which we denote by Σ. For all three families under consideration, the root system Σ is of type A_{n-1}. We denote the simple roots for Σ by $\alpha_1^\Sigma, \ldots, \alpha_{n-1}^\Sigma$ and explain below how these are related to the simple roots for Φ.

The restricted root system is contained in a vector space spanned by orthonormal basis vectors that live inside the set $\{\tilde{\beta} |\ \beta \in \sum_{i=1}^{n} \mathbb{Z}\epsilon_i\}$. We denote this orthonormal basis by $\epsilon_1^\Sigma, \ldots, \epsilon_n^\Sigma$. Write $\eta_1, \ldots, \eta_{n-1}$ for the fundamental weights associated to Σ and write $\hat{\eta}_1, \ldots, \hat{\eta}_n$ for the fundamental partitions. Let P_Σ denote the weight lattice and P_Σ^+ denote the dominant integral weights defined by the root system Σ. In other words P_Σ (resp. P_Σ^+) is just the set of $\mathbb{Z}$ (resp. $\mathbb{N}$) linear combinations of the fundamental weights $\eta_1, \ldots, \eta_{n-1}$. Let Λ_Σ^+ denote the $\mathbb{N}$ span of the fundamental partitions $\hat{\eta}_i, i = 1, \ldots, n$. Let W_Σ denote the Weyl group for the root system Σ. We identify the ϵ_i^Σ, η_i and $\hat{\eta}_i$ with elements of $\sum_{i=1}^{n} \mathbb{Z}\epsilon_i$ below for each of the three families under consideration. With respect to this identification, it is straightforward to check that the longest element of the Weyl group for Φ also acts as the longest element in the restricted Weyl group W_Σ. Abusing notation slightly, we refer to w_0 as the longest element of W_Σ with the understanding that w_0 is the restriction of the longest element of the Weyl group for Φ to restricted weights. In particular, $w_0\hat{\eta}_i = \epsilon_{n-i+1}^\Sigma + \cdots + \epsilon_n^\Sigma$ for each i and for each of the three families.

Type AI: For Type AI, $\tilde{\alpha}_i = (\alpha_i - \theta(\alpha_i))/2 = \alpha_i$ for each $i = 1, \ldots, n-1$. Hence $\alpha_i^\Sigma = \tilde{\alpha}_i = \alpha_i$ for $i = 1, \ldots, n-1$ and the set of positive simple roots for the root system Σ is just the set

$\{\alpha_1, \ldots, \alpha_{n-1}\}$. A similar straightforward computation yields $\tilde{\epsilon}_i = \epsilon_i$ for each i. In other words, the restricted root system Σ in this case equals Φ. Thus, $\epsilon_j^\Sigma = \epsilon_j$, $\hat{\eta}_j = \hat{\omega}_j$, $\eta_k = \omega_k$, and $\hat{\eta}_k = \hat{\omega}_k$ for $j = 1, \ldots, n$ and $k = 1, \ldots, n-1$. Note further that $\tilde{\omega}_k = (\omega_k - \theta(\omega_k))/2 = \omega_k$ and so η_k equals $\tilde{\omega}_k$, the restricted weight associated to ω_k. Similarly, $\hat{\eta}_k = (\hat{\omega}_k - \theta(\hat{\omega}_k))/2$, the restricted weight associated to $\hat{\omega}_k$, for each $k = 1, \ldots, n$.

Type AII: For Type AII, $\tilde{\alpha}_{2i-1} = 0$ for $i = 1, \ldots, n$ and $\tilde{\alpha}_{2j} = (\alpha_{2j-1} + 2\alpha_{2j} + \alpha_{2j+1})/2$ for $j = 1, \ldots, n-1$. The set of positive simple roots for Σ is $\{\tilde{\alpha}_{2j} |\ j = 1, \ldots, n-1\}$ and $\alpha_j^\Sigma = \tilde{\alpha}_{2j}$ for $j = 1, \ldots, n-1$. Note that the inner product on Φ needs to be scaled differently for the restricted root system. This is because

$$(\tilde{\alpha}_{2j}, \tilde{\alpha}_{2j}) = (\alpha_{2j-1} + 2\alpha_{2j} + \alpha_{2j+1}, \alpha_{2j-1} + 2\alpha_{2j} + \alpha_{2j+1})/4 = 1.$$

Hence the inner product for Σ takes the form $(\cdot,\cdot)_\Sigma = 2(\cdot,\cdot)$ where $(\cdot,\cdot)$ is the usual Cartan inner product for the root system of $\mathfrak{gl}_{2n}$. In other words, $(\cdot,\cdot)_\Sigma$ is the normalization of $(\cdot,\cdot)$ chosen so that $(\tilde{\alpha}_{2j}, \tilde{\alpha}_{2j}) = 2$ for each j. Now $\tilde{\epsilon}_{2j} = \tilde{\epsilon}_{2j-1} = (\epsilon_{2j-1} + \epsilon_{2j})/2$ and so $2(\tilde{\epsilon}_{2i}, \tilde{\alpha}_{2j}) = \delta_{ij}$. It follows that the corresponding set of orthonormal vectors is $\{\tilde{\epsilon}_2, \tilde{\epsilon}_4, \ldots, \tilde{\epsilon}_{2n}\}$ and hence $\epsilon_i^\Sigma = \tilde{\epsilon}_{2i}$ for $i = 1 \ldots, n$. The fundamental partitions associated to this restricted root system take the form

$$\hat{\eta}_r = \epsilon_1^\Sigma + \cdots \epsilon_r^\Sigma = \tilde{\epsilon}_2 + \cdots + \tilde{\epsilon}_{2r} = (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2r})/2 = \hat{\omega}_{2r}/2$$

for $r = 1, \ldots, n$. It follows that the fundamental weights are

$$\eta_r = \hat{\eta}_r - (r/N)\hat{\eta}_N = (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2r})/2 - (r/N)(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{2n})/2 = \omega_{2r}/2$$

for $r = 1, \ldots, n-1$. Note also that

$$(\hat{\omega}_{2j} - \theta(\hat{\omega}_{2j}))/2 = (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_{2j}) = 2(\tilde{\epsilon}_2 + \tilde{\epsilon}_4 + \cdots + \tilde{\epsilon}_{2j}) = 2\hat{\eta}_j$$

for each $j = 1, \ldots, n$. Similarly,

$$(\hat{\omega}_{2j-1} - \theta(\hat{\omega}_{2j-1}))/2 = (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_{2j-1}) = 2(\tilde{\epsilon}_2 + \tilde{\epsilon}_4 + \cdots + \tilde{\epsilon}_{2j-2}) + \tilde{\epsilon}_{2j}$$

for $j = 1, \ldots, n$. Hence $(\hat{\omega}_1 - \theta(\hat{\omega}_1))/2 = \hat{\eta}_1$ and

$$(\hat{\omega}_{2j-1} - \theta(\hat{\omega}_{2j-1}))/2 = \hat{\eta}_j + \hat{\eta}_{j-1}$$

for each $j = 2, \ldots, n$. It follows that $\omega_1 - \theta(\omega_1) = \eta_1$, $\omega_{2j} - \theta(\omega_{2j}) = 2\eta_j$ and $\omega_{2j-1} - \theta(\omega_{2j-1}) = \eta_j + \eta_{j-1}$ for $j = 2, \ldots, n$.

Type A diagonal case: For the diagonal case, we have $\tilde{\alpha}_i = (\alpha_i + \alpha_{n+i})/2 = \tilde{\alpha}_{n+i}$. Thus the set of positive simple roots for the root system Σ is $\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{n-1}\}$ and $\alpha_i^\Sigma = \tilde{\alpha}_i$ for $i = 1, \ldots, n-1$. Note that $(\tilde{\alpha}_i, \tilde{\alpha}_i) = ((\alpha_i + \alpha_{n+i})/2, (\alpha_i + \alpha_{n+i})/2) = 1$. Hence, the inner product for the restricted root system in the diagonal case is $(\cdot,\cdot)_\Sigma = 2(\cdot,\cdot)$. Here, the corresponding orthonormal basis is $\{\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n\}$ where $\tilde{\epsilon}_i = (\epsilon_i + \epsilon_{n+i})/2$ for each i. In particular, we have $\epsilon_i^\Sigma = \tilde{\epsilon}_i$ for $i = 1, \ldots, n$. The fundamental partitions in this case are

$$\hat{\eta}_r = \epsilon_1^\Sigma + \cdots \epsilon_r^\Sigma = \tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_r = (\epsilon_1 + \epsilon_{1+n} + \cdots + \epsilon_r + \epsilon_{r+n})/2$$

for $r = 1, \ldots, n$. It follows that the fundamental weights associated to Σ satisfy

$$\eta_r = \hat{\eta}_r - (r/n)\hat{\eta}_n = (\hat{\omega}_r + \hat{\omega}_{r+n})/2 - (r/n)(\hat{\omega}_n + \hat{\omega}_{2n})/2 = \omega_r/2 + \omega_{r+n}/2$$

for $r = 1, \ldots, n-1$. We have

$$(\hat{\omega}_j - \theta(\hat{\omega}_j))/2 = \tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_j = \hat{\eta}_j$$

and, similarly, $(\hat{\omega}_{n+j} - \theta(\hat{\omega}_{n+j}))/2 = \hat{\eta}_j$. Thus $\eta_j = \tilde{\omega}_j = \tilde{\omega}_{n+j}$ for $j = 1, \ldots, n$.

A finite-dimensional simple highest weight module is called spherical if it contains a nonzero $\mathcal{B}_\theta$ invariant vector, i.e., a vector v such that $x \cdot v = \epsilon(x)v$ for $x \in \mathcal{B}_\theta$. Note that $2\Lambda_\Sigma^+ = \{2\lambda, \lambda \in \Lambda_\Sigma^+\}$ is a subset of Λ_N^+ where $N = n$ in Type AI and $N = 2n$ in Type AII. Moreover, given $\gamma \in \Lambda_N^+$, the module $L(\gamma)$ is spherical if and only if $\gamma = 2\lambda + s\hat{\eta}_n$ for some $\lambda \in \Lambda_\Sigma^+$ and $s \in \mathbb{N}$ in Type AI and $\gamma = 2\lambda$ for $\lambda \in \Lambda_\Sigma^+$ in Type AII ([21], (3.12)).

Now consider the diagonal case. For $\gamma, \gamma' \in \Lambda_n^+$, $L(\gamma \oplus \gamma')$ is spherical if and only if $\gamma = \gamma'$. This follows from the classification of spherical modules using $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$ (see [14], Section 7) along with the fact that weight vectors admitting a trivial $K_{\epsilon_i}K_{\epsilon_{i+n}}^{-1}$ action for each $i = 1, \dots, n$ must have weights of the form $\gamma \oplus \gamma$. Note that the set of $\gamma \oplus \gamma$ with $\gamma \in \Lambda_n^+$ is precisely $2\Lambda_\Sigma^+$. Thus $L(\gamma \oplus \gamma')$ is spherical if and only if $\gamma \oplus \gamma' = 2\lambda$ for some $\lambda \in \Lambda_\Sigma^+$.

3. Quantized function algebras

3.1. **Quantized matrix functions.** Let Mat_N denote the space of $N \times N$ matrices with basis e_{ij}, $1 \le i, j \le N$. The matrix R defined by (3) can be written as $R = \sum_{i,j,k,l} r_{kl}^{ij} e_{ik} \otimes e_{jl}$ where

- $r_{ii}^{ii} = q$, $r_{ij}^{ij} = 1$ for all i, j, with $i \neq j$.
- $r_{ji}^{ij} = (q - q^{-1})$ for all $j < i$.
- $r_{kl}^{ij} = 0$ for all other choices of i, j, k, l.

We can view R as the matrix in $\mathrm{Mat}_N \times \mathrm{Mat}_N$ with $(i, k) \times (j, l)$ entry r_{kl}^{ij}. Given a matrix $A \in \mathrm{Mat}_N \times \mathrm{Mat}_N$, write A^{t_s} for the transpose in the first term when $s = 1$ and the second when $s = 2$. For example, $(R^{t_1})_{kl}^{ij} = r_{il}^{kj}$.

The quantized function algebra $\mathcal{O}_q(\mathrm{Mat}_N)$ is the bialgebra over $\mathbb{C}(q)$ generated by $t_{ij}, 1 \le i, j \le N$ where the t_{ij} satisfy the following relations

(i) $t_{ki}t_{kj} = qt_{kj}t_{ki}$, $t_{ik}t_{jk} = qt_{jk}t_{ik}$ $(i < j)$
(ii) $t_{il}t_{kj} = t_{kj}t_{il}, t_{ij}t_{kl} - t_{kl}t_{ij} = (q - q^{-1})t_{kj}t_{il}$ $(i < k; j < l)$

Set $T = (t_{ij})$, the $N \times N$ matrix with ij entry equal to t_{ij} and set $T_1 = T \otimes I$ and $T_2 = I \otimes T$ where I is the $N \times N$ identity matrix. These relations can be written in matrix form as $RT_1T_2 = T_2T_1R$, or equivalently, as the set of equations

$$\sum_{j,k} r_{jk}^{ld} t_{ja} t_{kb} = \sum_{j,k} t_{dk} t_{lj} r_{ab}^{jk}.$$

for all a, b, d, l. A straightforward computation shows that the map ι defined by

$$\iota(t_{ij}) = t_{ji} \tag{4}$$

for all $i, j = 1, \dots, n$ defines an algebra automorphism of $\mathcal{O}_q(\mathrm{Mat}_N)$.

The algebra $\mathcal{O}_q(\mathrm{Mat}_N)$ admits the structure of a $U_q(\mathfrak{gl}_N)$-bimodule algebra where the left action is determined by

$$E_k \cdot t_{ij} = \delta_{i-1,k} t_{i-1,j}, \quad F_k \cdot t_{ij} = \delta_{ik} t_{i+1,j}, \quad K_{\epsilon_r} \cdot t_{ij} = q^{\delta_{ir}} t_{ij}$$

and the right action by

$$t_{ij} \cdot E_k = \delta_{jk} t_{i,j+1}, \quad t_{ij} \cdot F_k = \delta_{j-1,k} t_{i,j-1}, \quad t_{ij} \cdot K_{\epsilon_r} = q^{\delta_{jr}} t_{ij}$$

for $r = 1, \dots, N$, $i, j = 1, \dots, N$ and $k = 1, \dots, N-1$. Here, we are using the notation $t_{uv} = 0$ for $u \in \{0, N+1\}$ or $v \in \{0, N+1\}$.

Let $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$ denote the bialgebra with the same coalgebra structure and opposite algebra structure of the bialgebra $\mathcal{O}_q(\mathrm{Mat}_N)$. Write $\partial_{ij}, 1 \le i, j \le N$ for the generators of $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$ so

that the algebra map defined by $t_{ij} \mapsto \partial_{ij}$ is an anti-automorphism. The algebra $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$ is also a $U_q(\mathfrak{gl}_N)$-bimodule algebra with action defined by

$$E_k \cdot \partial_{ij} = -\delta_{ik} q^{-1} \partial_{i+1,j}, \quad F_k \cdot \partial_{i,j} = -\delta_{i-1,k} q \partial_{i-1,j}, \quad K_{\epsilon_r} \cdot \partial_{ij} = q^{-\delta_{ir}} \partial_{ij}$$

and right action defined by

$$\partial_{ij} \cdot E_k = -\delta_{j-1,k} q \partial_{i,j-1}, \quad \partial_{ij} \cdot F_k = -\delta_{jk} q^{-1} \partial_{i,j+1}, \quad \partial_{ij} \cdot K_{\epsilon_r} = q^{-\delta_{jr}} \partial_{ij}$$

for $r = 1, \dots, N$, $i, j = 1, \dots, N$ and $k = 1, \dots, N-1$. Just as for the t_{uv}, we are using the notation $\partial_{uv} = 0$ for $u \in \{0, N+1\}$ or $v \in \{0, N+1\}$.

Let $\mathcal{O}_q(\mathrm{Mat}_N) \otimes \mathcal{O}_q(\mathrm{Mat}_N)$ denote the algebra generated by two copies of $\mathcal{O}_q(\mathrm{Mat}_N)$. The first is generated by $t_{ij}, 1 \leq i, j \leq N$ and the second is generated by $t_{i+N,j+N}, 1 \leq i, j \leq N$ and t_{ij} commutes with $t_{k+N,l+N}$ for all $i, j, k, l \in \{1, \dots, N\}$. Formally, t_{ij} can be identified with $t_{ij} \otimes 1$ and $t_{i+N,j+N}$ with $1 \otimes t_{ij}$. The algebra $\mathcal{O}_q(\mathrm{Mat}_N) \otimes \mathcal{O}_q(\mathrm{Mat}_N)$ is a $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N) \cong U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$-bimodule algebra. Here, the left action is given by $(g \otimes h) \cdot (a \otimes b) = (g \cdot a) \otimes (h \cdot b)$ for all $g \otimes h \in U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$ and $a \otimes b \in \mathcal{O}_q(\mathrm{Mat}_N) \otimes \mathcal{O}_q(\mathrm{Mat}_N)$. The right action is defined in the same way with the action on the right instead of the left. Similar notions apply for the algebra $\mathcal{O}_q(\mathrm{Mat}_N)^{op} \otimes \mathcal{O}_q(\mathrm{Mat}_N)^{op}$.

Consider the three families as described in Section 2.4. Set $\mathscr{P} = \mathcal{O}_q(\mathrm{Mat}_N)$ in Type AI with $N = n$ and in Type AII with $N = 2n$. For the diagonal case, set $\mathscr{P} = \mathcal{O}_q(\mathrm{Mat}_n) \otimes \mathcal{O}_q(\mathrm{Mat}_n)$. Similarly, set $\mathscr{D} = \mathcal{O}_q(\mathrm{Mat}_N)^{op}$ in Type AI with $N = n$ and in Type AII with $N = 2n$. For the diagonal case, set $\mathscr{D} = \mathcal{O}_q(\mathrm{Mat}_n)^{op} \otimes \mathcal{O}_q(\mathrm{Mat}_n)^{op}$.

3.2. **Functions on homogeneous spaces.** Let $\mathfrak{g}, \mathfrak{k}$ be a symmetric pair corresponding to one of the three families described in Section 2.4. Recall that J is the associated solution to the reflection equation. Write $J_{r,s}$ for the coefficient of e_{rs} in J. Define elements x_{ij} and d_{ij} by

$$x_{ij} = \sum_{r,s} t_{ir} J_{r,s} t_{js} \text{ and } d_{ij} = \sum_{r,s} q^{-2\hat{s}} \partial_{ir} J_{r,s} \partial_{js}$$

for $1 \leq i, j \leq N$ where $N = n$ in Type AI, $N = 2n$ in Type AII and diagonal type, $\hat{s} = s$ in Types AI and AII, and for the diagonal type $\hat{s} = s$ for $s \leq n$, and $\hat{s} = s - n$ when $s \geq n + 1$.

Let $\mathscr{P}_\theta$ be the subalgebra of $\mathscr{P}$ generated by the $x_{ij}, 1 \leq i, j \leq N$, and let $\mathscr{D}_\theta$ be the subalgebra of $\mathscr{D}$ generated by the $d_{ij}, 1 \leq i, j \leq N$. As explained in [21] and [19], the quantum homogeneous space $\mathcal{O}_q[G/K]$ associated to $\mathfrak{g}, \mathfrak{k}$ is generated by $\mathscr{P}_\theta$ and powers of quantum determinants. Set $X = (x_{ij})_{1 \leq i,j \leq N}$ and $D = (d_{ij})_{1 \leq i,j \leq N}$. Define X_1, X_2, D_1, and D_2 in the same way as T_1, T_2 (i.e. $X_1 = X \otimes I$, etc.). The following theorem summarizes properties of $\mathscr{P}_\theta$ from [19].

Theorem 3.1. *The relations satisfied by the generators x_{ij} of $\mathscr{P}_\theta$ are determined by the following equations:*

(i) *The matrix relation $R_\mathfrak{g} X_1 R_\mathfrak{g}^{t_1} X_2 = X_2 R_\mathfrak{g}^{t_1} X_1 R_\mathfrak{g}$.*
(ii) *The linear relations $x_{ij} = \gamma x_{ji}$ for $i < j$ and $x_{ab} = 0$ for all $(a, b) \in \mathcal{S}$ where*
- *$\gamma = q$ in Type AI, $\gamma = -q^{-1}$ in Type AII, $\gamma = 1$ in the diagonal type*
- *$\mathcal{S}$ is the empty set in Type AI, $\mathcal{S} = \{(i, i), i = 1, \dots, 2n\}$ in Type AII, and $\mathcal{S} = \{(i, j), 1 \leq i, j \leq n\} \cup \{ij, n + 1 \leq i, j \leq 2n\}$ in diagonal type.*

Moreover, $\mathscr{P}_\theta$ is a left $U_q(\mathfrak{g})$-module and (trivial) right $\mathcal{B}_\theta$-module subalgebra of $\mathscr{P}$.

In the diagonal case, it turns out that the matrix relations for $\mathscr{P}_\theta$ reduce to $R\hat{X}_1\hat{X}_2 = \hat{X}_2\hat{X}_1 R$ where $\hat{X} = (x_{i,j+n})_{1 \leq i,j \leq n}$ and R is the matrix defined by (3) with $N = n$ ([19], Lemma 5.12). In particular, $\mathscr{P}_\theta \cong \mathcal{O}_q(\mathrm{Mat}_n)$ as algebras via the map that sends $x_{i,j+n}$ to t_{ij} all $1 \leq i, j \leq n$. For

the other two families, Types AI and AII, explicit relations are given in [19], Lemma 5.8. Here we provide some of these relations in special cases that will be needed later in this paper. In particular, for Type AI we have

$$x_{en}x_{nn} = q^2 x_{nn}x_{en} \quad \text{and} \quad x_{an}x_{en} = qx_{en}x_{an} \tag{5}$$

for all $a < e < n$. Similarly, for Type AII, we have

$$x_{a,2n}x_{e,2n} = qx_{e,2n}x_{a,2n} \tag{6}$$

for $a < e < 2n$. More generally, it follows from [19], Lemma 5.8 that the defining relations are q-analogs of commutativity relations and take the form

$$x_{ij}x_{kl} = q^s x_{kl}x_{ij} + \sum_{\{a,b,c,d\}=\{i,j,k,l\}} (q - q^{-1})w x_{ab}x_{cd} \tag{7}$$

for some integer s and some element $w \in \mathbb{C}[q, q^{-1}]$ where neither $x_{ij}x_{kl}$ nor $x_{kl}x_{ij}$ appear in the final sum of the right hand side. It follows from the formulas for the relations satisfied by the t_{ij} and the fact that $\mathscr{P}_\theta \cong \mathcal{O}_q(\mathrm{Mat}_n)$ as algebras in the diagonal case, that (7) holds for the diagonal family as well.

The following result, also from [19], holds for $\mathscr{D}_\theta$ and shows that as an algebra, $\mathscr{D}_\theta$ is isomorphic to $\mathscr{P}_\theta^{op}$. In analogy to the situation for $\mathscr{P}_\theta$, the map sending $d_{i,j+n}$ to ∂_{ij} for $1 \leq i, j \leq n$ defines an isomorphism of $\mathscr{D}_\theta$ onto $\mathcal{O}_q(\mathrm{Mat}_n)^{op}$ for the diagonal family.

Theorem 3.2. *The relations satisfied by the generators d_{ij} of $\mathscr{D}_\theta$ are determined by the following equations:*

(i) *The matrix relation $R_\mathfrak{g} D_2 R_\mathfrak{g}^{t_1} D_1 = D_1 R_\mathfrak{g}^{t_1} D_2 R_\mathfrak{g}$.*
(ii) *The linear relations $d_{ij} = \gamma d_{ji}$ for $i < j$ and $d_{ab} = 0$ for all $(a, b) \in \mathcal{S}$ where*
- *$\gamma = q^{-1}$ in Type AI, $\gamma = -q$ in Type AII, $\gamma = 1$ in the diagonal type*
- *is the empty set in Type AI, $\mathcal{S} = \{ii, i = 1, \ldots, 2n\}$ in Type AII, and $\mathcal{S} = \{ij, 1 \leq i, j \leq n\} \cup \{ij, n + 1 \leq i, j \leq 2n\}$ in diagonal type.*

Moreover, $\mathscr{D}_\theta$ is a left $U_q(\mathfrak{g})$-module and (trivial) right $\mathcal{B}_\theta$-module subalgebra of $\mathscr{D}$.

Note that the defining relations for $\mathscr{P}_\theta$ are homogeneous and thus $\mathscr{P}_\theta$ has a natural degree function defined by $\deg(x_{ij}) = 1$ for all i, j. Let $\mathcal{J}$ be the filtration on $\mathscr{P}_\theta$ defined by deg. In particular, for each r, we have

$$\mathcal{J}_r(\mathscr{P}_\theta) = \sum_{j \leq r} \{x \in \mathscr{P}_\theta | \ \deg(x) \leq r\}.$$

Since all the relations for $\mathscr{P}_\theta$ are homogeneous with respect to degree, $\mathscr{P}_\theta$ is isomorphic to the graded algebra defined by this filtration. Just as for $\mathscr{P}_\theta$, we can define a degree function on $\mathscr{D}_\theta$ such that $\deg(d_{ij}) = 1$ for all i, j. The resulting filtration yields a graded algebra isomorphic to $\mathscr{D}_\theta$. For each r, set $\mathscr{P}_\theta^r$ equal to the homogeneous subspace of $\mathscr{P}_\theta^r$ consisting of elements of exactly degree r and $\mathscr{D}_\theta^r$ equal to the homogeneous subspace consisting of elements of exactly degree r.

The left $U_q(\mathfrak{g})$-module structures can be deduced directly from the left actions of $U_q(\mathfrak{g})$ on $\mathscr{P}$ and $\mathscr{D}$. The action of the generators of $U_q(\mathfrak{g})$ on the generators of $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ is explicitly given in [19], Lemma 5.4. As noted in [19], this action can be described as follows. The vector space spanned by the generators x_{ij} of $\mathscr{P}_\theta$ forms a simple left $U_q(\mathfrak{g})$-module generated by a highest weight vector x_{1r} of weight $L(\epsilon_1 + \epsilon_r)$ where $r = 1$ in Type AI, $r = 2$ in Type AII, and $r = n + 1$ in the diagonal type. Similarly, $\sum_{i,j} \mathbb{C}(q)d_{ij}$ is a simple left $U_q(\mathfrak{g})$-module generated by the lowest weight vector

d_{1r} of weight $-\epsilon_1 - \epsilon_r$ where $r = 1$ in Type AI, $r = 2$ in Type AII, and $r = n + 1$ in the diagonal type. Moreover, the action of the Cartan elements on $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ is determined by

$$K_{\epsilon_s} \cdot x_{ij} = q^{\delta_{is}+\delta_{js}} x_{ij} \qquad K_{\epsilon_s} \cdot d_{ij} = q^{-\delta_{is}-\delta_{js}} d_{ij} \tag{8}$$

for all i, j, s.

Note that the isomorphisms $\mathscr{P}_\theta \cong \mathcal{O}_q(\mathrm{Mat}_n)$ and $\mathscr{D}_\theta \cong \mathcal{O}_q(\mathrm{Mat}_n)^{op}$ in the diagonal setting as described above are actually $U_q(\mathfrak{gl}_n)$-bimodule isomorphisms where the left action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{O}_q(\mathrm{Mat}_n)$ (resp. $\mathcal{O}_q(\mathrm{Mat}_n)^{op}$) is the same as the action of the first copy of $U_q(\mathfrak{gl}_n)$ inside $U_q(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$ on $\mathscr{P}_\theta$ (resp. $\mathscr{D}_\theta$). Similarly, the right action for $\mathscr{P}_\theta$ (resp. $\mathscr{D}_\theta$) goes over to the action of the second copy of $U_q(\mathfrak{gl}_n)$ on $\mathcal{O}_q(\mathrm{Mat}_n)$ (resp. $\mathcal{O}_q(\mathrm{Mat}_n)^{op}$). (See [19], Lemma 5.12 for details.)

It is well-known that the algebra $\mathcal{O}_q(\mathrm{Mat}_N)$ admits a PBW basis using monomials in the t_{ij}. The next result from [19] shows that the same is true for $\mathscr{P}_\theta$. Using the fact that $x_{ij} \mapsto d_{ij}$ defines an antiautomorphism from $\mathscr{P}_\theta$ to $\mathscr{D}_\theta$, the next lemma also holds for $\mathscr{D}_\theta$ with each x_{ij} term replaced by d_{ij}.

Lemma 3.3. *The following monomials form a basis for $\mathscr{P}_\theta$:*

(i) *Type AI:*

$$x_{11}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1n}^{m_{1n}} x_{22}^{m_{22}} x_{23}^{m_{23}} \cdots x_{2n}^{m_{2n}} \cdots x_{n-1,n-1}^{m_{n-1,n-1}} x_{n-1,n}^{m_{n-1,n}} x_{nn}^{m_{nn}}$$

(ii) *Type AII:*

$$x_{12}^{m_{12}} x_{13}^{m_{13}} \cdots x_{1,2n}^{m_{1,2n}} x_{23}^{m_{23}} x_{24}^{m_{24}} \cdots x_{2,2n}^{m_{2,2n}} \cdots x_{2n-1,2n}^{m_{2n-1,2n}}$$

(iii) *Diagonal type:*

$$x_{1,n+1}^{m_{11}} x_{12}^{m_{12}} \cdots x_{1,2n}^{m_{1n}} x_{2,n+1}^{m_{21}} x_{2,n+2}^{m_{22}} \cdots x_{2,2n}^{m_{2n}} \cdots x_{n,n+1}^{m_{n1}} \cdots x_{n,2n}^{m_{nn}}$$

as each m_{ij} runs over nonnegative integers. Moreover, we also get a basis if the order of the monomials in the terms above are reversed.

3.3. **Module structure.** It is well-known that $\mathcal{O}_q(\mathrm{Mat}_N)$ admits a decomposition as $U_q(\mathfrak{gl}_N)$-bimodules

$$\mathcal{O}_q(\mathrm{Mat}_N) \cong \bigoplus_{\lambda \in \Lambda_N^+} L(\lambda) \otimes L(\lambda)^* \tag{9}$$

where $L(\lambda)$ is a left $U_q(\mathfrak{gl}_N)$-module and $L(\lambda)^*$ is a right $U_q(\mathfrak{gl}_N)$-module. By restriction, we can also view all these modules as $U_q(\mathfrak{sl}_N)$-modules. Note that $\mathcal{O}_q(\mathrm{Mat}_N)$ contains a nontrivial bi-invariant submodule with respect to the action of $U_q(\mathfrak{sl}_N)$. In particular, the submodule of $\mathcal{O}_q(\mathrm{Mat}_N)$ consisting of $U_q(\mathfrak{sl}_N)$-bi-invariants corresponds to

$$\bigoplus_{m \in \mathbb{N}} L(m\hat{\omega}_N) \otimes L(m\hat{\omega}_N)^*.$$

Decomposition (9) implies

$$\mathcal{O}_q(\mathrm{Mat}_N) \otimes \mathcal{O}_q(\mathrm{Mat}_N) \cong \bigoplus_{(\lambda,\lambda') \in \Lambda_N^+ \times \Lambda_N^+} L(\lambda \oplus \lambda') \otimes L(\lambda \oplus \lambda')^* \tag{10}$$

as $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$-bimodules. Moreover, the submodule of $\mathcal{O}_q(\mathrm{Mat}_N) \otimes \mathcal{O}_q(\mathrm{Mat}_N)$ consisting of $U_q(\mathfrak{sl}_N \oplus \mathfrak{sl}_N)$-bi-invariants is

$$\bigoplus_{m,m' \in \mathbb{N}} L(m\hat{\omega}_N \oplus m'\hat{\omega}_N) \otimes L(m\hat{\omega}_N \oplus m'\hat{\omega}_N)^*.$$

Setting $N = n$ in Type AI and the diagonal case and $N = 2n$ in Type AII, we can read off of (9) and (10) the $U_q(\mathfrak{g})$-module decomposition of $\mathscr{P}$. It follows from the description of spherical weights in Section 2.5 that the right $\mathcal{B}_\theta$ invariants $\mathscr{P}^{\mathcal{B}_\theta}$ of $\mathscr{P}$ admits the following decomposition as left $U_q(\mathfrak{g})$-modules

$$\bigoplus_{\lambda\in\Lambda_\Sigma^+, s\in\mathbb{N}} L(2\lambda + s\hat{\omega}_n) \text{ in Type AI,}$$

$$\bigoplus_{\lambda\in\Lambda_\Sigma^+} L(2\lambda) \text{ in Type AII, and}$$

$$\bigoplus_{\lambda\in\Lambda_\Sigma^+} L(2\lambda) = \bigoplus_{\gamma\in\Lambda_n^+} L(\gamma\oplus\gamma) \text{ in the diagonal type.}$$

By [31] Lemma 5.3, the fact that $\mathscr{P}_\theta$ is generated by right $\mathcal{B}_\theta$ invariant elements ensures that $\mathscr{P}_\theta$ is a subalgebra and submodule of $\mathscr{P}^{\mathcal{B}_\theta}$. We see in Section 4 that $\mathscr{P}^{\mathcal{B}_\theta}$ agrees with $\mathscr{P}_\theta$ in both Type AII and the diagonal case while it is slightly larger in Type AI.

4. Detailed module decompositions

4.1. **Quantum determinants.** There is a quantum analog of the determinant, denoted by $\det_q(T)$, which is a central element in $\mathcal{O}_q(\mathrm{Mat}_N)$. The quantum determinant can be expressed explicitly in terms of the t_{ij} as

$$\det_q(T) = \sum_{s\in\mathcal{S}_n} (-q)^{l(s)} t_{s(1),1}\cdots t_{s(N),N} \tag{11}$$

and satisfies $\iota(\det_q(T)) = \det_q(T)$ where ι is the antiautomorphism defined in Section 3.1 (see (1.26) of [21]). It is straightforward to check that the quantum determinant $\det_q(T)$ satisfies $\det_q(T)\cdot K_i = \det_q(T)$, $\det_q(T)\cdot F_i = 0$ and $\det_q(T)\cdot E_i = 0$ for $i = 1,\dots,N-1$. Hence $\det_q(T)$ is right invariant with respect to the action of $U_q(\mathfrak{sl}_N)$. The same is true with respect to the left action of $U_q(\mathfrak{sl}_N)$ and can be easily verified with a similar computation using the fact that $\iota(\det_q(T)) = \det_q(T)$. Hence $\det_q(T)$ is both a right and left invariant element with respect to the action of $U_q(\mathfrak{sl}_N)$. However, the same is not true upon passing to $U_q(\mathfrak{gl}_N)$. Indeed, we have

$$\det_q(T)\cdot K_{\epsilon_i} = K_{\epsilon_i}\cdot\det_q(T) = q\det_q(T). \tag{12}$$

Let $\det_q(T')$ be the quantum determinant defined using the elements $t_{i+N,j+N}$ instead of the t_{ij} viewed as elements of $\mathcal{O}_q(\mathrm{Mat}_n)\otimes\mathcal{O}_q(\mathrm{Mat}_n)$. The properties for $\det_q(T)$ carry over to $\det_q(T')$ where each subscript i is replaced by $i+N$.

Note that the weight of $\det_q(T)$ with respect to the action of $U_q(\mathfrak{gl}_N)$ is $\hat{\omega}_N = \epsilon_1+\cdots+\epsilon_N$. Hence by (9), the submodule of $U_q(\mathfrak{sl}_N)$-bi-invariants inside $\mathcal{O}_q(\mathrm{Mat}_N)$ is the polynomial ring $\mathbb{C}(q)[\det_q(T)]$. Similarly, the submodule of $\mathcal{O}_q(\mathrm{Mat}_N)\otimes\mathcal{O}_q(\mathrm{Mat}_N)$ consisting of $U_q(\mathfrak{sl}_N\oplus\mathfrak{sl}_N)$-bi-invariants is the polynomial ring $\mathbb{C}(q)[\det_q(T),\det_q(T')]$.

Lemma 4.1. *The intersection of* $\mathbb{C}(q)[\det_q(T)]$ *and* $\mathscr{P}_\theta$ *is* $\mathbb{C}(q)[(\det_q(T))^2]$ *in Type AI and equals* $\mathbb{C}(q)[\det_q(T)]$ *in type AII. The intersection of* $\mathbb{C}(q)[\det_q(T),\det_q(T')]$ *and* $\mathscr{P}_\theta$ *is* $\mathbb{C}(q)[\det_q(T)\det_q(T')]$ *in the diagonal case.*

Proof. Consider Type AI. Note that any element in $\mathscr{P}_\theta$ can be written as a linear combination of monomials, say $t_{i_1,j_1}\cdots t_{i_m,j_m}$. Moreover, by the form of the elements x_{ab}, each right index j_k must appear an even number of times in a particular monomial. Thus, examining $\det_q(T)$, we see

that $\det_q(T) \notin \mathscr{P}_\theta$ since each right index j_k shows up exactly once in $t_{s(1),1} \cdots t_{s(n),n}$. The same holds for $(\det_q(T))^m$ for m odd. On other hand, by Remark 4.12 of [21], $(\det_q(T))^2 \in \mathscr{P}_\theta$. Hence the first assertion for Type AI holds. For Type AII, note that Remark 4.12 of [21] also shows that $\det_q(T) \in \mathscr{P}_\theta$ in this case. This completes the proof of the first assertion of the lemma.

For the diagonal case, note that $\det_q(T)\det_q(T')$ is right invariant with respect to action of $U_q(\mathfrak{sl}_n \oplus \mathfrak{sl}_n)$ as well as with respect to the action of $K_{\epsilon_i}K_{\epsilon_{n+i}}^{-1}$ for $i = 1, \ldots, n$. Since $\mathcal{B}_\theta$ is a subalgebra of the algebra generated by both $U_q(\mathfrak{sl}_n \oplus \mathfrak{sl}_n)$ and the $K_{\epsilon_i}K_{\epsilon_{n+i}}^{-1}$ for $i = 1, \ldots, n$, it follows that $\det_q(T)\det_q(T')$ is an element of $\mathscr{P}^{\mathcal{B}_\theta}$. Now $\det_q(T)\det_q(T')$ has weight $\hat{\omega}_n \oplus \hat{\omega}_n$ with respect to the left action of $U_q(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$ and, moreover, generates a trivial left $U_q(\mathfrak{sl}_n \oplus \mathfrak{sl}_n)$-module. Hence from the description of $\mathscr{P}^{\mathcal{B}_\theta}$ in the diagonal case given in Section 3.3, we see that $(\det_q(T)\det_q(T'))^m$ is a basis vector for the one dimensional left module $L(m\hat{\omega}_n \oplus m\hat{\omega}_n)$. Thus $\mathscr{P}^{\mathcal{B}_\theta} \cap \mathbb{C}(q)[\det_q(T), \det_q(T')] = \mathbb{C}(q)[\det_q(T)\det_q(T')]$. Since $\mathscr{P}_\theta \subseteq \mathscr{P}^{\mathcal{B}_\theta}$, we also have that the intersection $\mathscr{P}_\theta \cap \mathbb{C}(q)[\det_q(T), \det_q(T')]$ is subset of $\mathbb{C}(q)[\det_q(T)\det_q(T')]$.

Now consider the element $\det_q(X)$ defined by replacing each t_{ij} in the definition of $\det_q(T)$ by x_{ij}, again in the diagonal case. Recall that $\mathscr{P}_\theta$ is isomorphic as an algebra and $U_q(\mathfrak{gl}_n)$-bimodule to $\mathcal{O}_q(\mathrm{Mat}_N)$ (see the discussions following Theorems 3.1 and 3.2). It follows that $\det_q(X)$ is an element of $\mathscr{P}^{\mathcal{B}_\theta}$ invariant with respect to the left action of $U_q(\mathfrak{sl}_n \oplus \mathfrak{sl}_n)$. Moreover, it is straightforward to see that the weight of $\det_q(X)$ is $\hat{\omega}_n \oplus \hat{\omega}_n$. By the previous paragraph, $\det_q(X)$ must be a nonzero scalar multiple of $\det_q(T)\det_q(T')$. This guarantees the inclusion $\mathbb{C}(q)[\det_q(T)\det_q(T')] \subseteq \mathscr{P}_\theta$ which combined with the previous paragraph yields the desired equality. □

Set $H_n = \det_q(T)\det_q(T')$ in the diagonal case, $H_n = \det_q(T)^2$ in Type AI, and $H_n = \det_q(T)$ in Type AII. By Lemma 4.1, $H_n \in \mathscr{P}_\theta$. Moreover, since $\det_q(T)$ is a central element in $\mathcal{O}_q(\mathrm{Mat}_N)$, we must have H_n is central in $\mathscr{P}_\theta$.

4.2. **Chains of algebras.** Consider the chain of quantized enveloping algebras

$$U_q(\mathfrak{g}_1) \subset U_q(\mathfrak{g}_2) \subset \cdots \subset U_q(\mathfrak{g}_n) = U_q(\mathfrak{g})$$

where $\mathfrak{g}_r = \mathfrak{gl}_r$ in Type AI, $\mathfrak{g}_r = \mathfrak{gl}_{2r}$ in Type AII, $\mathfrak{g}_r = \mathfrak{gl}_r \oplus \mathfrak{gl}_r$ in the diagonal case and $U_q(\mathfrak{g}_r)$ is identified with the subalgebra of $U_q(\mathfrak{g})$ generated by E_i, F_i, K_{ϵ_j} where

- $i \in \{1, \ldots, r-1\}$ and $j \in \{1, \ldots, r\}$ in Type AI
- $i \in \{1, \ldots, 2r-1\}$ and $j \in \{1, \ldots, 2r\}$ in Type AII
- $i \in \{1, \ldots, r-1\} \cup \{n+1, \ldots, n+r-1\}$, $j \in \{1, \ldots, r\} \cup \{n+1, \ldots, n+r\}$ in the diagonal case.

Note that $U_q(\mathfrak{g}_1)$ is just a commutative Laurent polynomial ring over $\mathbb{C}(q)$ in Types AI and the diagonal case. In Types AI, this Laurent polynomial ring is generated by $K_{\epsilon_1}$and in the diagonal case, it is generated by K_{ϵ_1} and $K_{\epsilon_{n+1}}$. In Type AII, $U_q(\mathfrak{g}_1)$ is the copy of the quantized enveloping algebra of $\mathfrak{gl}_2$ generated by E_1, F_1, and $K_{\epsilon_1}^{\pm 1}, K_{\epsilon_2}^{\pm 1}$.

Similarly, we have a chain of quantized function algebras

$$\mathscr{P}(\mathfrak{g}_1) \subset \mathscr{P}(\mathfrak{g}_2) \subset \cdots \subset \mathscr{P}(\mathfrak{g}_n) = \mathscr{P}$$

where $\mathscr{P}(\mathfrak{g}_r) \cong \mathcal{O}_q(\mathrm{Mat}_r)$ in Type AI, $\mathscr{P}(\mathfrak{g}_r) \cong \mathcal{O}_q(\mathrm{Mat}_{2r})$ in Type AII, and $\mathscr{P}(\mathfrak{g}_r) \cong \mathcal{O}_q(\mathrm{Mat}_r) \otimes \mathcal{O}_q(\mathrm{Mat}_r)$ in the diagonal case. Moreover, this isomorphism is an equality for $r = n$ and so $\mathscr{P}(\mathfrak{g}_n) = \mathscr{P}$. For $r < n$, $\mathscr{P}(\mathfrak{g}_r)$ is identified with the subalgebra of $\mathscr{P}$ generated by

- t_{ij} for $1 \leq i, j \leq r$ in Type AI
- t_{ij} for $1 \leq i, j \leq 2r$ in Type AII
- t_{ij} for $1 \leq i, j \leq r$ and $n+1 \leq i, j \leq n+r$ in the diagonal case.

Note that $\mathscr{P}(\mathfrak{g}_r)$ is a $U_q(\mathfrak{g}_r)$-bimodule and, moreover, this bimodule structure is compatible with the $U_q(\mathfrak{g})$-bimodule structure on $\mathscr{P}$.

Set $\mathcal{B}_\theta^r = \mathcal{B}_\theta \cap U_q(\mathfrak{g}_r)$ for $r = 1, \dots, n$. Note that this gives us a chain of subalgebras

$$\mathcal{B}_\theta^1 \subset \mathcal{B}_\theta^2 \subset \cdots \subset \mathcal{B}_\theta^n = \mathcal{B}_\theta.$$

For each r, it is straightforward to see that θ restricts to an involution on $\mathfrak{g}_r$ with fixed Lie subalgebra $\mathfrak{k}_r = \mathfrak{g}_r^\theta$. We write $\mathcal{B}_\theta^r$ for the right coideal subalgebra of $U_q(\mathfrak{g}_r)$ that is a quantum analog of $U(\mathfrak{k}_r)$. Note that $\mathcal{B}_\theta^r$ belongs to the same family as $\mathcal{B}_\theta$ with the only difference being the rank of the underlying Lie algebra $\mathfrak{g}_r$.

Using $\mathcal{B}_\theta^r$, $\mathscr{P}(\mathfrak{g}_r)$, and $U_q(\mathfrak{g}_r)$, one can define the quantized function algebra $\mathscr{P}_\theta(\mathfrak{g}_r)$ generated by elements $x(r)_{ij}$ where $1 \le i, j \le r$ in Type AI, $1 \le i, j \le 2r$ in Type AII, and $1 \le i, j - n \le r$ or $1 \le i - n, j \le r$ in the diagonal case. Here, we use the notation x_{ij} for the generators when $r = n$ (i.e. $x(n)_{ij} = x_{ij}$). Note that the difference between $x(r)_{ij}$ and x_{ij} has to do with which t_{kl} are involved in the expression of these elements in terms of elements of $\mathscr{P}$. For example, in Type AI,

$$x(r)_{ij} = \sum_{k=1}^{r} t_{ik} t_{jk}$$

while

$$x_{ij} = \sum_{k=1}^{n} t_{ik} t_{jk}.$$

Nevertheless, we see from the next lemma that this distinction is not important.

Lemma 4.2. *For each r, s with $1 \le r < s \le n$, the map $\psi_{r,s} : \mathscr{P}_\theta(\mathfrak{g}_r) \to \mathscr{P}_\theta(\mathfrak{g}_s)$ defined by $\psi_{r,s}(x(r)_{ij}) = x(s)_{ij}$ induces an algebra embedding which preserves the left $U_q(\mathfrak{g}_r)$-module and (trivial) right $\mathcal{B}_\theta^r$-module structures.*

Proof. Note that the relations for both algebras are given by Theorem 3.1. Moreover, there are two types of relations: linear and quadratic. The linear relations take the form $x_{ab} = 0$ for various conditions on a, b and $x_{ij} = \gamma x_{ji}$ for an appropriate scalar γ and all i, j. Clearly these agree for the two algebras and so $x(r)_{ab} = 0$ if and only if $x(s)_{ab} = 0$ and $x_{ij}(r) = \gamma x_{ji}(r)$ if and only if $x_{ij}(s) = \gamma x_{ji}(s)$.

By (7), the quadratic relations correspond to q-analogs of commutativity relations between two generators, say x_{ij} and x_{ld}. Moreover, the only terms showing up in these relations are of the form $x_{ab}x_{cd}$ where $\{a, b, c, d\} = \{i, j, l, d\}$. Now if i, j and l, d satisfy the necessary conditions for $x(r)_{ij}$ and $x(r)_{ld}$ to be generators of $\mathscr{P}_\theta(\mathfrak{g}_r)$, then $x(r)_{ab}$ is also a valid generator whenever $\{a, b\} \in \{i, j, l, d\}$. In other words, when i, j, l, d are chosen so that $x(r)_{ij}, x(r)_{ld}$ are among the generators for $\mathscr{P}_\theta(\mathfrak{g}_r)$, then the quadratic relations involving $x(r)_{ij}$ and $x(r)_{ld}$ inside $\mathscr{P}_\theta(\mathfrak{g}_r)$ take exactly the same form as the relations satisfied by x_{ij} and x_{ld} inside of $\mathscr{P}_\theta$. The same holds with r replaced by s. Thus, the quadratic relations satisfied by the $x(r)_{ij}$ of $\mathscr{P}_\theta(\mathfrak{g}_r)$ agree with the relations coming from the larger algebra $\mathscr{P}_\theta(\mathfrak{g}_s)$ for the corresponding elements $x(s)_{ij}$.

The module structures for both algebras can be deduced directly from the actions of $U_q(\mathfrak{g})$ and $\mathcal{B}_\theta$ on $\mathscr{P}$ and these actions do not depend on r or s, Thus, these module actions agree which yields the desired module isomorphisms. □

An immediate consequence of Lemma 4.2 is that $\mathscr{P}_\theta(\mathfrak{g}_r)$ is isomorphic to a subalgebra and left $U_q(\mathfrak{g}_r)$-submodule of $\mathscr{P}_\theta$ where each generator $x(r)_{ij}$ is mapped to x_{ij}. Moreover, it is straightforward to see that the embeddings of this lemma are all compatible with each other and so $\psi_{r,s} \circ \psi_{s,m} = \psi_{r,m}$ for all $1 \leq r < s < m \leq n$.

4.3. **Highest weight generators.** Let $T_{(r)}$ be the submatrix of T with entries t_{ij} where $1 \leq i, j \leq r$ in Type AI and Type AII. Similarly, let $T'_{(r)}$ be the submatrix of T' with entries $t_{n+i,n+j}$ where again $1 \leq i, j \leq r$. Set

- $\hat{H}_r = (\det_q T_{(r)})^2$ for Type AI
- $\hat{H}_r = \det_q T_{(2r)}$ in Type AII and
- $\hat{H}_r = (\det_q T_{(r)})(\det_q T'_{(r)})$ in the diagonal case.

Note that $\hat{H}_r \in \mathscr{P}(\mathfrak{g}_r)$. Moreover, by Lemma 4.1, we have $\hat{H}_r \in \mathscr{P}_\theta(\mathfrak{g}_r)$. Set $H_r = \psi_{r,n}(\hat{H}_r)$ for $r = 1, \ldots, n$.

Note that for each r, $\mathscr{P}_\theta(\mathfrak{g}_r)$ has a natural degree function compatible with the degree function on $\mathscr{P}_\theta$. In particular, we have $\deg x(r)_{ij} = 1$ all r, i, j.

Proposition 4.3. *The elements $H_1, \ldots, H_n$ generate a commutative subring of $\mathscr{P}_\theta$ that is isomorphic to a polynomial ring in these n generators. Moreover, H_r is a highest weight vector of weight $2\hat{\eta}_r$ with respect to the left action of $U_q(\mathfrak{g})$ and $H_r \in \mathcal{J}_r(\mathscr{P}_\theta)$.*

Proof. Since $\det_q(T_{(r)})$ is a central element of $\mathcal{O}_q(\mathrm{Mat}_r)$, it follows that $\hat{H}_r$ is in the center of $\mathscr{P}(\mathfrak{g}_r)$. Hence $\hat{H}_r$ is in the center of $\mathscr{P}_\theta(\mathfrak{g}_r)$. It follows that $\hat{H}_r$ commutes with each $\psi_{m,r}(\hat{H}_m)$ for $m \leq r$. By induction, we get $\psi_{1,r}(\hat{H}_r), \ldots, \psi_{r-1,r}(\hat{H}_r), \hat{H}_r$ generates a commutative subring of $\mathscr{P}_\theta(\mathfrak{g}_r)$. When $r = n$, this sequence is simply $H_1, \ldots, H_n$ and so these elements generate a commutative subring of $\mathscr{P}_\theta$.

Note that for each of the three families, H_r has degree $2r$ viewed as a (noncommutative) polynomial in the t_{ij}. Since the $x_{ij}(r)$ are quadratic in the t_{ij} terms, it follows that $\hat{H}_r$ has degree r viewed as a (noncommutative) polynomial in the $x_{ij}(r)$. Hence H_r has degree r as a polynomial in the x_{ij} and so $H_r \in \mathcal{J}_r(\mathscr{P}_\theta)$.

The fact that H_r is left $U_q(\mathfrak{g}_r)$ invariant follows from the definition of $\hat{H}_r$ and properties of quantum determinants (see Section 4.1). Since $\hat{H}_r$ only contains terms t_{kl} with $1 \leq k, l \leq r$, it follows that H_r only contains terms x_{ij} for $1 \leq i, j \leq r$. The formulas for the left action of $U_q(\mathfrak{gl}_N)$ on $\mathcal{O}_q(\mathrm{Mat}_N)$ (in Section 3.1) ensure that $E_i \cdot t_{jk} = 0$ for all i with $i > j$. Hence, $E_s \cdot a = 0$ for all $a \in \mathscr{P}(\mathfrak{g}_r)$ where $s > r$ in Type AI, $s > 2r$, and $r < s \leq n, n + r < s \leq 2n$ in the diagonal case. This guarantees that $E_s \cdot H_r = 0$ for all $E_s \notin U_q(\mathfrak{g}_r)$. But we also know that $E_s \cdot \hat{H}_r = 0$ for all $E_s \in U_q(\mathfrak{g}_r)$ because of the left $U_q(\mathfrak{g}_r)$ invariant property of H_r. This proves that $E_s \cdot H_r = 0$ for all $E_s \in U_q(\mathfrak{g})$.

Note that $\det_q(T_{(r)})$ has weight $\epsilon_1 + \cdots + \epsilon_r$ in terms of the left action of $U_q(\mathfrak{gl}_r)$. Thus, it is straightforward to see from the definitions of $\hat{H}_r$ and of the restricted weight $\hat{\eta}_r$ (see Section 2.5) that $\hat{H}_r$ has weight $2\hat{\eta}_r$ for each r. The weight of H_r is the same as that of $\hat{H}_r$, hence by the previous paragraph, H_r is a highest weight vector of weight $2\hat{\eta}_r$ with respect to the left action of $U_q(\mathfrak{g})$.

We finish the proof by arguing that the $H_1, \ldots, H_n$ are algebraically independent and hence the ring they generate is a polynomial ring in these n generators. Suppose

$$\sum_m a_m H_1^{m_1} \cdots H_n^{m_n} = 0$$

where $m = (m_1, \dots, m_n)$. Since the monomials $H_1^{m_1} \cdots H_n^{m_n}$ have distinct weights $\sum_i 2m_i\hat{\eta}_i$, we can separate the monomials using the action of $U_q(\mathfrak{g})$. Thus the above equality implies

$$a_m H_1^{m_1} \cdots H_n^{m_n} = 0$$

and hence $a_m = 0$ each m. □

We frequently write $H_{2\mu}$ for the element $H_1^{m_1} \cdots H_n^{m_n}$ for each $\mu = \sum_i m_i\hat{\eta}_i$, thus labeling this element by its weight. In particular, by the above proposition, $H_{2\mu}$ is a highest weight vector of weight 2μ with respect to the action of $U_q(\mathfrak{g})$ on $\mathscr{P}_\theta$.

4.4. **Explicit module descriptions.** The decompositions of Section 3.3 combined with Lemma 4.1 ensure that

$$\mathscr{P}_\theta \cong \bigoplus_{\lambda \in \Lambda_\Sigma^+} L(2\lambda) \tag{13}$$

as left $U_q(\mathfrak{g})$-modules and trivial right $\mathcal{B}_\theta$-modules. (Note that this only differs in Type AI from the decompositions in Section 3.3.)

Theorem 4.4. *We have*

$$\mathscr{P}_\theta = \bigoplus_{\lambda \in \Lambda_\Sigma^+} (U_q(\mathfrak{g})) \cdot H_{2\lambda}$$

where $(U_q(\mathfrak{g})) \cdot H_{2\lambda}$ is isomorphic to the simple (left) $U_q(\mathfrak{g})$-module generated by the highest weight vector $H_{2\lambda}$ with weight 2λ and is a trivial right $\mathcal{B}_\theta$-module.

Proof. We see from (13) that $\mathscr{P}_\theta$ is the direct sum of finite-dimensional simple modules with respect to the left action of $U_q(\mathfrak{g})$. Moreover, the decomposition is multiplicity free with exactly one finite-dimensional simple module of the form $L(2\lambda)$ for each $\lambda \in \Lambda_\Sigma^+$. By Proposition 4.3, $H_{2\lambda}$ generates a finite-dimensional simple (left) $U_q(\mathfrak{g})$-module with highest weight 2λ for each $\lambda \in \Lambda_\Sigma^+$. Thus the module $L(2\lambda)$ in the decomposition (13) must correspond to $(U_q(\mathfrak{g})) \cdot H_{2\lambda}$. □

The entire construction of this section can be transferred to the differential parts $\mathscr{D}_\theta$ by using the antiautomorphisms sending $\mathscr{P}$ to $\mathscr{D} = \mathscr{P}^{op}$. Let $H_{2\mu}^*$ be the image of $H_{2\mu}$ via this antiautomorphism. A comparison of the action of $U_q(\mathfrak{g})$ on $\mathscr{P}$ and on $\mathscr{D}$ yields that $H_{2\lambda}^*$ is a lowest weight vector of weight -2λ. Hence, we have a similar decomposition as above for $\mathscr{D}_\theta$, again as left $U_q(\mathfrak{g})$-modules and trivial right $\mathcal{B}_\theta$-modules

$$\mathscr{D}_\theta \cong \oplus_{\lambda \in \Lambda_\Sigma^+} (U_q(\mathfrak{g})) \cdot H_{2\lambda}^*.$$

Note that $(U_q(\mathfrak{g})) \cdot H_{2\lambda}^*$ can be viewed as the left dual of $(U_q(\mathfrak{g})) \cdot H_{2\lambda}$.

Remark 4.5. *The generators of each module $L(2\lambda)$ is expressed using formulas in terms of the t_{ij} for Types AI and AII in* [21] *(see Lemma 4.10.A), Our approach yields another concrete identification of these generators that also applies to the diagonal case. Our methods, which rely directly on quantum determinants also lead to formulas in the t_{ij}. However, because these highest weight terms are elements of $\mathscr{P}_\theta(\mathfrak{g}_r)$ for various choices of r, it is easier to read off the possible x_{ij} that may appear. This will be helpful in describing and analyzing the quantum Capelli operators in Section 9.1 of this paper.*

5. Quantum Weyl Algebras

5.1. **Generators and relations.** We associate a quantum Weyl algebra $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$ with polynomials corresponding to $\mathcal{O}_q(\mathrm{Mat}_N)$ and constant term differentials corresponding to $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$ as defined and studied in [31], [3], [2], and [19]. This Weyl algebra $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$ is generated by t_{ij} and ∂_{ij} for $1 \leq i, j \leq N$. The algebra $\mathcal{O}_q(\mathrm{Mat}_N)$ (resp. $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$) embeds inside $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$ and can be identified with the subalgebra generated by the t_{ij} (resp. ∂_{ij}). Moreover, the t_{ij} and ∂_{ij} satisfy the following relation

$$\partial_{ab} t_{ef} = \sum_{r,l,j,k} (R^{t_2})^{rl}_{ea} (R^{t_2})^{jk}_{fb} t_{rj} \partial_{lk} + \delta_{ae}\delta_{bf}$$

for all a, b, e, f in $\{1, \ldots, N\}$. The quantum Weyl algebra $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$ inherits the structure of a $U_q(\mathfrak{gl}_N)$-bimodule from the bimodules $\mathcal{O}_q(\mathrm{Mat}_N)$ and $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$.

We may view $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ as right $\mathscr{B}_\theta$ invariant subalgebras of $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$. However, together, they generate an algebra inside of $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$ that is too large to be taken for a quantum analog of the Weyl algebra with polynomial part equal to $\mathscr{P}_\theta$. In particular, the subset $\mathscr{P}_\theta\mathscr{D}_\theta$ consisting of sums of terms of the form pd with $p \in \mathscr{P}_\theta$ and $d \in \mathscr{D}_\theta$ is strictly smaller than the subalgebra generated by $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ (see [19] for more details). Instead, we use the construction of [19] which starts with a twisted tensor product of $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ and deforms it so as to add constant terms to some of the relations. The result is the quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$ associated to θ for each of the three settings of this paper. As an algebra, $\mathscr{P}\mathscr{D}_\theta$ is generated by x_{ij}, d_{ij}, $1 \leq i, j \leq N$ where $N = n$ in Type AI and $N = 2n$ in Type AII and the diagonal case. The algebra $\mathscr{P}_\theta$ (resp. $\mathscr{D}_\theta$) embeds inside $\mathscr{P}\mathscr{D}_\theta$ and can be identified with the subalgebra generated by the x_{ij} (resp. d_{ij}). Moreover, the x_{ij} and d_{ij} satisfy the following relation

$$d_{ab} x_{ef} = \sum_{w,r,x,q,p,m,y,l} (R^{t_2}_{\mathfrak{g}})^{wr}_{xq} (R^{t_2}_{\mathfrak{g}})^{pq}_{ma} (R^{t_2}_{\mathfrak{g}})^{xy}_{fl} (R^{t_2}_{\mathfrak{g}})^{ml}_{eb} x_{pw} d_{ry} + q^{-\delta_{ef}} \delta_{ae}\delta_{bf} \tag{14}$$

for all $a, b, e, f \in 1, \ldots, N$. The map $\mathscr{P}_\theta \otimes \mathscr{D}_\theta$ to $\mathscr{P}\mathscr{D}_\theta$ defined by multiplication is an isomorphism of left $U_q(\mathfrak{g})$-module algebra and (trivial) right $\mathcal{B}_\theta$-module algebras. Relations (14) in the diagonal case are equivalent to the simpler relations

$$d_{a,b+n} x_{e,f+n} = \sum_{r,l,j,k} (R^{t_2})^{rl}_{ea} (R^{t_2})^{jk}_{fb} x_{r,j+n} d_{l,k+n} + \delta_{ae}\delta_{bf} \tag{15}$$

for all $1 \leq a, b, e, f \leq n$ where here we are taking into account linear relations satisfied by the d_{ij} and by the x_{ij}. In particular, this relation combined with results from Section 3.2 ensure that in the diagonal case $\mathscr{P}\mathscr{D}_\theta$ and $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_n)$ are isomorphic as algebras via the map sending $x_{i,j+n}$ to t_{ij} and $d_{i,j+n}$ to ∂_{ij} for each $1 \leq i, j \leq n$. (For additional details, see [19].)

The following result from [19] gives insight into the overall form of the relations coming from the twisting map.

Theorem 5.1. *([19], Corollary 8.11) For each of the three families, the following inclusions hold for the quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$*

$$d_{ab} x_{ef} - q^{\delta_{af}+\delta_{ae}+\delta_{bf}+\delta_{be}} x_{ef} d_{ab} - q^{-\delta_{ef}} \delta_{ae}\delta_{bf} \in \sum_{(e',f',a',b')>(e,f,a,b)} \mathbb{C}(q) x_{e'f'} d_{a'b'}$$

for all $a, b, e, f \in \{1, \ldots, \mathrm{rank}(\mathfrak{g})\}$ where

- *$a \leq b$ and $e \leq f$ in Type AI*
- *$a < b$ and $e < f$ in Type AII*

- *$a \leq n < b$ and $e \leq n < f$ in diagonal type*

and $(e', f', a', b') > (e, f, a, b)$ if and only if $e' \geq e, f' \geq f, a' \geq a, b' \geq b$ and at least one of these inequalities is strict.

It is also helpful for arguments later in the paper to express these relations in special cases. We do this in the next lemma.

Lemma 5.2. *In Type AI and for $a < n$, we have*

(i) $d_{an}x_{en} = q^{1+\delta_{ae}}x_{en}d_{an} - \delta_{ae}\sum_{a'>a} q^{2+\delta_{a'n}}(q^{-2}-1)x_{a'n}d_{a'n} + \delta_{ae}$ *where $e < n$,.*

(ii) $d_{nn}x_{ef} = q^{\delta_{nf}+\delta_{ne}}x_{ef}d_{nn} + q^{-\delta_{ef}}\delta_{ne}\delta_{nf}$ *where $e \leq f$.*

In Type AII and for $a < 2n, e < 2n$, we have

(iii) $d_{a,2n}x_{e,2n} = q^{1+\delta_{ae}}x_{e,2n}d_{a,2n} - \delta_{ae}\sum_{a'>a} q^{2+\delta_{a',2n}}(q^{-2}-1)x_{a',2n}d_{a',2n} + \delta_{ae}$.

Proof. Consider Type AI. Using the explicit formulas for the entries of R (see Section 3.1) $(R^{t_2})^{ul}_{vn} = r^{un}_{vl} \neq 0$ implies that $n = l$ and $u = v$. Similarly, $(R^{t_2})^{ul}_{nv} \neq 0$ implies that $n = v = u = l$. Moreover, if $v < n$, then $(R^{t_2})^{vn}_{vn} \neq 1$ while if $v = n$, then $(R^{t_2})^{nn}_{nn} = q$.

Suppose that a and e are both strictly less than n. By the above information about the entries for R, we get

$$\begin{aligned} d_{an}x_{en} &= (R^{t_2})^{na}_{na}(R^{t_2})^{na}_{na}(R^{t_2})^{ea}_{ea}(R^{t_2})^{nn}_{nn}(R^{t_2})^{en}_{en}x_{en}d_{an} \\ &\quad + \delta_{ae}\sum_{a'>a}(R^{t_2})^{na'}_{na'}(R^{t_2})^{a'a'}_{aa}(R^{t_2})^{nn}_{nn}(R^{t_2})^{en}_{en}x_{a'n}d_{a'n} + \delta_{ae} \\ &= q^{1+\delta_{ae}}x_{en}d_{an} - \delta_{ae}\sum_{a'>a} q^{2+\delta_{a'n}}(q^{-2}-1)x_{a'n}d_{a'n} + \delta_{ae}. \end{aligned} \tag{16}$$

This proves (i).

Using (14) and the above information about the entries for R, we see that

$$\begin{aligned} d_{nn}x_{ef} &= (R^{t_2})^{fn}_{fn}(R^{t_2})^{en}_{en}(R^{t_2})^{fn}_{fn}(R^{t_2})^{en}_{en}x_{ef}d_{nn} + q^{-\delta_{ef}}\delta_{ne}\delta_{nf} \\ &= q^{2\delta_{nf}+2\delta_{ne}}x_{ef}d_{nn} + q^{-\delta_{ef}}\delta_{ne}\delta_{nf} \end{aligned} \tag{17}$$

for $e \leq f$. This proves (ii).

The argument for (iii) is the same as for (i) with n replaced by $2n$ everywhere. □

We can define a degree function and related filtration on $\mathscr{P}\mathscr{D}_\theta$ that is compatible with the degree functions on $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ (see Section 3.2). In particular, the degree function for the quantum Weyl algebra $\mathscr{P}\mathscr{D}_\theta$ is $\deg d_{ij} = \deg x_{ij} = 1$ for all i, j. We use the same notation, namely $\mathcal{J}$, to denote the filtration on $\mathscr{P}\mathscr{D}_\theta$ defined by this degree function.

Recall that multiplication induces an isomorphism from $\mathscr{P}_\theta \otimes \mathscr{D}_\theta$ to $\mathscr{P}\mathscr{D}_\theta$. Inductively define subspaces of $\mathscr{P}\mathscr{D}_\theta$ by $\tilde{\mathcal{J}}_0 = \mathbb{C}(q)$ and

$$\tilde{\mathcal{J}}_r = \sum_{u+v=r} \mathcal{J}_u(\mathscr{P}_\theta)\mathcal{J}_v(\mathscr{D}_\theta).$$

Since the map defined by multiplication from $\mathscr{P} \otimes \mathscr{D}$ to $\mathscr{P}\mathscr{D}_\theta$ is a vector space isomorphism, we see that $\tilde{\mathcal{J}}_{r-1}$ is a proper subspace of $\tilde{\mathcal{J}}_r$ for all $r \geq 0$. We get a filtration on $\mathscr{P}\mathscr{D}_\theta$ by setting

$$\mathcal{J}_r(\mathscr{P}\mathscr{D}_\theta) = \sum_{u \leq r} \tilde{\mathcal{J}}_r(\mathscr{P}\mathscr{D}_\theta).$$

Note that the filtration $\mathcal{J}$ is preserved by the action of $U_q(\mathfrak{g})$. Moreover, it agrees with the filtrations on $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ defined via degree as presented in Section 3.2. Since $\mathscr{P}\mathscr{D}_\theta$ is a PBW deformation of a twisted tensor product of $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$, the associated graded algebra defined by $\mathcal{J}$ is simply this twisted tensor product.

5.2. **Action on polynomials.** Let $\mathcal{L}$ be the left ideal of $\mathscr{P}\mathscr{D}_\theta$ generated by the elements d_{ij} for $1 \leq i \leq j \leq N$ where $N = n$ in Type AI and $N = 2n$ in the other two cases. Note that $\mathscr{P}\mathscr{D}_\theta$ admits a direct sum decomposition $\mathscr{P}\mathscr{D}_\theta = \mathscr{P}_\theta \oplus \mathcal{L}$. Define an action of $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$ via this projection. In other words $a \cdot b = c$ where c is the unique element of $\mathscr{P}_\theta$ such that $ab - c \in \mathcal{L}$.

Recall (Section 3.2) that $\mathscr{P}_\theta^r$ equal to the homogeneous space of degree r with respect to the degree filtration $\mathcal{J}$. Note that $\mathscr{P}_\theta^0 = \mathbb{C}(q)$ and so $\mathscr{P}_\theta = \mathbb{C}(q) \oplus \sum_{r>0} \mathscr{P}_\theta^r$. This decomposition can be used to define a projection of $\mathscr{P}_\theta$ onto the scalars. Define a bilinear form $\langle \cdot, \cdot \rangle$ from $\mathscr{D}_\theta \times \mathscr{P}_\theta$ to $\mathbb{C}(q)$ by

$$\langle d, p \rangle = (d \cdot p)_0 \tag{18}$$

where $(a)_0$ denotes the image of a in $\mathbb{C}(q)$ via this projection.

Lemma 5.3. *The bilinear form $\langle \cdot, \cdot \rangle$ on $\mathscr{D}_\theta \times \mathscr{P}_\theta$ satisfies*

$$\sum \langle u_{(1)} \cdot d, u_{(2)} \cdot p \rangle = \epsilon(u) \langle d, p \rangle$$

for all $u \in U_q(\mathfrak{g})$, $d \in \mathscr{D}_\theta$ and $p \in \mathscr{P}_\theta$ and hence is (left) $U_q(\mathfrak{g})$ invariant.

Proof. Write $dp = \langle d, p \rangle + a$ where $a \in \sum_{r>0} \mathscr{P}_\theta^r$. Since $\mathscr{P}\mathscr{D}_\theta$ is a left $U_q(\mathfrak{g})$-module, we have

$$u \cdot (dp) = \sum (u_{(1)} \cdot d)(u_{(2)} \cdot p) = u \cdot (dp).$$

On the other hand

$$u \cdot (dp) = u \cdot (\langle d, p \rangle + a) = u \cdot (\langle d, p \rangle) + u \cdot a = \epsilon(u) \langle d, p \rangle + u \cdot a$$

since $\langle d, p \rangle$ is a scalar. Since the action of $U_q(\mathfrak{g})$ on $\mathscr{P}_\theta$ preserves degree, $\sum_{r>0} \mathscr{P}_\theta^r$ is a $U_q(\mathfrak{g})$-module. Thus, $u \cdot a \in \sum_{r>0} \mathscr{P}_\theta^r$. Hence putting together the two formulas for $u \cdot (dp)$ yields

$$\sum (u_{(1)} \cdot d) \cdot (u_{(2)} \cdot p) \in \epsilon(u) \langle d, p \rangle + \sum_{r>0} \mathscr{P}_\theta^r$$

which implies the desired property. Thus $\langle \cdot, \cdot \rangle$ is (left) $U_q(\mathfrak{g})$ invariant. □

The action of $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$ can be viewed as a map of algebras, say ϕ, from $\mathscr{P}\mathscr{D}_\theta$ into End $\mathscr{P}_\theta$. (Here we write End $\mathscr{P}_\theta$ for $\mathrm{End}_{\mathbb{C}(q)} \mathscr{P}_\theta$ which are endomorphisms over the scalars. The field $\mathbb{C}(q)$ is dropped since it can be understood from context.) Given $a \in \mathscr{P}\mathscr{D}_\theta$ we frequently write ϕ_a for $\phi(a)$ in order to make the exposition below clearer. Since $\mathscr{P}_\theta$ is a left $U_q(\mathfrak{g})$-module, End $\mathscr{P}_\theta$ inherits the structure of a $U_q(\mathfrak{g})$-bimodule in the standard way. Thus End $\mathscr{P}_\theta$ is an (ad $U_q(\mathfrak{g})$)-module via

$$((\mathrm{ad}\ a) \cdot b)(p) = \left(\sum a_{(1)} b S(a_{(2)}) \right)(p) = \sum a_{(1)} b \left(S(a_{(2)})(p) \right)$$

for all $a \in U_q(\mathfrak{g})$, $b \in$ End $\mathscr{P}_\theta$ and $p \in \mathscr{P}_\theta$.

Proposition 5.4. *The map ϕ is a $U_q(\mathfrak{g})$-module map with respect to the left action on $\mathscr{P}\mathscr{D}_\theta$ and the left adjoint action on* End $\mathscr{P}_\theta$ *and so*

$$\phi_{u \cdot a} = (\mathrm{ad}\ u) \cdot \phi_a \tag{19}$$

for all $a \in \mathscr{P}\mathscr{D}_\theta$ and $u \in U_q(\mathfrak{g})$. Thus the image of $\mathscr{P}\mathscr{D}_\theta$ under ϕ is an (ad $U_q(\mathfrak{g})$)*-submodule of* End $\mathscr{P}_\theta$.

Proof. Consider $a \in \mathscr{P}\mathscr{D}_\theta$, $b \in \mathscr{P}_\theta$, and $c \in \mathscr{P}_\theta$ such that $ab - c \in \mathcal{L}$ and so $\phi_a(b) = a \cdot b = c$. We prove the proposition by showing that (19) holds for all generators of $U_q(\mathfrak{g})$ playing the role of u. Given K in the Cartan part of $U_q(\mathfrak{g})$, we have

$$((\text{ad } K) \cdot \phi_a)(K \cdot b) = (K \cdot \phi_a \cdot K^{-1})(K \cdot b) = (K \cdot \phi_a)(b) = K \cdot \phi_a(b).$$

Since $\phi_a(b) = c$, it follows that $K \cdot (\phi_a(b)) - K \cdot (ab) \in \mathcal{L}$. On the other hand, $K \cdot \mathcal{L} = \mathcal{L}$ and $K \cdot (ab) = (K \cdot a)(K \cdot b)$. Hence

$$((\text{ad } K) \cdot \phi_a)(K \cdot b) - (K \cdot a)(K \cdot b) \in \mathcal{L}$$

for all $b \in U_q(\mathfrak{g})$. Since $(K \cdot a)(K \cdot b) - \phi_{K \cdot a}(K \cdot b) \in \mathcal{L}$, it follows that

$$((\text{ad } K) \cdot \phi_a)(K \cdot b) - \phi_{K \cdot a}(K \cdot b) \in \mathcal{L}$$

for all $b \in U_q(\mathfrak{g})$. Replacing b with $K^{-1} \cdot b$ in the above expression shows that $\phi_{K \cdot a}$ and $(\text{ad } K) \cdot \phi_a$ agree on all $b \in \mathscr{P}_\theta$. Thus $\phi_{K \cdot a} = (\text{ad } K) \cdot \phi_a$ for all $a \in \mathscr{P}\mathscr{D}_\theta$.

Now fix i and consider the action of E_i on a and ϕ_a. Recall the description of $(\text{ad } E_i)$ from Section 2.3. Since $\mathscr{P}\mathscr{D}_\theta$ is a left $U_q(\mathfrak{g})$- module algebra, using the form of $\Delta(E_i)$ from Section 2.2, we have $E_i \cdot ab = (E_i \cdot a)b + (K_i \cdot a)(E_i \cdot b)$ for all $a, b \in \mathscr{P}\mathscr{D}_\theta$. Hence, for all $b \in \mathscr{P}_\theta$ we have

$$\begin{aligned}((\text{ad } E_i) \cdot \phi_a)(b) &= (E_i \cdot \phi_a)(b) - (K_i \cdot \phi_a \cdot K_i^{-1} E_i)(b) = E_i \cdot \phi_a(b) - K_i \cdot (\phi_a(K_i^{-1} E_i \cdot b)) \\ &\in E_i \cdot ab - (K_i \cdot \phi_a \cdot K_i^{-1})(E_i \cdot b) + \mathcal{L} \\ &= E_i \cdot ab - (K_i \cdot a)(E_i \cdot b) + \mathcal{L} = (E_i \cdot a)(b) + \mathcal{L}.\end{aligned}$$

Hence $(\text{ad } E_i) \cdot \phi_a = \phi_{(E_i \cdot a)}$. A similar arguent holds for E_i replaced by F_i. □

5.3. **Orthogonality conditions.** As explained in Section 5.1, as an algebra, $\mathscr{P}\mathscr{D}_\theta$ is isomorphic to $\mathscr{P}\mathscr{D}_q(\text{Mat}_n)$ in the diagonal case. Thus, the second assertion of the next result is a generalization of [3], Proposition 1 to include the other two families.

Proposition 5.5. *For each r and s with $r \neq s$, $\mathscr{D}_\theta^r$ is equal to the vector space dual of $\mathscr{P}_\theta^r$ and $\mathscr{D}_\theta^r$ is orthogonal to $\mathscr{P}_\theta^s$ with respect to the bilinear form defined by (18). Moreover, $\mathscr{P}_\theta$ is a faithful $\mathscr{P}\mathscr{D}_\theta$-module with respect to the action defined above.*

Proof. Note that the relation between the bilinear form and the action ensures that the moreover part of the proposition is an immediate consequence of the main assertion. Hence, we focus on proving the duality result.

By Theorem 5.1

$$\mathscr{D}_\theta^{r+s} \mathscr{P}_\theta^s \subseteq \sum_{s \geq k \geq 0} \mathscr{P}_\theta^k \mathscr{D}_\theta^{r+k} \text{ and } \mathscr{D}_\theta^r \mathscr{P}_\theta^{r+s} \subseteq \sum_{s \geq k \geq 0} \mathscr{P}_\theta^{r+k} \mathscr{D}_\theta^k$$

for all $r \geq 0$ and $s \geq 0$. It follows that $(d \cdot p)_0 = 0$ for $d \in \mathscr{D}_\theta^r, p \in \mathscr{P}_\theta^s$ with $r \neq s$.

By Lemma 3.3, the set of monomials of the form

$$x_{e_1,f_1}^{m_1} x_{e_2,f_2}^{m_2} \cdots x_{e_r,f_r}^{m_r}$$

form a PBW basis for $\mathscr{P}_\theta^m$ where $m = m_1 + \cdots + m_r$ and $(e_1, f_1) > (e_2, f_2) > \cdots > (e_r, f_r)$. As explained preceding Lemma 3.3, the analogous result holds for $\mathscr{D}_\theta$ with each x_{ij} replaced by d_{ij}. Moreover, by Lemma 3.3, we can switch the ordering of the subscripts and still get a PBW basis. We do this below for $\mathscr{D}_\theta$. Consider a sequence of ordered pairs $(e_1, f_1), \ldots, (e_r, f_r)$ satisfying

$$(a, b) \geq (e_1, f_1) > (e_2, f_2) > \cdots > (e_r, f_r)$$

where here " $>$ " is the standard lexicographic ordering (from left to right). It follows from Theorem 5.1 that

$$d_{ab}x_{e_1,f_1}^{m_1}x_{e_2,f_2}^{m_2}\cdots x_{e_r,f_r}^{m_r} \in c\ \delta_{a,e_1}\delta_{b,f_1}x_{e_1.f_1}^{m_1-1}x_{e_2,f_2}^{m_2}\cdots x_{e_r,f_r}^{m_r} + \mathscr{P}_\theta^{m-1} + \sum_{(a',b')\geq(a,b)} \mathscr{P}_\theta^m d_{a'b'}$$

where $m = m_1 + \cdots + m_r$ and c is a nonzero scalar. Hence given $h \in \mathscr{D}_\theta^{m-1}$, we have

$$(hd_{ab}\cdot x_{e_1,f_1}^{m_1}x_{e_2,f_2}^{m_2}\cdots x_{e_r,f_r}^{m_r})_0 \neq 0$$

implies that $(a,b) = (e_1,f_1)$. Using induction, we obtain

$$(d_{a_k,b_k}^{s_k}d_{a_{k-1},b_{k-1}}^{s_{k-1}}\cdots d_{a_1,b_1}^{s_1}\cdot x_{e_1,f_1}^{m_1}x_{e_2,f_2}^{m_2}\cdots x_{e_r,f_r}^{m_r})_0 \neq 0$$

with

$$(a_k,b_k) < (a_{k-1},b_{k-1}) < \cdots < (a_1,b_1)$$

if and only if $r = k, s_i = m_i, e_i = a_i$, and $f_i = b_i$ for $i = 1,\ldots,r$. This proves the desired duality result. □

5.4. **Action on highest weight terms.** By Proposition 4.3, the highest weight vector $H_{2\mu}, \mu = \sum_i m_i\hat{\eta}_i$ is homogeneous of degree $\sum_r m_r r$. Note that this degree is just the size of μ viewed as a partition. In other words, $\sum_r m_r r = \sum_r \mu_r = |\mu|$ where μ is expressed as $\sum_i \mu_i\epsilon_i^\Sigma$, a linear combination in terms of the orthonormal basis in the restricted root setting. Moreover, the action of $U_q(\mathfrak{g})$ preserves the degree. Thus, for each $\mu \in \Lambda_\Sigma^+$, the module $(\mathrm{ad}\ U_q(\mathfrak{g}))\cdot H_{2\mu}$ sits inside the homogeneous component $\mathscr{P}_\theta^m$ of degree $m = |\mu|$. Recall the definition of the augmentation ideals U_+^+ and U_+^- given in Section 2.2.

Proposition 5.6. *For all μ and γ in Λ_Σ^+ with $\mu \neq \gamma$, the space $U_q(\mathfrak{g})\cdot H_{2\mu}$ is equal to the $U_q(\mathfrak{g})$-module dual of $U_q(\mathfrak{g})\cdot H_{2\mu}^*$ and orthogonal to $U_q(\mathfrak{g})\cdot H_{2\gamma}^*$ with respect to the bilinear form defined by (18). Moreover,*

$$\langle H_{2\mu}^*, H_{2\mu}\rangle \neq 0$$

whereas

$$\langle E\cdot H_{2\mu}^*, H_{2\mu}\rangle = 0 = \langle H_{2\mu}^*, F\cdot H_{2\mu}\rangle$$

for all $E \in U_+^+$ and $F \in U_+^-$.

Proof. Let $\mu \in \Lambda_\Sigma^+$ and set $m = |\mu|$. Since the left action of $U_q(\mathfrak{g})$ preserves degree, both $\mathscr{P}_\theta^m$ and $\mathscr{D}_\theta^m$ are left $U_q(\mathfrak{g})$- modules. By Lemma 5.3, the dualities of vector spaces in Proposition 5.5 are actually dualities of left $U_q(\mathfrak{g})$-modules.

Note that there is only one way to express a weight $\mu \in \Lambda_\Sigma^+$ as a linear combination of the $\hat{\eta}_r$. This means that $U_q(\mathfrak{g})\cdot H_{2\mu}$ is the unique simple module with highest weight 2μ inside the decomposition of $\mathscr{P}_\theta$, and thus inside of $\mathscr{P}_\theta^m$. Similarly, $U_q(\mathfrak{g})\cdot H_{2\mu}^*$ is the unique simple module with lowest weight -2μ inside the decomposition of $\mathscr{D}_\theta$, and thus inside of $\mathscr{D}_\theta^m$. Hence, by the previous paragraph, $U_q(\mathfrak{g})\cdot H_{2\mu}$ is equal to the $U_q(\mathfrak{g})$-module dual of $U_q(\mathfrak{g})\cdot H_{2\mu}^*$ with respect to bilinear form defined by (18).

Since $H_{2\mu}^*$ is a lowest weight generating vector for $U_q(\mathfrak{g})\cdot H_{2\mu}^*$, it follows that

$$U_q(\mathfrak{g})\cdot H_{2\mu}^* = H_{2\mu}^* \oplus U_+^+\cdot H_{2\mu}^*.$$

The fact that $H_{2\mu}$ is a highest weight vector combined with the $U_q(\mathfrak{g})$ invariance of the bilinear form $\langle\cdot,\cdot\rangle$ (Lemma 5.3) ensures that $H_{2\mu}$ is perpendicular to $U_+^+\cdot\mathscr{D}_\theta$. The argument showing

$H^*_{2\mu}$ is perpendicular to $U^-_+ \cdot \mathscr{P}_\theta$ follows in a similar fashion. This completes the proof of the proposition. □

Note that by the above proposition, the action of $H^*_{2\mu}$ on $H_{2\mu}$ is nonzero. In the next result, we obtain more information as to when such an action (and related ones) are possibly nonzero.

Lemma 5.7. *Given μ and γ in Λ^+_Σ such that $|\gamma| \geq |\mu|$ and $\mu \neq \gamma$, we have $(u \cdot H^*_{2\gamma})H_{2\mu} \in \mathcal{L}$ and so $(u \cdot H^*_{2\gamma}) \cdot H_{2\mu} = 0$ for any $u \in U^+_+$.*

Proof. Consider $u \in U^+$ and note that $u - \epsilon(u)$ is in U^+_+. Moreover, it follows from the form the coproduct takes on the generators E_i of U^+ that

$$\Delta(u - \epsilon(u)) \in u \otimes 1 + U_q(\mathfrak{g}) \otimes U^+_+.$$

Since $H_{2\mu}$ is a highest weight vector, it is annihilated by elements of U^+_+. Hence

$$u \cdot \left(H^*_{2\gamma}H_{2\mu}\right) = (u \cdot H^*_{2\gamma})H_{2\mu} + \epsilon(u)H^*_{2\gamma}H_{2\mu}. \tag{20}$$

Assume that

$$H^*_{2\gamma}H_{2\mu} \in \mathcal{L}. \tag{21}$$

Since $\mathcal{L}$ is a left $U_q(\mathfrak{g})$-module, It follows that $u \cdot (H^*_{2\gamma}H_{2\mu}) \in \mathcal{L}$ for any $u \in U^+$. Hence by (20), the same holds true for $(u \cdot H^*_{2\gamma})H_{2\mu}$. Thus it is sufficient to prove (21) given the assumptions on γ and μ.

Recall that $H^*_{2\gamma}$ has degree $|\gamma|$ and $H_{2\mu}$ has degree $|\mu|$. It follows from the defining relations of $\mathscr{P}\mathscr{D}_\theta$ that when we move the d_{ij} terms to the right pass the x_{ij} terms in the expression of the left hand side of (21) we end up an expression of the form $\sum_j u_j v_j$ where each v_j is in $\mathscr{D}^{r_j}_\theta$ and each u_j is in $\mathscr{P}^{w_j}_\theta$ and $r_j - w_j = |\gamma| - |\mu|$. In other words, the relations cancel out the same number of x_{ij} and d_{ij} terms.

If $|\gamma|$ is strictly greater than $|\mu|$, then each v_j has degree at least 1 and so we end up with an element in $\sum_{i,j} \mathscr{P}\mathscr{D}_\theta d_{ij} = \mathcal{L}$. Now assume that $|\gamma| = |\mu|$. In this case, we end up with $\deg v_j = \deg u_j$ each j and so

$$H^*_{2\gamma}H_{2\mu} \in \mathbb{C}(q) + \mathcal{L}.$$

Note that this scalar term must be $\langle H^*_{2\gamma}, H_{2\mu}\rangle$. By Proposition 5.6, this term is zero for $\gamma \neq \mu$. □

6. Action of Cartan elements

6.1. **Special Cartan elements.** We turn our attention to understanding the action of the various elements of the Cartan subalgebra on the generators of $\mathscr{P}_\theta$ and then identifying them with elements of the appropriate quantum Weyl algebra.

Lemma 6.1. *Let $N = n$ in Type AI, $N = 2n$ in Type AII and $N \in \{n, 2n\}$ in the diagonal type. The element $K_{2\epsilon_N} - 1$ acts the same on $\mathscr{P}_\theta$ as the element X in $\mathscr{P}\mathscr{D}_\theta$ where*

$$X = \begin{cases} (q^2-1)((q^3+q)x_{nn}d_{nn} + \sum_{a=1}^{n-1} x_{an}d_{an}) \text{ in Type AI} \\ (q^2-1)(\sum_{a=1}^{2n-1} x_{a,2n}d_{a,2n}) \text{ in Type AII} \\ (q^2-1)\sum_{a=1}^{n} x_{a,2n}d_{a,2n} \text{ in diagonal type } (N=n) \\ (q^2-1)\sum_{a=1}^{n} x_{n,a+n}d_{n,a+n} \text{ in diagonal type } (N=2n) \end{cases}$$

Proof. Consider first Type AI. It follows from Lemma 5.2 and the relations (5) and (6) satisfied by the x_{ij} that for any scalars $c_1, \dots, c_n$,

$$\begin{aligned}\sum_{a=1}^{e} c_a x_{an} d_{an} x_{en} &= c_e x_{en} d_{en} x_{en} + \sum_{a<e} c_a x_{an} d_{an} x_{en} \\ &= q^{-\delta_{en}} c_e x_{en} + q^{2+2\delta_{en}} c_e x_{en} x_{en} d_{en} + \sum_{a>e} \mathscr{P}_\theta d_{an} + q^{2+2\delta_{en}} c_a x_{en} \sum_{a<e} x_{an} d_{an}\end{aligned}$$

and so

$$\begin{aligned}\sum_{a=1}^{e} c_a x_{an} d_{an} x_{en}^2 &= q^{-\delta_{en}} c_e x_{en}^2 + q^{2+2\delta_{en}} c_e x_{en} x_{en} d_{en} x_{en} \\ &\quad + \sum_{a>e} \mathscr{P}_\theta d_{an} + q^{2+2\delta_{en}} x_{en}^2 \sum_{a<e} c_a x_{an} d_{an} \\ &= q^{-\delta_{en}} (q^{2+2\delta_{en}} + 1) c_e x_{en}^2 + q^{4+4\delta_{en}} c_e x_{en}^3 d_{en} \\ &\quad + \sum_{a>e} \mathscr{P}_\theta d_{an} + q^{4+4\delta_{en}} x_{en}^2 \sum_{a<e} c_a x_{an} d_{an}.\end{aligned}$$

By induction, we have

$$\sum_{a=1}^{e} c_a x_{an} d_{an} x_{en}^s = q^{-\delta_{en}} \frac{(q^{(2+2\delta_{en})s} - 1)}{(q^{2+2\delta_{en}} - 1)} c_e x_{en}^s + \sum_{a \geq e} \mathscr{P}_\theta d_{an} + q^{(2+2\delta_{en})s} x_{en}^s \sum_{a<e} c_a x_{an} d_{an}.$$

Now consider $c_n = q^3 + q$ and $c_a = 1$ for $a < n$. We have

$$\begin{aligned}\sum_{a=1}^{n} c_a x_{an} d_{an} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} &= \frac{(q^{4s_n} - 1)}{(q^2 - 1)} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} \\ &\quad + q^{4s_n} x_{nn}^{s_n} \left(\sum_{a=1}^{n-1} c_a x_{an} d_{an} \right) x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} + \mathscr{P} d_{nn} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1}.\end{aligned}$$

For the final term, we can move d_{nn} to the right, past all the x_{ij}, yielding an element of $\mathscr{P}_\theta d_{nn}$. The penultimate term equals

$$\begin{aligned}q^{4s_n} \frac{(q^{2s_{n-1}} - 1)}{(q^2 - 1)} x_{nn}^{s_n} \cdots x_{1n}^{s_1} + q^{4s_n + 2s_{n-1}} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \left(\sum_{a=1}^{n-2} c_a x_{an} d_{an} \right) x_{n-2,n}^{s_{n-2}} \cdots x_{1n}^{s_1} \\ + \left(\mathscr{P}_\theta d_{n-1,n} + \mathscr{P}_\theta d_{nn} \right) x_{n-2,n}^{s_n - 2} \cdots x_{1n}^{s_1}.\end{aligned}$$

Noting that this final term is in $\mathscr{P}_\theta d_{n-1,n} + \mathscr{P}_\theta d_{nn}$ (here, we can once again move the d terms all the way to the right), and summing up the above two expressions yields

$$\begin{aligned}\sum_{a=1}^{n} c_a x_{an} d_{an} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} &= \frac{(q^{4s_n + 2s_{n-1}} - 1)}{(q^2 - 1)} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} \\ &\quad + q^{4s_n + 2s_{n-1}} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \left(\sum_{a=1}^{n-2} c_a x_{an} d_{an} \right) x_{n-2,n}^{s_{n-2}} \cdots x_{1n}^{s_1} + \mathscr{P}_\theta d_{n-1,n} + \mathscr{P}_\theta d_{nn}.\end{aligned}$$

Using induction, we get

$$\sum_{a=1}^{n} c_a x_{an} d_{an} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} = \frac{(q^{4s_n+2s_{n-1}+\cdots 2s_1}-1)}{(q^2-1)} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} + \sum_a \mathscr{P}_\theta d_{an}.$$

By Theorem 5.1, $\sum_a \mathscr{P}_\theta d_{an} x_{ef} \in \sum_{a'} \mathscr{P}_\theta d_{a'n}$ for all e, f strictly less than n. Hence

$$\sum_{a=1}^{n} c_a x_{an} d_{an} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} x_{e_1,f_1} \cdots x_{e_r,f_r}$$
$$= \frac{(q^{4s_n+2s_{n-1}+\cdots 2s_1}-1)}{(q^2-1)} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} x_{e_1,f_1} \cdots x_{e_r,f_r} + \sum_a \mathscr{P}_\theta d_{an}.$$

for all $(e_1, f_1) \geq \cdots \geq (e_r, f_r)$ with $e_j \leq f_j < n$ each j. It follows from the definition of $\mathcal{L}$ that $\mathscr{P}_\theta d_{an} \subset \mathcal{L}$. Thus, we get

$$\left(\sum_{a=1}^{n} c_a x_{an} d_{an}\right) \cdot x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} x_{e_1,f_1} \cdots x_{e_r,f_r}$$
$$= \frac{(q^{4s_n+2s_{n-1}+\cdots 2s_1}-1)}{(q^2-1)} x_{nn}^{s_n} x_{n-1,n}^{s_{n-1}} \cdots x_{1n}^{s_1} x_{e_1,f_1} \cdots x_{e_r,f_r}$$

for all $(e_i, f_i), i = 1, \ldots, r$ as described above. In other words, $\sum_{a=1}^{n} c_a x_{an} d_{an}$ acts the same as $(K_{2\epsilon_n} - 1)/(q^2 - 1)$ on a basis for $\mathscr{P}_\theta$, and thus on $\mathscr{P}_\theta$, as desired. The argument in Type AII is exactly the same where we omit any terms involving d_{nn} and x_{nn} and replace n with $2n$ everywhere.

Now consider the diagonal case. Using the relations for $\mathscr{P}\mathscr{D}_\theta$, which in this case is the same as that for $\mathscr{P}\mathscr{D}_q[\mathrm{Mat}_n]$ with t_{ij} replaced by $x_{i,j+n}$ and ∂_{uv} replaced by $d_{u,v+n}$ (see (15) and [19], Sections 6.3 and 8.3), gives us

$$x_{a,2n} d_{a,2n} x_{e,f+n} \in \sum_{a' \geq a} \mathscr{P} d_{a',2n}$$

whenever $f < n$ or $e < a$. Hence, by induction,

$$x_{a,2n} d_{a,2n} x_{e_1,f_1+n} \cdots x_{e_r,f_r+n} \in \sum_{a' > a} \mathscr{P}_\theta d_{a',2n}$$

for any choice of $(e_1, f_1), \cdots, (e_r, f_r)$ with $(e_i, f_i) < (a, n)$ each i. On the other hand,

$$\sum_{a \leq e} x_{a,2n} d_{a,2n} x_{e,2n} = q^2 x_{e,2n}^2 d_{e,2n} + x_{e,2n} + q^2 x_{e,2n} \sum_{a<e} x_{a,2n} d_{a,2n} + \sum_{a' \geq e} \mathscr{P}_\theta d_{a',2n}.$$

By induction, we have

$$x_{a,2n} d_{a,2n} x_{a,2n}^s = q^{2s} x_{a,2n}^2 d_{a,2n} + \frac{(q^{2s}-1)}{(q^2-1)} x_{a,2n} + q^{2s} x_{e,2n}^s \sum_{a<e} x_{a,2n} d_{a,2n} + \sum_{a'>a} \mathscr{P}_\theta d_{a'n}.$$

Hence

$$\sum_{a=1}^{n} x_{a,2n} d_{a,2n} x_{n,2n}^{s_n} x_{n-1,2n}^{s_{n-1}} \cdots x_{1,2n}^{s_1} x_{e_1,f_1+n} \cdots x_{e_r,f_r+n}$$
$$= \frac{(q^{2s_n+2s_{n-1}+\cdots 2s_1}-1)}{(q^2-1)} x_{n,2n}^{s_n} x_{n-1,2n}^{s_{n-1}} \cdots x_{1,2n}^{s_1} x_{e_1,f_1+n} \cdots x_{e_r,f_r+n} + \sum_a \mathscr{P}_\theta d_{a,2n}$$

and so

$$\left(\sum_{a=1}^{n} x_{a,2n}d_{a,2n}\right)\cdot x_{n,2n}^{s_n}x_{n-1,2n}^{s_{n-1}}\cdots x_{1,2n}^{s_1}x_{e_1,f_1+n}\cdots x_{e_r,f_r+n}$$
$$=\frac{(q^{2s_n+2s_{n-1}+\cdots+2s_1}-1)}{(q^2-1)}x_{n,2n}^{s_n}x_{n-1,2n}^{s_{n-1}}\cdots x_{1,2n}^{s_1}x_{e_1,f_1+n}\cdots x_{e_r,f_r+n}$$

for any sequence $(e_1,f_1),\ldots,(e_r,f_r)$ with $f_i<n$ for each i. Thus $\sum_{a=1}^{n}x_{a,2n}d_{a,2n}$ acts on $\mathscr{P}_\theta$ the same way as $(K_{2\epsilon_n}-1)/(q^2-1)$. This proves the first assertion in the diagonal case. The proof for the second case is the same with the order of the subscripts switched. □

6.2. **Relationships between Cartan elements.** In the next lemma, we write $K_{2\epsilon_r+\cdots+2\epsilon_N}$ in terms of elements in the $U_q(\mathfrak{g})$-module generated by $K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}$ and $K_{2\epsilon_N}$ where N is either n or $2n$ depending on $\mathfrak{g}$. This is the crucial step in showing that $K_{2\epsilon_r+\cdots+2\epsilon_N}$ acts the same as an element in $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$.

The proof of this lemma is very similar to the proof of [18], Proposition 4.2.1. However, in [18], the coproduct used is the opposite of the one used here which in turn impacts the formulas for the adjoint action. We include a proof here so that the reader will not need to adjust the proof of [18] with respect to subtle differences between the setups in the two papers.

Lemma 6.2. *The element $K_{2\epsilon_r+\cdots+2\epsilon_N}$ can be expressed as a linear combination of the following three terms, each with nonzero coefficient*

- $((\text{ad } E_rE_{r+1}\cdots E_{N-1})\cdot K_{2\epsilon_N})\left((\text{ad } F_{N-1}\cdots F_r)\cdot K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}\right)$
- $\left((\text{ad } F_{N-1}\cdots F_r)\cdot K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}\right)((\text{ad } E_rE_{r+1}\cdots E_{N-1})\cdot K_{2\epsilon_N})$
- $K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N}$

where $N=n$, $r=1,\ldots,n$ in Type AI, $N=2n$ and $r=1,\ldots,2n$ for Type AII, and $N=n$, $r=1,\ldots,n$ and $N=2n$, $r=n+1,\ldots,2n$ in the diagonal case.

Proof. Set $\beta_r=\alpha_r+\cdots+\alpha_{N-1}$. Straightforward calculations show that

$$(\text{ad } E_{N-1})\cdot K_{2\epsilon_N}=(1-q^{-2})E_{N-1}K_{2\epsilon_N}$$

and

$$(\text{ad } E_j)\cdot X_{\beta_{j-1}}K_{2\epsilon_N}=(E_jX_{\beta_{j-1}}-qX_{\beta_{j-1}}E_j)K_{2\epsilon_N}$$

where X is a sum of monomials in the E_i of weight β_{j-1}. It follows that

$$(\text{ad } E_rE_{r+1}\cdots E_{N-2}E_{N-1})\cdot K_{2\epsilon_N}=(1-q^{-2})E_{\beta_r}K_{2\epsilon_N}$$

where E_{β_r} is the root vector associated to the root β_r (see [10], Section 7.3.1 or [19], Section 3.3).

We have a similar result for the $F_j's$. In particular, we have

$$(\text{ad } F_r)\cdot K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}=(1-q^2)F_rK_{2\epsilon_{r+1}+\cdots+2\epsilon_N}$$

and

$$(\text{ad } F_j)\cdot Y_{\beta'_j}K_{\beta'_j}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}=(F_jY_{\beta'_j}-q^{-1}Y_{\beta'_j}F_j)K_{\beta'_j}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}$$

where Y is a sum of monomials in the F_i of weight $\alpha_{N-1}+\cdots+\alpha_{j-1}$. By induction, we produce the q^{-1} commutator $[F_{N-1},[F_{N-2},\cdots,[F_{r+1},F_r]_{q^{-1}}]_{q^{-1}}]\cdots]_{q^{-1}}$ which is just a (nonzero) scalar multiple of the root vector F_{β_r} ([10], Section 7.3.1). Hence

$$(\text{ad } F_{N-1}\cdots F_r)\cdot K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}=(1-q^2)\gamma F_{\beta_r}K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}$$

for some nonzero scalar γ.

Since E_{β_r} and F_{β_r} are root vectors under one of Lusztig's automorphisms ([10], Section 7.3.1), it follows that $[E_{\beta_r}, F_{\beta_r}] = (K_{\beta_r} - K_{\beta_r}^{-1})(q - q^{-1})$. Hence

$$\begin{aligned} q^{-2}(E_{\beta_r}K_{2\epsilon_N})(F_{\beta_r}K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}) - (F_{\beta_r}K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N})(E_{\beta_r}K_{2\epsilon_N}) \\ = (E_{\beta_r}F_{\beta_r} - F_{\beta_r}E_{\beta_r})K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N} \\ = (q-q^{-1})^{-1}(K_{\beta_r}^2 - 1)K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N}. \end{aligned}$$

Note that $2\beta_r = 2\epsilon_r - 2\epsilon_N$. Thus the above simplifies to

$$(q-q^{-1})^{-1}(K_{2\epsilon_r+2\epsilon_{r+1}+\cdots+2\epsilon_N} - K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N}).$$

Therefore, $K_{2\epsilon_r+\cdots+2\epsilon_N}$ can be written as a linear combination of the elements $K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N}$, $(E_{\beta_r}K_{2\epsilon_N})(F_{\beta_r}K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N})$, and $(F_{\beta_r}K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N})(E_{\beta_r}K_{2\epsilon_N})$ with each term appearing with nonzero coefficient. □

6.3. **Acting as quantum Weyl Algebra elements.** Let ψ denote the map from $U_q(\mathfrak{g})$ to End $\mathscr{P}_\theta$ that agrees with the action of $U_q(\mathfrak{g})$ on $\mathscr{P}_\theta$. We show that $\psi(K_{2\epsilon_r+\cdots+2\epsilon_m})$ agrees with the image under ϕ of an element of $\mathscr{P}\mathscr{D}_\theta$. Moreover, we determine a bound on the degree of these elements using the degree function defined in Section 5.1.

Proposition 6.3. *The image $\psi(K_{2\epsilon_r+\cdots+2\epsilon_N})$ of $K_{2\epsilon_r+\cdots+2\epsilon_N}$ inside* End $\mathscr{P}_\theta$ *is equal to the image ϕ_{a_r} for some $a_r \in \mathscr{P}\mathscr{D}_\theta$ where*

- *$N = n$, $r = 1, \ldots, n$ in Type AI*
- *$N = 2n$ and $r = 1, \ldots, 2n$ for Type AII*
- *$N = n$, $r = 1, \ldots, n$ and $N = 2n$, $r = n+1, \ldots, 2n$ in the diagonal case.*

Moreover, $\deg a_r \leq 2(N - r + 1)$.

Proof. By Lemma 6.1, $\psi(K_{2\epsilon_N}) = \phi_{a_N}$ for an appropriate element $a_N \in \mathscr{P}\mathscr{D}_\theta$. Moreover, one sees from the formulas for $\psi(K_{2\epsilon_N})$ given in this lemma that $\deg a_N = 2 = 2(N - N + 1)$. Now assume that for some $r \leq N$, $\psi(K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}) = \phi_{a_{r+1}}$ for an element a_{r+1} of degree at most $2(N-r)$ where $N = n$ in Type AI, $N = 2n$ in Type AII, and $N = n$ or $N = 2n$ in the diagonal case with $N - n \leq r \leq N - 1$.

By Lemma 6.2, $K_{2\epsilon_r+\cdots+2\epsilon_N}$ is a linear combination of products of the form xy where $x \in (\text{ad } U_q(\mathfrak{g})) \cdot K_{2\epsilon_N}$ and $y \in (\text{ad } U_q(\mathfrak{g})) \cdot K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}$. Since $\phi(\mathscr{P}\mathscr{D}_\theta)$ is a subalgebra and, by Proposition 5.4, an (ad $U_q(\mathfrak{g})$)-submodule of End $\mathscr{P}_\theta$, it follows that $\psi(K_{2\epsilon_r+\cdots+2\epsilon_N}) \in \phi(\mathscr{P}\mathscr{D}_\theta)$, and so the main assertion of the lemma follows by induction.

We now turn our attention to understanding the degree assertion at the conclusion of the lemma. As explained above, $\deg a_N = 2$. Now assume that $\deg a_{r+1} = 2(N - (r+1) + 1)$. It follows that $\deg(a_{r+1}a_N) \leq \deg(a_{r+1}) + \deg(a_N) = 2(N - (r+1) + 1) + 2 = 2(N - r + 1)$. Since the filtration $\mathcal{J}$ is preserved by the action of $U_q(\mathfrak{g})$, we also have $\deg(E_{\beta_r}K_{2\epsilon_N}) \leq 2$ and $\deg(F_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}) \leq 2(N-r)$. Therefore

$$\deg(E_{\beta_r}F_{\beta_r} - F_{\beta_r}E_{\beta_r})K_{\beta_r}K_{2\epsilon_{r+1}+\cdots+2\epsilon_N}K_{2\epsilon_N} \leq 2(N - r + 1)$$

and so $\deg a_r \leq 2(N - r + 1)$.. The final assertion of the proposition now follows by induction. □

7. The locally finite subalgebra

7.1. **The locally finite subalgebra for $U_q(\mathfrak{gl}_N)$.** In [6] and [7], a complete description of the locally finite subalgebra of $U_q(\mathfrak{sl}_N)$ as a direct sum of (ad $U_q(\mathfrak{sl}_N)$)-modules is given. This result is then generalized to the simply connected quantized enveloping algebra (see Section 2.2) in [7] (see also [5], 7.1). In particular, we have

$$\mathcal{F}(\check{U}_q(\mathfrak{sl}_N)) = \bigoplus_{\lambda \in -P^+} (\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda}. \tag{22}$$

Note that $-P^+ = \{-\lambda |\ \lambda \in P^+\} = \{w_0\lambda |\ \lambda \in P^+\}$. Thus, $-P^+$ is the $\mathbb{N}$ linear combination of the $w_0\omega_i, i = 1, \dots, N-1$. As explained in Section 2.1, the fundamental weight ω_i is equal to $\hat{\omega}_i = \epsilon_1 + \cdots + \epsilon_i$ plus a scalar multiple of $\hat{\omega}_N = \epsilon_1 + \cdots + \epsilon_N$. We see from (1) that this scalar is i/N which is not an integer and so, the simply connected quantized enveloping algebra $\check{U}_q(\mathfrak{sl}_N)$ is not a subalgebra of $U_q(\mathfrak{gl}_N)$. However, the two algebras are closely related. For instance, we can extend $U_q(\mathfrak{gl}_N)$ in a similar manner to the construction of $\check{U}_q(\mathfrak{sl}_N)$ so that the resulting algebra contains both $\check{U}_q(\mathfrak{sl}_N)$ and $U_q(\mathfrak{gl}_N)$. To do this,define an algebra $\check{U}_q(\mathfrak{gl}_N)$ generated by $U_q(\mathfrak{gl}_N)$ and the Laurent polynomial ring $\mathbb{C}(q)[K_{\hat{\omega}_N/N}^{\pm 1}]$ subject to the additional relations

- $(K_{\hat{\omega}_N/N})^N = K_{\hat{\omega}_N}$
- $K_{\hat{\omega}_N/N}$ commutes with all elements of $U_q(\mathfrak{gl}_N)$ and thus is in the center of $\check{U}_q(\mathfrak{gl}_N)$.

The algebra $\check{U}_q(\mathfrak{gl}_N)$ can be given a Hopf structure by insisting that $K_{\hat{\omega}_N/N}$ satisfies the same formulas for coproduct, counit, and antipode as an element $K \in U_q(\mathfrak{gl}_N)$ (as given in Section 2.2). By (1), it follows that $\check{U}_q(\mathfrak{sl}_N)$ can be identified with a subalgebra of $\check{U}_q(\mathfrak{gl}_N)$. Though much of the discussion below focuses on either $\check{U}_q(\mathfrak{sl}_N)$ or $U_q(\mathfrak{gl}_N)$, it will sometimes be useful to pass to the larger algebra $\check{U}_q(\mathfrak{gl}_N)$.

Since $K_{\hat{\omega}_N}$ is in the center of $U_q(\mathfrak{gl}_N)$, and, similarly, $K_{\hat{\omega}_N/N}$ is in the center of $\check{U}_q(\mathfrak{gl}_N)$, we have

$$(\text{ad } U_q(\mathfrak{sl}_N)) \cdot u K_{\hat{\omega}_N}^s = [(\text{ad } U_q(\mathfrak{sl}_N)) \cdot u] K_{\hat{\omega}_N}^s \tag{23}$$

for any s with $Ns \in \mathbb{Z}$ and any $u \in U_q(\mathfrak{gl}_N)$. Consider $\lambda = -\sum_i \lambda_i \omega_i$ and $\lambda' = -\sum_i \lambda_i \hat{\omega}_i$, we have the following $U_q(\mathfrak{sl}_N)$-module isomorphism

$$(\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda} \cong (\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda'} \tag{24}$$

via the map sending $(\text{ad } a) \cdot K_{2\lambda}$ to $(\text{ad } a) \cdot K_{2\lambda'}$ for all $a \in U_q(\mathfrak{sl}_N)$.

Note that $w_0\Lambda_N^+$ is equal to the $\mathbb{N}$ linear combination of the weights $w_0\hat{\omega}_i = \epsilon_{N-i+1} + \cdots + \epsilon_N$, $i = 1, \dots, N-1$. It follows from (22) and (24) that

$$\mathcal{F}(\check{U}_q(\mathfrak{sl}_N)) \cong \bigoplus_{\lambda \in w_0\Lambda_N^+} (\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda} \subseteq \bigoplus_{\lambda \in w_0\Lambda_N^+, r \in \mathbb{Z}} (\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda + r\hat{\omega}_N} \subseteq \mathcal{F}(U_q(\mathfrak{gl}_N)).$$

Note that the above equalities, isomorphisms, and inequalities all hold when (ad $U_q(\mathfrak{sl}_N)$) is replaced by (ad $U_q(\mathfrak{gl}_N)$), (ad $\check{U}_q(\mathfrak{sl}_N)$) or (ad $\check{U}_q(\mathfrak{gl}_N)$). This is because the adjoint action of additional elements of the form K_μ in these Hopf algebras is semisimple with the same eigenspaces as that of the original Cartan subalgebra of $U_q(\mathfrak{sl}_N)$. Thus the action of these extra elements preserve the decomposition into (ad $U_q(\mathfrak{sl}_N)$)-modules.

Recall [7] that $U_q(\mathfrak{sl}_N)$ admits an (ad $U_q(\mathfrak{sl}_N)$) filtration $\mathcal{G}$ defined by the degree function $\deg_{\mathcal{G}}(K_i) = -1$, $\deg_{\mathcal{G}}(F_i) = 1$, and $\deg_{\mathcal{G}}(E_i) = 0$ for $i = 1, \dots, N-1$. Moreover this extends to a filtration on $\check{U}_q(\mathfrak{sl}_N)$ by setting $\deg_{\mathcal{G}} K_\beta = m$ where $\beta = m_1\alpha_1 + \cdots + m_{N-1}\alpha_{N-1}$ and $m = m_1 + \cdots + m_{N-1}$. This filtration can also be extended to $U_q(\mathfrak{gl}_N)$ by setting $\deg_{\mathcal{G}} K_{\epsilon_i} = N - i$.

Indeed, this extension restricts to the desired degree function on $U_q(\mathfrak{sl}_N)$ since $K_i = K_{\epsilon_i}K_{\epsilon_{i+1}}^{-1}$ and $\deg_{\mathcal{G}} K_{\epsilon_i} - \deg_{\mathcal{G}} K_{\epsilon_{i+1}} = N - i - (N - i + 1) = -1$. Note that on passage to the graded setting, the relations stay the same except for the ones relating E_i and F_i. Indeed, we have

$$E_iF_i - F_iE_i = (q - q^{-1})^{-1}K_i^{-1} + \text{lower degree terms.}$$

Let G^- be the subalgebra of $U_q(\mathfrak{sl}_N)$ generated by the $F_iK_i, i = 1, \dots, N-1$. Note that all elements of U^+ and all elements of G^- have degree 0 with respect to $\mathcal{G}$. As in [7] and Chapter 7 of [5] $\mathrm{gr}_{\mathcal{G}}U^+G^-K_\beta$ is an (ad $U_q(\mathfrak{sl}_N)$)-submodule of $\mathrm{gr}_{\mathcal{G}}(\check{U}_q(\mathfrak{sl}_N))$ (resp. $\mathrm{gr}_{\mathcal{G}}U_q(\mathfrak{gl}_N)$) for all $\beta \in P^+$ (resp. $\beta \in \sum_{i=1}^N \mathbb{Z}\epsilon_i$). Moreover,

$$\mathrm{gr}_{\mathcal{G}}U^+G^-K_\beta \ \cap \ \mathrm{gr}_{\mathcal{G}}U^+G^-K_{\beta'} = 0$$

for $\beta \neq \beta'$. Hence both $\mathrm{gr}_{\mathcal{G}}(\check{U}_q(\mathfrak{sl}_N))$ and $\mathrm{gr}_{\mathcal{G}}U_q(\mathfrak{gl}_N)$ can be written as a direct sum of modules with respect to the adjoint action of the form $\mathrm{gr}_{\mathcal{G}}U^+G^-K_\beta$.

Write $\mathbb{Z}_{odd}$ for the set of odd integers.

Theorem 7.1. *The locally finite subalgebra of $U_q(\mathfrak{gl}_N)$ admits the following decomposition into a direct sum of* (ad $U_q(\mathfrak{gl}_N)$)*- modules*

$$\mathcal{F}(U_q(\mathfrak{gl}_N)) = \bigoplus_{\lambda \in w_0\Lambda_N^+, s \in \mathbb{Z}} (\text{ad } U_q(\mathfrak{sl}_N)) \cdot K_{2\lambda + s\hat{\omega}_N} = \bigoplus_{\lambda \in w_0\Lambda_N^+, s \in \mathbb{Z}_{odd}} [(\text{ad } U_q(\mathfrak{gl}_N)) \cdot K_{2\lambda}]K_{\hat{\omega}_N}^s. \tag{25}$$

Similarly, the locally finite subalgebra of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ can be written as

$$\begin{aligned} \mathcal{F}(U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)) &= \bigoplus_{\lambda \in w_0\Lambda_N^+ \times w_0\Lambda_N^+, s, s' \in \mathbb{Z}} (\text{ad } U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)) \cdot K_{2\lambda + s\hat{\omega}_N + s'\hat{\omega}_{2n}} \\ &= \bigoplus_{\lambda \in w_0\Lambda_N^+ \times w_0\Lambda_N^+, s, s' \in \mathbb{Z}_{odd}} [(\text{ad } U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)) \cdot K_{2\lambda}]K_{\hat{\omega}_N}^s K_{\hat{\omega}_{2N}}^{s'}. \end{aligned} \tag{26}$$

Moreover, the above equalities all hold with (ad $U_q(\mathfrak{gl}_N)$) *replaced by* (ad $U_q(\mathfrak{sl}_N)$) *and* (ad $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$) *replaced by* (ad $U_q(\mathfrak{sl}_N \oplus \mathfrak{sl}_N)$).

Proof. Note that the second equality of (25) follows from (23). Also, (26) follows directly from (25) and the final assertion follows from the discussion preceding the theorem. Hence, we focus on proving the first equality of (25).

By [6], Lemma 6.1, K_β admits a locally finite action if and only if (β, α_i) is a nonpositive even integer for $i = 1, \dots, N-1$. (Here, we are taking into account the slightly different definition of the quantized enveloping algebra used in [6].) Hence

$$K_\beta \in \mathcal{F}(U_q(\mathfrak{gl}_N))$$

if and only if $\beta = 2\gamma + s\hat{\omega}_N$ for some $\gamma \in w_0\Lambda_N^+$ and $s \in \mathbb{Z}$. The same result holds upon passage to the graded setting. Set

$$\hat{\Lambda}_N = \{2w_0\lambda + s\hat{\omega}_N |\ \lambda \in \Lambda_N^+, s \in \mathbb{Z}\}.$$

It follows as in [5], Section 7, that the intersection of the locally finite part of $\mathrm{gr}_{\mathcal{G}}U_q(\mathfrak{gl}_N)$ with $\mathrm{gr}_{\mathcal{G}}U^+G^-K_\beta$ is (ad $U_q(\mathfrak{gl}_N)$)$\mathrm{gr}_{\mathcal{G}}K_\beta$ if $\beta \in \hat{\Lambda}_N$ and equal to 0 otherwise. Moreover, for $\beta \in \hat{\Lambda}_N$, we have (ad $U_q(\mathfrak{gl}_N)$)$\mathrm{gr}_{\mathcal{G}}K_\beta \cong$ (ad $U_q(\mathfrak{gl}_N)$)K_β.

Note that by the definition of the degree function corresponding to the filtration $\mathcal{G}$ that $\deg_{\mathcal{G}}(K_\beta) \geq 0$ for $\beta \in \hat{\Lambda}_N$. Hence, if $\mathrm{gr}_{\mathcal{G}}(u)$ is in the locally finite part of $\mathrm{gr}_{\mathcal{G}}U_q(\mathfrak{gl}_N)$, then $\deg_{\mathcal{G}} u \geq 0$. Now suppose that $u \in \mathcal{F}(U_q(\mathfrak{gl}_N))$ with $\deg_{\mathcal{G}} u = m$. We can write $u = \sum_{i=1}^s u_iK_{\beta_i} +$ lower degree

terms where $u_i \in G^-U^+$ and $\deg_{\mathcal{G}} K_{\beta_i} = m$ for each $i = 1, \ldots, s$. By the decomposition of $\mathcal{G}$ into a direct sum of adjoint modules of the form $\mathrm{gr}_{\mathcal{G}} U^+G^-K_\beta$, we must have that for each i, $\mathrm{gr}_{\mathcal{G}} u_i K_{\beta_i}$ generates a finite dimensional module. Hence, for each i, there exists a $v_i \in U_q(\mathfrak{gl}_N))$ such that $\mathrm{gr}_{\mathcal{G}} u_i K_{\beta_i} = (\mathrm{ad}\ v_i)\mathrm{gr}_{\mathcal{G}} K_{\beta_i}$. It follows that $u - \sum_i (\mathrm{ad}\ v_i)K_{\beta_i}$ s an element in $\mathcal{F}(U_q(\mathfrak{gl}_N))$ of degree strictly less than m. The theorem now follows by reverse induction on m. □

7.2. **A special subalgebra.** Let $\mathcal{F}_2(U_q(\mathfrak{gl}_N))$ be the subspace of $\mathcal{F}(U_q(\mathfrak{gl}_N))$ equal to

$$\mathcal{F}_2(U_q(\mathfrak{gl}_N)) = \bigoplus_{\lambda \in w_0\Lambda_N^+} (\mathrm{ad}\ U_q(\mathfrak{gl}_N)) \cdot K_{2\lambda}$$

and let $\mathcal{F}_2(U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N))$ be the subspace of $\mathcal{F}(U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N))$ equal to

$$\mathcal{F}_2(U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)) = \bigoplus_{\lambda \in w_0\Lambda_N^+ \times w_0\Lambda_N^+} (\mathrm{ad}\ U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)) \cdot K_{2\lambda}.$$

Let $U_q^2(\mathfrak{gl}_N)$ denote the subalgebra of $U_q(\mathfrak{gl}_N)$ generated by E_i, F_iK_i, K_i^2, $i = 1, \ldots, N-1$ and $K_{2w_0\hat{\omega}_j}$ for $j = 1, \ldots, N$. Recall (Section 2.1) that Q^+ is the $\mathbb{N}$ linear combination of the positive simple roots. Note that $E_iF_iK_i - q^{-2}F_iK_iE_i$ is a scalar multiple of $(K_i^2 - 1)$. Hence $U_q^2(\mathfrak{gl}_N) \cap U^0$ is the Laurent polynomial ring

$$\mathbb{C}(q)[K_i^2, K_{2w_0\hat{\omega}_j}|, i = 1, \ldots, N-1, j = 1, \ldots, N] = \mathbb{C}(q)[K_\lambda, \lambda \in Q^+ + w_0\Lambda_N^+].$$

It further follows that

$$U_q^2(\mathfrak{gl}_N) = \bigoplus_{\lambda \in 2Q^+ + 2w_0\Lambda_N^+} U^+G^-K_\lambda. \tag{27}$$

One checks from the formulas for the adjoint action that $U_q^2(\mathfrak{gl}_N)$ is an $(\mathrm{ad}\ U_q(\mathfrak{gl}_N))$-submodule of $U_q(\mathfrak{gl}_N)$.

Lemma 7.2. *The space $\mathcal{F}_2(U_q(\mathfrak{gl}_N))$ equals the intersection $U_q^2(\mathfrak{gl}_N) \cap \mathcal{F}(U_q(\mathfrak{gl}_N))$. Hence $\mathcal{F}_2(U_q(\mathfrak{gl}_N))$ is a subalgebra of $\mathcal{F}(U_q(\mathfrak{gl}_N))$.*

Proof. Suppose that $-\lambda + c\hat{\omega}_N$ is in $2Q^+ + 2w_0\Lambda_N^+$ for some dominant integral weight $\lambda \in P^+$ and scalar c. We can express $-\lambda + c\hat{\omega}_N$ as a linear combination, say

$$-\lambda + c\hat{\omega}_N = \sum_{i=1}^{N} 2c_i w_0\hat{\omega}_N \tag{28}$$

of the $w_0\hat{\omega}_N$. Since $\alpha_i = w_0\hat{\omega}_i - w_0\hat{\omega}_{i+1}$ for each $i = 1, \ldots, N-1$ and $-\lambda + c\hat{\omega}_N \in 2Q^+ + 2w_0\Lambda_N^+$, each $c_i, i = 1, \ldots, N$ must be an even integer. Moreover, the fact that $-\lambda \in P^+$ forces $c_i \geq 0$ for $i = 1, \ldots, N-1$. Since $-\lambda + c\hat{\omega}_N \in 2Q^+ + 2w_0\Lambda_N^+$, we can also express $-\lambda + c\hat{\omega}_N$ as a sum of the following form:

$$-\lambda + c\hat{\omega}_N = \sum_{i=1}^{N-1} 2d_i\alpha_i + \sum_{i=1}^{N} 2d_i' w_0\hat{\omega}_i \tag{29}$$

for nonnegative integers $d_{N-1}, d_1', \ldots, d_N'$ and an integer c_N. Note that as sums in terms of the orthonormal basis $\epsilon_1, \ldots, \epsilon_N$, the only places that ϵ_1 appears are α_1 and $\hat{\omega}_N$. Thus, the coefficient of ϵ_1 in the right hand side of (29) is $2d_1 + 2d_N'$ which is a nonnegative even integer. The coefficient of ϵ_1 in the right hand side of (28) is c_N. Hence, $c_N = 2d_1 + 2d_N'$ and so c_N is a nonnegative even

integer. Thus $c_1, \dots, c_N$ are all even nonnegative integers. This forces $-\lambda + c\hat{\omega}_N$ to be an element of $2\Lambda_N^+$. Thus, we have

$$(-P^+ + \mathbb{Q}\hat{\omega}_N) \cap (2Q^+ + 2w_0\Lambda_N^+) = 2w_0\Lambda_N^+.$$

Because of the relationship between elements in P^+ and those in Λ_N^+, we have $2w_0\Lambda_N^+ + \mathbb{Z}\hat{\omega}_N \subset -P^+ + \mathbb{Q}\hat{\omega}_N$. Hence

$$(2w_0\Lambda_N^+ + \mathbb{Z}\hat{\omega}_N) \cap (2Q^+ + 2w_0\Lambda_N^+) = 2w_0\Lambda_N^+ \tag{30}$$

where here inclusion follows from the previous equality and, moreover, we get an equality because $2w_0\Lambda_N^+$ is a subset of both $2w_0\Lambda_N^+ + \mathbb{Z}\hat{\omega}_N$ and $(2Q^+ + 2w_0\Lambda_N^+)$.

By the definition of $U_q^2(\mathfrak{gl}_N)$ and the previous paragraph, if $K_\beta \in U_q^2(\mathfrak{gl}_N)$ and $\beta \in 2w_0\Lambda_N^+ + \mathbb{Z}\hat{\omega}_N$, then $\beta = 2\lambda$ for some $\lambda \in w_0\Lambda_N^+$. Thus, the first assertion follows from (25), the definition of $\mathcal{F}_2(U_q(\mathfrak{gl}_N))$, and the fact that $U_q^2(\mathfrak{gl}_N)$ is an (ad $U_q(\mathfrak{gl}_N)$)-module. The second assertion follows from the fact that the intersection under consideration is an intersection of algebras. □

7.3. **Mapping to the quantum Weyl algebra.** Define $U_q^2(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ as the subalgebra of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ generated by $E_i, F_i, K_{2\hat{\omega}_j}^{\pm 1}$ for $i = 1, \dots, N-1, N+1, \dots, 2N-1$ and $j = 1, \dots, 2N$. Note that $U_q^2(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ can be identified with $U^2(\mathfrak{gl}_N) \otimes U_q^2(\mathfrak{gl}_N)$. Using this identification, Lemma 7.2 ensures that analogous results holds for $U_q^2(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$.

Set U_2 equal to $U_q^2(\mathfrak{gl}_n)$ in Type AI, $U_q(\mathfrak{gl}_{2n})$ in Type AII, and $U_q^2(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)$ in the diagonal case. Similarly, set $\mathcal{F}_2(U_q(\mathfrak{g})) = \mathcal{F}_2(U_q(\mathfrak{gl}_N))$ for $N = n$ in type AI and $N = 2n$ in Type AII and set $\mathcal{F}_2(U_q(\mathfrak{g})) = \mathcal{F}_2(U_q(\mathfrak{gl}_n \oplus \mathfrak{gl}_n))$ in the diagonal setting. By Lemma 7.2 and subsequent discussion, we have that $\mathcal{F}(U_q(\mathfrak{g})) \cap U^2 = \mathcal{F}_2(U_q(\mathfrak{g}))$.

Theorem 7.3. *The image $\psi(\mathcal{F}_2(U_q(\mathfrak{g}))$ in* End $\mathscr{P}_\theta$ *is an* (ad $U_q(\mathfrak{g})$)*-submodule algebra of $\phi(\mathscr{P}\mathscr{D}_\theta)$.*

Proof. By Lemma 6.2, $\psi(K) \in \phi(\mathscr{P}\mathscr{D}_\theta)$ for $K = K_{2\epsilon_i + \dots + 2\epsilon_N}$ for $i = 1, \dots N$ where $N = n$ in Type AI and $N = 2n$ in Type AII. For the diagonal case, $\psi(K) \in \phi(\mathscr{P}\mathscr{D}_\theta)$ for $K = K_{2\epsilon_i + \dots + 2\epsilon_n}$ and $K = K_{2\epsilon_{i+n} + \dots + 2\epsilon_{2n}}$ for $i = 1, \dots n$. The proof now follows from the fact that for each of the three families, $\mathcal{F}_2(U_q(\mathfrak{g}))$ is an (ad $U_q(\mathfrak{g})$)-module generated by these elements, $\mathcal{F}_2(U_q(\mathfrak{g}))$ is an algebra (this is just Lemma 7.2), that ψ is an (ad $U_q(\mathfrak{g})$)-module algebra map, and that, by Proposition 5.4, $\phi(\mathscr{P}\mathscr{D}_\theta)$ is an (ad $U_q(\mathfrak{g})$)- submodule algebra of End $\mathscr{P}_\theta$. □

By Proposition 5.5, $\mathscr{P}_\theta$ is a faithful $\mathscr{P}\mathscr{D}_\theta$-module. It follows that ϕ is an injective algebra map and so as algebras, $\mathscr{P}\mathscr{D}_\theta$ is isomorphic to $\phi(\mathscr{P}\mathscr{D}_\theta)$. This allows us to define an algebra map directly from $\mathcal{F}_2(U_q(\mathfrak{g})$ to $\phi(\mathscr{P}\mathscr{D}_\theta)$ that is compatible with the action on $\mathscr{P}_\theta$. Thus we have the following consequence of the previous theorem.

Corollary 7.4. *There is a unique $U_q(\mathfrak{g})$-module algebra homomorphism Υ from $\mathcal{F}_2(U_q(\mathfrak{g}))$ to $\phi(\mathscr{P}\mathscr{D}_\theta)$ such that*

$$\phi(\Upsilon(a)) = \psi(a) \quad \text{and} \quad \Upsilon((\text{ad } u) \cdot a) = u \cdot \Upsilon(a)$$

for all $a \in \mathcal{F}_2(U_q(\mathfrak{g}))$ and $u \in U_q(\mathfrak{g})$.

8. The Center of $U_q(\mathfrak{g})$

8.1. **Basis for the Center.** We recall here basic properties of $U_q(\mathfrak{g})$ and its center (a good reference is [5]). Note that these properties are mostly stated and proved in terms of the semisimple case, (i.e. $U_q(\mathfrak{sl}_n)$ instead of $U_q(\mathfrak{gl}_n)$); we make the necessary translations here.

By Section 7.1 of ([5]), the (ad $U_q(\mathfrak{sl}_N)$)-modules (ad $U_q(\mathfrak{sl}_N)) \cdot K_{2\mu}$ for $\mu \in -P^+$ contains a unique (up to nonzero scalar multiple) central element which we denote by $z_{2\mu}$. Moreover, the set $\{z_{2\mu} |\ \mu \in -P^+\}$ forms a basis for the center $Z(\check{U}_q(\mathfrak{sl}_N))$ of $\check{U}_q(\mathfrak{sl}_N)$.

The arguments in [5] also apply to the (ad $U_q(\mathfrak{sl}_N)$)-modules of the form (ad $U_q(\mathfrak{sl}_N)) \cdot K_{2\mu+c\hat{\omega}_N}$ where $\mu \in w_0\Lambda_N^+$ and $c \in \mathbb{Z}$. In particular, the (ad $U_q(\mathfrak{sl}_N)$)-module (ad $U_q(\mathfrak{sl}_N)) \cdot K_{2\mu+c\hat{\omega}_N}$ for $\mu \in w_0\Lambda_N^+$ and $c \in \mathbb{Z}$ contains a unique (up to nonzero scalar multiple) central element of $U_q(\mathfrak{gl}_N)$ which we denote by $z_{2\mu+c\hat{\omega}_N}$. Moreover, it follows from (23) that

$$z_{2\lambda+c\hat{\omega}_N} = z_{2\lambda} K_{\hat{\omega}_N}^c$$

for all $\lambda \in w_0\Lambda_N^+$ and $c \in \mathbb{Z}$. Hence, the decomposition of the locally finite subalgebra in Theorem 7.1 ensures that the set $\{z_{2\lambda} K_{\hat{\omega}_N}^c |\ \lambda \in w_0\Lambda_N^+, c \in \mathbb{Z}\}$ forms a basis for the center, $Z(U_q(\mathfrak{gl}_N))$, of $U_q(\mathfrak{gl}_N)$.

We can view both sets $\{z_{2\mu} |\ \mu \in -P^+\}$ and $\{z_{2\lambda} K_{\hat{\omega}_N}^c |\ \lambda \in w_0\Lambda_N^+, c \in \mathbb{Z}\}$ as elements in the algebra $\check{U}_q(\mathfrak{gl}_N)$. Given $\mu \in -P^+$, we can find $\mu' \in w_0\Lambda_N^+$ and a rational number $s \in \frac{1}{N}\mathbb{Z}$ such that $2\mu = 2\mu' + s\hat{\omega}_N$. It follows from (23) that $z_{2\mu} = z_{2\mu'} K_{\hat{\omega}_N}^s$.

Note that the results described so far in this section extend in a straightforward manner to the center of $U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$ with P^+ replaced by $P^+ \times P^+$, $w_0\Lambda_N^+$ replaced by $w_0\Lambda_N^+ \times w_0\Lambda_N^+$, s replaced by a pair of rational numbers s and s', and $K_{\hat{\omega}_N}^s$ replaced by $K_{\hat{\omega}_N}^s K_{\hat{\omega}_{2N}}^{s'}$.

The description of the basis for the center of $U_q(\mathfrak{gl}_N)$ combined with Lemma 7.2 and the subsequent discussion implies that $Z(U_2) = \sum_\lambda \mathbb{C}(q) z_{2\lambda}$ where λ runs over elements in $w_0\Lambda_n^+$ in Type AI and $w_0\Lambda_{2n}^+$ in Type AII and $w_0\Lambda_n^+ \times w_0\Lambda_n^+$ in the diagonal setting.

8.2. **Harish-Chandra Maps.** The Harish-Chandra map is the projection φ_{HC} of $U_q(\mathfrak{g})$ onto U^0 defined by the direct sum decomposition

$$U_q(\mathfrak{g}) = U^0 \oplus (U_+^- U_q(\mathfrak{g}) + U_q(\mathfrak{g}) U_+^+).$$

where U^0, U_+^-, U_+^+ are defined in Section 2.2. Note that the above direct sum decomposition restricts to a direct sum decomposition on U_2 with U^0 replaced with U_2^0 and $(U_+^- U_q(\mathfrak{g}) + U_q(\mathfrak{g}) U_+^+)$ replaced with $(U_+^- U_2 + U_2 U_+^+)$. Hence the restriction of φ_{HC} to U_2 yields a projection onto U_2^0.

Let $\mathcal{A}$ be the group generated by

$$K_{2\epsilon_i^\Sigma}^{\pm 1}, i = 1, \ldots, n.$$

Recall that $\epsilon_i^\Sigma = \tilde{\epsilon}_i$ in Type AI, $\epsilon_i^\Sigma = \tilde{\epsilon}_{2i-1} = \tilde{\epsilon}_{2i}$ in Type AII, and $\epsilon_i^\Sigma = \tilde{\epsilon}_i = \tilde{\epsilon}_{n+i}$ in the diagonal case. Hence $\mathcal{A}$ can also be viewed as the group generated by the

$$K_{2\tilde{\epsilon}_i}, i = 1, \ldots, N$$

for $N = n$ in Type AI and $N = 2n$ in Type AII and the diagonal type.

Set $U_\theta^0 = \mathcal{B}_\theta \cap U^0$. Note that U_θ^0 is $\mathbb{C}(q)$ in Type AI, equal to the $\mathbb{C}(q)$ algebra generated by $K_i^{\pm 1}, i = 1, 3, \ldots, 2n-1$ in Type AII, and the $\mathbb{C}(q)$ algebra generated by $(K_{\epsilon_i} K_{\epsilon_{n+i}}^{-1})^{\pm 1}, i = 1, \ldots, n$ in the diagonal case. In each case, we get a sub-Hopf algebra of $U_q(\mathfrak{g})$. Set $(U_\theta^0)_+$ equal to the augmentation ideal of U_θ^0.

Lemma 8.1. *For all three families, there is an inclusion*

$$\mathbb{C}(q)[K_{2\epsilon_1}^{\pm 1}, \ldots, K_{2\epsilon_N}^{\pm 1}] \subseteq \mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+ \mathbb{C}(q)[\mathcal{A}]. \tag{31}$$

Moreover, in Type AI, $(U_\theta^0)_+ = 0$ and $\mathbb{C}(q)[K_{2\epsilon_1}^{\pm 1}, \ldots, K_{2\epsilon_N}^{\pm 1}] = \mathbb{C}(q)[\mathcal{A}]$.

Proof. It is straightforward to see that $\mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}]$ is closed under multiplication and hence is a subalgebra of U^0. Thus, all we need to show is that $K_{2\epsilon_i}^{\pm 1}$ is in this subalgebra for each choice of i.

In Type AI, $U_\theta^0 = \mathbb{C}(q)$ and so its augmentation ideal is 0. This proves the lemma in Type AI. Now consider Type AII. Using the description of the restricted root system in Section 2.5, we have $2\epsilon_i^\Sigma = 2\tilde{\epsilon}_{2i} = \epsilon_{2i-1} + \epsilon_{2i}$ and so $\mathbb{C}(q)[\mathcal{A}]$ is the Laurent polynomial ring generated by $K_{\epsilon_{2i-1}}K_{\epsilon_{2i}}$ for $i = 1, \ldots, n$. On the other hand, U_θ^0 is the Laurent polynomial ring generated by $K_{2i-1} = K_{\epsilon_{2i-1}}K_{\epsilon_{2i}}^{-1}$ for $i = 1, \cdots, n$. It follows that

$$K_{2\epsilon_{2i-1}} = K_{\epsilon_{2i-1}}K_{\epsilon_{2i}} + (K_{\epsilon_{2i-1}}K_{\epsilon_{2i}}^{-1} - 1)K_{\epsilon_{2i-1}}K_{\epsilon_{2i}}$$

with the first summand on the right hand side contained in $\mathbb{C}(q)[\mathcal{A}]$ and the second in $(U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}]$. Similarly,

$$K_{2\epsilon_{2i}} = K_{\epsilon_{2i-1}}K_{\epsilon_{2i}} + (K_{\epsilon_{2i-1}}^{-1}K_{\epsilon_{2i}} - 1)K_{\epsilon_{2i-1}}K_{\epsilon_{2i}}$$

where the right hand side is again contained in $\mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}]$. Replacing each K term with its inverse in the above two expressions shows that both $K_{2\epsilon_{2i-1}}^{-1}$ and $K_{2\epsilon_{2i}}^{-1}$ are also elements of $\mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}]$. This proves the lemma in Type AII.

The argument in the diagonal case is similar to that of Type AII. Here, we have $\mathbb{C}(q)[\mathcal{A}]$ is the Laurent polynomial ring generated by $K_{\epsilon_i}K_{\epsilon_{n+i}}$ for $i = 1, \ldots, n$ and U_θ^0 is the Laurent polynomial ring generated by $K_{\epsilon_i}K_{\epsilon_{n+i}}^{-1}$ for $i = 1, \cdots, n$. Hence

$$K_{2\epsilon_i} = K_{\epsilon_i}K_{\epsilon_{n+i}} + (K_{\epsilon_i}K_{\epsilon_{n+i}}^{-1} - 1)K_{\epsilon_i}K_{\epsilon_{n+i}} \in \mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}].$$

Similar equalities hold for $K_{2\epsilon_{n+i}}, K_{2\epsilon_i}^{-1}$ and $K_{2\epsilon_i}^{-1}$. □

Using (31), we define a projection map of U_2^0 onto $\mathbb{C}(q)[\mathcal{A}]$. Note that this projection map can be described in the following alternative way:

$$\sum_\mu c_\mu K_\mu \mapsto \sum_\mu c_\mu K_{\tilde{\mu}} \tag{32}$$

where μ are elements in $\sum_i \mathbb{Z}\epsilon_i$, the c_μ are scalars and $\tilde{\mu}$ is the restricted weight defined by μ. Note that this projection map extends to U^0 using either (32) directly or enlarging $\mathcal{A}$ and U_θ^0 so as to create a direct sum along the lines of (31) that contains U^0.

By definition, $U_2^0 = U_2 \cap U^0$ is a subalgebra of the Laurent polynomial ring generated by $K_{2\epsilon_i}^{\pm 1}$ for all i. Hence by the above lemma, we have

$$U_2^0 \subseteq \mathbb{C}(q)[\mathcal{A}] \oplus (U_\theta^0)_+\mathbb{C}(q)[\mathcal{A}]. \tag{33}$$

Let $\tilde{\varphi}_{HC}$ denote the projection of U_2 onto $\mathbb{C}(q)[\mathcal{A}]$ defined by taking the composition of the Harish-Chandra map φ_{HC} with this projection. It should be noted that the decomposition (31) is similar to (3.5) of [17] and $\tilde{\varphi}_{HC}$ corresponds to $\tilde{\mathcal{P}} \circ \mathcal{P}$ of this reference. The map $\mathcal{P}$ is the Harish-Chandra map for the simply connected quantized enveloping algebra (i.e. using the semisimple Lie algebra related to $\mathfrak{g}$) and $\tilde{\mathcal{P}}$ is defined using (32) with μ in the weight lattice. To make the exposition easier below, we also refer to $\tilde{\mathcal{P}} \circ \mathcal{P}$ as $\tilde{\varphi}_{HC}$ and call this map the restricted Harish-Chandra map.

It should also be noted that a more general restricted Harish-Chandra map is defined in [17] using a quantum analog of the Iwasawa decomposition. Upon restriction to the center, this more general restricted Harish-Chandra map agrees with $\tilde{\varphi}_{HC}$. Since we are only worried about the image of central elements under the restricted Harish-Chandra map, we just use $\tilde{\varphi}_{HC}$ and do not consider the more general version. (See [17], Section 6 for more details.)

When $\mathfrak{g} = \mathfrak{gl}_N$ (i.e. in Types AI and AII), it is possible to lift this entire story to $\check{U}_q(\mathfrak{gl}_N)$ in such a way that all these maps are compatible. In particular, we have

$$\tilde{\varphi}_{HC}(z_{2\mu}) = \tilde{\varphi}_{HC}(z_{2\mu'})K_{\tilde{\omega}}^{s} = \tilde{\varphi}_{HC}(z_{2\mu'})K_{\hat{\eta}_N}^{ms} \tag{34}$$

where $\mu \in -P^+$, $\mu' \in w_0\Lambda_N^+$ with $2\mu = 2\mu' + s\hat{\omega}_N$, $\tilde{\omega} = (\hat{\omega}_N - \theta(\hat{\omega}_N))/2$ is the restricted weight for $\hat{\omega}_N$, $m = 1$ in Type AI and $m = 2$ for Type AII, and the final equality replacing $\tilde{\omega}$ with $\hat{\eta}_N$ follows from Section 2.5.

Similar results holds in the diagonal setting. In particular, we have

$$\tilde{\varphi}_{HC}(z_{2\mu}) = \tilde{\varphi}_{HC}(z_{2\mu'})K_{\tilde{\omega}}^{s}K_{\tilde{\omega}'}^{s'} = \tilde{\varphi}_{HC}(z_{2\mu'})K_{\hat{\eta}_n}^{s+s'} \tag{35}$$

where $\mu \in -P^+ \times P^+$, $\mu' \in w_0\Lambda_N^+ \times w_0\Lambda_N^+$ with $2\mu = 2\mu' + s\hat{\omega}_n + s'\hat{\omega}_n$, $\tilde{\omega} = (\hat{\omega}_n - \theta(\hat{\omega}_n))/2$ and $\tilde{\omega}' = (\hat{\omega}_{2n} - \theta(\hat{\omega}_{2n})/2$ are the restricted weights for $\hat{\omega}_n$ and $\hat{\omega}_{2n}$ respectively, and the final equality uses $\tilde{\omega} = \tilde{\omega}' = \hat{\eta}_n$ as explained in Section 2.5.

Note that for $z \in Z(U_q(\mathfrak{g}))$ we actually have $z \in U^0 \oplus U_+^- U^0 U_+^+$ and so $z \cdot v = \varphi_{HC}(z) \cdot v$ whenever v is a highest weight vector. Now consider a highest weight generating vector $v_{2\beta}$ for the simple module $L(2\beta)$ where $\beta \in \Lambda_\Sigma^+$. Since β is a restricted weight, it follows that $K_{2\mu} \cdot v_{2\beta} = K_{\tilde{\mu}} \cdot v_{2\beta}$ for all weights μ. Hence

$$z \cdot v_{2\beta} = \varphi_{HC}(z) \cdot v_{2\beta} = \tilde{\varphi}_{HC}(z) \cdot v_{2\beta}$$

for all $z \in Z(U_q(\mathfrak{g}))$. Since $H_{2\beta} \in \mathscr{P}_\theta$ is a highest weight vector of weight 2β that generates a $U_q(\mathfrak{g})$-module isomorphic to $L(2\beta)$, we also have

$$z \cdot H_{2\beta} = \tilde{\varphi}_{HC}(z) \cdot H_{2\beta}. \tag{36}$$

Since central elements act as scalars on all finite-dimensional simple $U_q(\mathfrak{g})$-modules, it follows that the restricted Harish-Chandra map can be used to determine the eigenvalues with respect to the action of $Z(U_q(\mathfrak{g}))$ on $\mathscr{P}_\theta$:

8.3. **Dotted Weyl Group Invariance.** By [17], the image of the center of $\check{U}_q(\mathfrak{sl}_n)$ in Type AI, $\check{U}_q(\mathfrak{sl}_{2n})$ in Type AII, and $\check{U}_q(\mathfrak{sl}_n \oplus \mathfrak{sl}_n)$ in the diagonal case under $\tilde{\varphi}_{HC}$ equals the invariants $\mathbb{C}(q)[\tau(2\gamma)|\ \gamma \in P_\Sigma^+]^{W_\Sigma \bullet}$ of $\mathbb{C}(q)[\tau(2\gamma)|\ \gamma \in P_\Sigma^+]$ with respect to the dotted action of the restricted Weyl group W_Σ. The dotted action of W_Σ is defined by

$$w \bullet q^{(\tilde{\rho}, 2\gamma)}K_{2\gamma} = q^{(\tilde{\rho}, 2w\gamma)}K_{2w\gamma} \tag{37}$$

for all $w \in W_\Sigma$ and $\gamma \in P_\Sigma^+$.

Recall that $\eta_1, \ldots, \eta_{n-1}$ denote the fundamental weights for the restricted root system and that $\hat{\eta}_1, \ldots, \hat{\eta}_n$ are the corresponding fundamental partitions (see Section 2.5). Note that $\mathbb{C}(q)[\tau(2\gamma)|\ \gamma \in P_\Sigma^+]^{W_\Sigma \bullet}$ is a polynomial ring with generators

$$m_{2w_0\eta_i} = \sum_{w \in W_\Sigma} w \bullet q^{(\tilde{\rho}, 2w_0\eta_i)}K_{2w_0\eta_i} \text{ for } i = 1, \ldots, n-1$$

where here w_0 denotes the longest element of W_Σ. Moreover, up to a nonzero scalar,

$$m_{2w_0\eta_i} = q^{(\tilde{\rho}, 2w\eta_i)}K_{2w_0\eta_i} + \sum_{\tilde{\beta} \geq w_0\eta_i} a_{\tilde{\beta}}K_{2\tilde{\beta}}.$$

Here, we are using the standard Bruhat ordering on the restricted root system Σ. In particular, $\tilde{\beta} \geq \tilde{\beta}'$ provided $\tilde{\beta} - \tilde{\beta}'$ can be expressed as a linear combination of the positive simple roots $\alpha_1^\Sigma, \ldots, \alpha_n^\Sigma$ with all non-negative integer coefficients.

We can translate these results to the setting of $U_q(\mathfrak{g})$ where $\mathfrak{sl}$ is replaced by $\mathfrak{gl}$ everywhere as follows. First, recall that $\hat{\eta}_n$ is orthogonal to all the simple restricted roots. Hence, $\hat{\eta}_n$ is invariant under the action of the restricted Weyl group. It follows from (34), (35), and (37) that $\tilde{\varphi}_{HC}(Z(U_q(\mathfrak{g})))$ is invariant with respect to dotted action of W_Σ. Note that

$$m_{2w_0\hat{\eta}_i} = \sum_{w\in W_\Sigma} w \bullet q^{(\tilde{\rho},2w_0\hat{\eta}_i)} K_{2w_0\hat{\eta}_i} \text{ for } i = 1,\dots,n-1 \quad \text{and} \quad K_{2\hat{\eta}_n}^{\pm 1}.$$

form a set of generators for $\mathbb{C}(q)[\mathcal{A}]^{W_\Sigma\bullet}$.

Set $\mathcal{A}_\geq$ equal to the submonoid of $\mathcal{A}$ generated by $K_{2\epsilon_1^\Sigma},\dots,K_{2\epsilon_n^\Sigma}$ and set $\mathbb{C}(q)[\mathcal{A}_\geq]$ equal to the polynomial algebra generated by these elements. Note that since the restricted root system is of Type A_{n-1}, the restricted Weyl group W_Σ acts as permutations on the set $\{K_{2\epsilon_1^\Sigma},\dots,K_{2\epsilon_n^\Sigma}\}$. Hence the dotted action of W_Σ on $\mathbb{C}(q)[\mathcal{A}]$ restricts to an action on $\mathbb{C}(q)[\mathcal{A}_\geq]$. Moreover, $\mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma\bullet}$ is the polynomial ring with generators

$$m_{2w_0\hat{\eta}_i} = \sum_{w\in W_\Sigma} w \bullet q^{(\tilde{\rho},2w_0\hat{\eta}_i)} K_{2w_0\hat{\eta}_i} \text{ for } i = 1,\dots,n-1 \quad \text{and} \quad K_{2w_0\hat{\eta}_n} = K_{2\hat{\eta}_n}.$$

Lemma 8.2. *The image of $Z(U_2)$ under the restricted Harish-Chandra map $\tilde{\varphi}_{HC}$ is a subalgebra of $\mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma\bullet}$.*

Proof. By (33), $\tilde{\varphi}_{HC}(Z(U_2)) \subseteq \mathbb{C}(q)[\mathcal{A}]$ and by the discussion preceding the lemma, this image of $Z(U_2)$ is invariant under the dotted action of W_Σ.

Consider first Type AI with $N = n$ and Type AII with $N = 2n$. Recall (Section 8.1) that $Z(U_2)$ has a basis consisting of central elements $z_{2\lambda}$ where λ runs over elements $2w_0\Lambda_N^+$. Recall also that the central element $z_{2\lambda}$ is in $(\text{ad } U_q(\mathfrak{g}))\cdot K_{2\lambda}$. It follows from the formulas for the adjoint action (Section 2.3) for $U_q(\mathfrak{g})$ that

$$\varphi_{HC}(z_{2\lambda}) \in ((\text{ad } U_q(\mathfrak{g}))\cdot K_{2\lambda}) \subseteq \sum_{\gamma\in Q^+} \mathbb{C}(q) K_{2\lambda+2\gamma}$$

where φ_{HC} is the ordinary Harish-Chandra map as defined in Section 8.2. It follows that

$$\tilde{\varphi}_{HC}(z_{2\lambda}) \in ((\text{ad } U_q(\mathfrak{g}))\cdot K_{2\tilde{\lambda}}) \subseteq \sum_{\gamma\in Q^+} \mathbb{C}(q) K_{2\tilde{\lambda}+2\tilde{\gamma}}.$$

Using the description of the restricted root systems in Section 2.5, we see that $\lambda \in 2w_0\Lambda_N^+$ ensures that $\tilde{\lambda} \in 2w_0\Lambda_\Sigma^+$. Arguing as in the proof of Lemma 7.2 and, in particular, translating relation (30) to the restricted root setting yields

$$(w_0\Lambda_\Sigma^+ + \mathbb{Q}\hat{\eta}_n) \cap (2w_0\Lambda_\Sigma^+ + 2Q_\Sigma^+) = 2w_0\Lambda_\Sigma^+.$$

It follows that

$$\left(\sum_{\gamma\in Q^+} \mathbb{C}(q) K_{2\tilde{\lambda}+2\tilde{\gamma}}\right)^{W_\Sigma\bullet} \subset \mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma\bullet}.$$

The lemma now follows from the fact that $\tilde{\varphi}_{HC}(z_{2\lambda})$ is invariant with respect to the dotted action of W_Σ. □

8.4. **Central generators.** Let $z_i, \dots, z_n$ be the elements of $Z(U_q(\mathfrak{g}))$ defined by

(i) $z_i = z_{2(\epsilon_{n+1-i}+\cdots+\epsilon_n)}$ for $i = 1, \dots, n$ in Type AI and in the diagonal setting.

(ii) $z_i = z_{2(\epsilon_{2n+1-i}+\cdots+\epsilon_{2n})}$ for $i = 1, \dots, n$ in Type AII

Note that these elements are contained in U_2 and hence are in $Z(U_2)$.

Theorem 8.3. *The algebra $\tilde{\varphi}_{HC}(Z(U_2))$ equals $\mathbb{C}(q)[\mathcal{A}_{\geq}]^{W_\Sigma \bullet}$ and is the polynomial ring on the n generators $\tilde{\varphi}_{HC}(z_1), \dots, \tilde{\varphi}_{HC}(z_n)$.*

Proof. By Lemma 8.2 it is sufficient to show that the subalgebra generated by $\tilde{\varphi}_{HC}(z_i)$, $i = 1, \dots, n$ contains the generators for $\mathbb{C}(q)[\mathcal{A}_{\geq}]^{W_\Sigma \bullet}$ as described right before Lemma 8.2..

Consider Type AI and the diagonal case. By the discussion in Section 2.5, $(\tilde{\epsilon}_{n+i-1} + \cdots + \tilde{\epsilon}_n) = w_0\hat{\eta}_i$, the image of the i^{th} fundamental partition under the longest element in W_Σ. This assertion is just the fact that the restriction of the fundamental partition $\hat{\omega}_i$ equals $\hat{\eta}_i$ for $i = 1, \dots, n$. Moreover, as explained in 2.5, the restriction $\tilde{\omega}_i$ of the fundamental weight ω_i equals the fundamental restricted weight η_i for $i = 1, \dots, n-1$. Recall that $z_{2\mu}$ is the unique up to nonzero scalar central element in $(\text{ad } U_q(\mathfrak{g}))K_{2\mu}$. When $\mu = \hat{\omega}_n$ for the two families Type AI and diagonal under consideration, the element $K_{2\hat{\omega}_n}$ is already a central element, so in this case, we have $z_{2\hat{\omega}_n} = K_{2\hat{\omega}_n}$. Hence $\tilde{\varphi}_{HC}(z_{2\hat{\omega}_n}) = K_{2\hat{\eta}_n}$. Thus the algebra generated by $\tilde{\varphi}_{HC}(z_i)$, $i = 1, \dots, n$ contains $K_{2\hat{\eta}_n}$.

Since the restricted root system is of Type A_{n-1}, the fundamental weights $\eta_1, \dots, \eta_{n-1}$ are minuscule. In other words, $\eta_j \not> \eta_k$ for any pair j, k and, in addition, $\eta_j \not> 0$ for any j where the inequalities are defined via the Bruhat order with respect to the restricted root system. As discussed in Section 8.2 above, it follows from [17], Section 6, that the map $\mathcal{P}_B$ of [17] agrees with $\tilde{\varphi}_{HC}$ upon restriction to the center. Moreover, up to a nonzero constant multiple

$$\tilde{\varphi}_{HC}(z_{2w_0\omega_i}) = m_{2w_0\eta_i} \tag{38}$$

for $i = 1, \dots, n-1$. Thus by (34) and (35), the same result holds for ω_i and η_i replaced by $\hat{\omega}_i$ and $\hat{\eta}_i$ for each i. Thus the algebra generated by $\tilde{\varphi}_{HC}(z_i)$, $i = 1, \dots, n$ contains $m_{2w_0\hat{\eta}_i}, i = 1, \dots, n-1$. We already showed this algebra contains $K_{2\hat{\eta}_n}$. Hence the theorem holds in Type AI and the diagonal setting.

Type AII is more complicated. As in [17], Chapter 8, we set $\eta_0 = \eta_n = 1 = w_0\eta_0 = w_0\eta_n$ and note that $m_{2w_0\eta_0} = m_{2w_0\eta_n} = |W_\Sigma|$, a nonzero scalar. By [17] Theorem 8.9 and its proof $\tilde{\varphi}_{HC}(Z(\check{U}_q(\mathfrak{sl}_{2n}))$ is generated by $\tilde{\varphi}_{HC}(z_{2w_0\omega_i})$ for $i = 1, \dots, n$. Moreover, by [17], Theorem 8.9, $\tilde{\varphi}_{HC}(Z(\check{U}_q(\mathfrak{sl}_{2n}))$ equals the dotted W_Σ invariants of $\mathbb{C}(q)[K_{2\mu}|\mu \in P_\Sigma]$ where recall Section 2.5 that P_Σ is the weight lattice for Σ. Hence, by [17] Lemma 8.8, $\tilde{\varphi}_{HC}(Z(\check{U}_q(\mathfrak{sl}_{2n}))$ is a polynomial ring of rank $n-1$ generated by $\tilde{\varphi}_{HC}(z_{2w_0\omega_i})$ for $i = 1, \dots, n-1$. In [17], it is shown that $\tilde{\varphi}_{HC}(z_{2w_0\omega_1}) = m_{2w_0\eta_1}$. The same reasoning applies to the fundamental partitions $\hat{\omega}_1$ and $\hat{\eta}_1$. Indeed, as explained in Section 2.5, the restricted weight for $w_0\hat{\omega}_1$ is $w_0\hat{\eta}_1$. Since in addition $\hat{\omega}_1$ is a minuscule weight, we must have $\tilde{\varphi}_{HC}(z_{2w_0\hat{\omega}_1}) = m_{2w_0\hat{\eta}_1}$ up to some nonzero scalar. Hence $m_{2w_0\hat{\eta}_1} \in \tilde{\varphi}_{HC}(Z(U_q(\mathfrak{gl}_{2n}))$.

Taking into account differences in notation from [17], Section 8, we have that $m_{2w_0\eta_k}$ can be written as a linear combination of elements in the set

$$\{\tilde{\varphi}_{HC}(z_{2w_0\omega_k}), m_{2w_0\eta_s}m_{2w_0\eta_{k-s}} |\ 1 \leq s < k\}$$

for $k = 2, \dots, n$. We recall here some facts about restricted weights from Section 2.5. The restriction to Σ of the longest element w_0 of the Weyl group for the root system of $\mathfrak{g}$ agrees with the longest element of W_Σ. Moreover, $w_0\omega_j = w_0\hat{\omega}_j - (j/2n)\hat{\omega}_{2n}$, $w_0\eta_j = w_0\hat{\eta}_j - (j/n)\hat{\eta}_n$ for $j = 1, \dots, n$. Note that the second equality in the $j = n$ case follows from the fact that $\eta_n = 1$ and $w_0\hat{\eta}_n = \hat{\eta}_n$. Hence

$$\tilde{\varphi}_{HC}(z_{2w_0\omega_k}) = \tilde{\varphi}_{HC}(z_{2w_0\hat{\omega}_k})K_{(-k/n)\hat{\omega}_{2n}}$$

for each $j = 1, \dots, n$. As explained in Section 2.5, $2\hat{\eta}_n = \hat{\omega}_{2n}$. Thus we also have

$$m_{2w_0\eta_j} = m_{2w_0\hat{\eta}_j} K_{(-j/n)2\hat{\eta}_{2n}} = m_{2w_0\hat{\eta}_j} K_{(-j/n)\hat{\omega}_{2n}}$$

for each $j = 1, \dots, n$ and so

$$\begin{aligned} m_{2w_0\eta_s} m_{2w_0\eta_{k-s}} &= m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{k-s}} K_{(-s/n)\hat{\omega}_{2n}} K_{((-k+s)/n)\hat{\omega}_{2n}} \\ &= m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{k-s}} K_{(-k/n)\hat{\omega}_{2n}} \end{aligned}$$

for each k and each s with $1 \leq s < k$. Thus a relation of the form

$$m_{2w_0\eta_k} = a_0 \tilde{\varphi}_{HC}(z_{2w_0\omega_k}) + \sum_{s=1}^{k-1} a_j m_{2w_0\eta_s} m_{2w_0\eta_{k-s}} \tag{39}$$

becomes

$$m_{2w_0\hat{\eta}_k} = a_0 \tilde{\varphi}_{HC}(z_{2w_0\hat{\omega}_k}) + \sum_{s=1}^{k-1} a_j m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{k-s}}.$$

by multiplying both sides by $K_{(k/n)\hat{\omega}_{2n}}$. Hence $m_{2w_0\hat{\eta}_k}$ can be written as a linear combination of elements in the set

$$\{\tilde{\varphi}_{HC}(z_{2w_0\hat{\omega}_k}), m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{k-s}} |\ 1 \leq s < k\}$$

for $k = 2, \dots, n$. Then arguing by induction yields the algebra generated by $\tilde{\varphi}_{HC}(z_i)$, $i = 1, \dots, n$ contains the elements $m_{2w_0\hat{\eta}_j}$, $j = 1, \dots, n$. The theorem follows from the facts that $w_0\hat{\eta}_n = \hat{\eta}_n$, $K_{2\hat{\eta}_n}$ is invariant with respect to the dotted action of W_Σ and, thus, $m_{2w_0\hat{\eta}_n}$ is a nonzero scalar multiple of $K_{2\hat{\eta}_n}$. □

Let Z denote the subring of $Z(U_2)$ generated by these elements. By the previous theorem, Theorem 8.3, the image under the restricted Harish-Chandra map $\tilde{\varphi}_{HC}$ of the $z_1, \dots, z_n$ is a polynomial ring of rank n. Hence, since Z is commutative, $z_1, \dots, z_n$ must also generate a polynomial ring of rank n. We see that the same is true for the image of Z with respect to Υ of Corollary 7.4.

Corollary 8.4. *The algebra $\Upsilon(Z)$ is a polynomial subring of* End $\mathscr{P}_\theta$ *with n generators $\Upsilon(z_i), i = 1, \dots, n$. Moreover, each $\Upsilon(z_r)$ is an element in $\mathscr{P}\mathscr{D}_\theta$ of degree less than or equal to $2r$.*

Proof. Just as in the definition of the $m_{2w_0\hat{\eta}_i}$, we set $m_{2\lambda} = \sum_{w \in W_\Sigma} q^{(\tilde{\rho}, 2w\lambda)} K_{2w\lambda}$ for each $\lambda \in \Lambda_\Sigma^+$. Note that $\tilde{\gamma} \in Q^+$ for all $\gamma \in Q^+$. Hence, given $\lambda \in \Lambda_\Sigma^+$ and $w \in W_\Sigma$, we have $w\lambda \in \lambda - Q^+$. Since (ρ, γ) is a positive integer for $\gamma \in Q^+$ so is $(\tilde{\rho}, \tilde{\gamma}) = (\rho, \tilde{\gamma})$. Hence, for each $\lambda \in \Lambda_\Sigma^+$, we have

$$m_{2\lambda} \in q^{(\tilde{\rho}, 2\lambda)} K_{2\lambda} + \sum_{\gamma \in Q^+} q^{(\tilde{\rho}, 2\lambda) - 2} \mathbb{C}[q^{-2}] K_{2\lambda - 2\gamma}.$$

Let $v_{2\beta}$ be a highest weight generating vector for $L(2\beta)$ where $\beta \in \Lambda_\Sigma^+$. It follows that

$$m_{2\lambda} \cdot v_{2\beta} \in \left(q^{(\tilde{\rho} + 2\beta, 2\lambda)} (1 + q^{-2}\mathbb{C}[q^{-2}]) \right) v_{2\beta}. \tag{40}$$

Note that any $a \in \mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma \bullet}$ can be expressed as a linear combination $a_1 m_{2\lambda_1} + \cdots + a_s m_{2\lambda_s}$ where each $\lambda_s \in \Lambda_\Sigma^+$. We argue that there exists $\beta \in \Lambda_\Sigma^+$ so that $a \cdot v_{2\beta} \neq 0$. Reordering and multiplying by a nonzero element of $\mathbb{C}[q]$ if necessary, we may assume that $|\lambda_1| \geq |\lambda_j|$ for each $2 \leq j \leq s$, $a_i \in \mathbb{C}[q]$ for each $1 \leq i \leq s$, and $a_1 = q^d +$ terms of lower degree in q. It is straightforward to check that there exists $\beta_1 \in \Lambda_\Sigma^+$ such that $(\lambda_1, \beta_1) > (\lambda, \beta_1)$ for all $\lambda \neq \lambda_1$

satisfying $|\lambda| \leq |\lambda_1|$. Thus, by (40), for a large enough positive integer r, $a \cdot v_{2r\beta} = q^{d+(\tilde{\rho}+2r\beta_1, 2\lambda_1)} +$ terms of degree strictly less than $d + (\tilde{\rho} + 2r\beta_1, 2\lambda_1)$. Thus $a \cdot v_{2\beta} \neq 0$ for $\beta = r\beta_1$.

Recall that $H_{2\beta}$ is a highest weight vector in the $U_q(\mathfrak{g})$-module $\mathscr{P}_\theta$ for each $\beta \in \Lambda_\Sigma^+$. By (36), $z \cdot H_{2\beta} = \tilde{\varphi}_{HC}(z) \cdot H_{2\beta}$ for all $z \in Z(U_q(\mathfrak{g}))$. By Theorem 8.3, $\tilde{\varphi}_{HC}$ defines an isomorphism from Z onto $\mathbb{C}(q)[\mathcal{A}_{\geq}]^{W_\Sigma \bullet}$. Hence, by the previous paragraph, given $z \in Z$, we can find $\beta \in \Lambda_\Sigma^+$ so that $\tilde{\varphi}_{HC}(z) \cdot H_{2\beta} \neq 0$. It follows that $z \cdot H_{2\beta} \neq 0$. By Corollary 7.4, $\Upsilon(z) \cdot H_{2\beta} = z \cdot H_{2\beta}$ and so $\Upsilon(z) \neq 0$. In other words, Υ is injective upon restriction to Z. This proves the first assertion of this corollary.

For the second assertion, note that z_r is in the (ad $U_q(\mathfrak{g})$)-module generated by $K_{2\epsilon_{N+1-r}+\cdots+2\epsilon_N}$ where $N = n$ in Type AI and the diagonal case and $N = 2n$ in Type AII. Hence, by Lemma 6.2, $\Upsilon(z_r)$ has degree less than or equal to $2r$. □

9. Quantum Capelli operators

9.1. **Definition and description.** The decompositions Section 4.4 combined together yields the following module decomposition and related isomorphisms of left $U_q(\mathfrak{g})$-modules:

$$\mathscr{P}\mathscr{D}_\theta = \bigoplus_{\mu,\xi \in \Lambda_\Sigma^+} (U_q(\mathfrak{g}) \cdot H_{2\mu}) \otimes \left(U_q(\mathfrak{g}) \cdot H_{2\xi}^*\right) \cong \bigoplus_{\mu,\xi \in \Lambda_\Sigma^+} L(2\mu) \otimes L^*(2\xi)$$

where $L^*(2\xi)$ is a left $U_q(\mathfrak{g})$- module dual of $L(2\xi)$. This last isomorphism can be made concrete by sending $H_{2\mu}$ to a pre-chosen highest weight generating vector $v_{2\mu}$ of $L(2\mu)$ and $H_{2\xi}^*$ to $\langle H_{2\xi}^*, H_{2\xi}\rangle v_{2\xi}^*$ where $v_{2\xi}^*$ is the lowest weight generating vector for $L(2\xi)^*$ satisfying $v_{2\xi}^*(v_{2\xi}) = 1$. Here, $\langle \cdot, \cdot \rangle$ is the bilinear form defined by (18) in Section 5.2. Note that by Proposition 5.6, $\langle H_{2\xi}^*, H_{2\xi}\rangle \neq 0$. The choice of this nonzero scalar is so that $U_q(\mathfrak{g}) \cdot H_{2\mu}^*$ is the dual of $U_q(\mathfrak{g}) \cdot H_{2\mu}^*$ via $\langle \cdot, \cdot \rangle$ in such a way so that $\langle H_{2\mu}^*, H_{2\mu}\rangle = \langle (\langle H_{2\mu}^*, H_{2\mu}\rangle) v_{2\mu}^*, v_{2\mu}\rangle$.

Recall that there is a natural isomorphism from $(L(2\mu) \otimes L^*(2\mu))$ to End $L(2\mu)$ as left $U_q(\mathfrak{g})$-modules and so $(L(2\mu) \otimes L^*(2\mu))^{U_q(\mathfrak{g})}$ is the one-dimensional subspace consisting of the scalars. Furthermore, $(L(2\mu) \otimes L^*(2\xi))^{U_q(\mathfrak{g})} = 0$ for $\mu \neq \xi$. Hence, the left $U_q(\mathfrak{g})$-module invariants of $\mathscr{P}\mathscr{D}_\theta$ satisfy

$$\mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})} \cong \bigoplus_{\mu,\xi \in \Lambda_\Sigma^+} (L(2\mu) \otimes L^*(2\xi))^{U_q(\mathfrak{g})} = \bigoplus_{\mu \in \Lambda_\Sigma^+} (L(2\mu) \otimes L^*(2\mu))^{U_q(\mathfrak{g})}.$$

Let C_μ be the basis vector for the space $(U_q(\mathfrak{g}) \cdot H_{2\mu}) \otimes \left(U_q(\mathfrak{g}) \cdot H_{2\mu}^*\right)^{U_q(\mathfrak{g})}$ corresponding to the identity element in End $L(2\mu)$ via the isomorphism between $(U_q(\mathfrak{g}) \cdot H_{2\mu}) \otimes \left(U_q(\mathfrak{g}) \cdot H_{2\mu}^*\right)$ and $L(2\mu) \otimes L^*(2\mu)$ described above. We refer to the set $\{C_\mu |\ \mu \in \Lambda_\Sigma^+\}$ as the Capelli operators. Note that the Capelli operators form a basis for the $U_q(\mathfrak{g})$ invariant subspace of $\mathscr{P}\mathscr{D}_\theta$.

We can associate a $2r$ tuple $(u_1, u_2, \ldots, u_{2r})$ to a monomial $x_{i_1,j_1} \cdots x_{i_r,j_r}$ of degree r in $\mathscr{P}_\theta$ so that

(i) $u_1 \leq u_2 \leq \cdots \leq u_{2r}$
(ii) $\{u_1, u_2, \ldots, u_{2r}\} = \{i_1, j_1 \cdots, i_r, j_r\}$
(iii) The number of times an integer k shows up in the sequence $u_1, \ldots, u_{2r}$ is the same as the number of times k shows up in $i_1, j_1, \ldots, i_r, j_r$.

By (8), the weight of $x_{i_1,j_1} \cdots x_{i_r,j_r}$ is $\mu = v_1\epsilon_1 + \cdots + v_n\epsilon_n$ where v_i equals the number of entries in $(u_1, u_2, \ldots, u_{2r})$ equal to i. Thus the $2r$ tuple $(u_1, u_2, \ldots, u_{2r})$ is completely and uniquely determined by the weight μ. We use the notation (μ) to refer to this unique tuple $(u_1, u_2, \ldots, u_{2r})$ associated to $x_{i_1,j_1} \cdots x_{i_r,j_r}$. Note that the size $|\mu|$ of the weight μ equals $v_1 + \cdots + v_n$ which is

also equal to r, the degree of the associated monomial $x_{i_1,j_1}\cdots x_{i_r,j_r}$. We can similarly associate a $2r$ tuple $(u_1,u_2,\ldots,u_{2r})$ to a monomial $d_{i_1,j_1}\cdots d_{i_r,j_r}$ of degree r in $\mathscr{D}_\theta$ that satisfies (i), (ii), and (iii). In this case, $(u_1,u_2,\ldots,u_{2r})=(\mu)$ where $-\mu$ is the weight of $d_{i_1,j_1}\cdots d_{i_r,j_r}$.

Given a pair of weights μ,μ' we write (μ,μ') for the tuple whose entries consist of those of (μ) followed by those of (μ'). Write $|(\mu,\mu')|=|(\mu)|+|(\mu')|$ and use the lexicographic order to give tuples of the same size a partial ordering. Set $(\mathscr{P}_\theta)_\mu$ equal to the μ weight space of $\mathscr{P}_\theta$ and $(\mathscr{D}_\theta)_{-\mu}$ for the $-\mu$ weight space of $\mathscr{D}_\theta$.

Lemma 9.1. *The Capelli operator C_μ for $\mu\in\Lambda_\Sigma^+$ satisfies*

$$C_\mu \in \langle H_{2\mu}^*, H_{2\mu}\rangle^{-1} H_{2\mu}H_{2\mu}^* + \sum_{\substack{(\nu,\nu)>(2\mu,2\mu)\\ |(\nu,\nu)|=|(2\mu,2\mu)|}} (\mathscr{P}_\theta)_\nu(\mathscr{D}_\theta)_{-\nu}. \tag{41}$$

Moreover, $C_\lambda\cdot H_{2\mu}=1$ for $\lambda=\mu$ and $C_\lambda\cdot H_{2\mu}=0$ for $|\mu|\le|\lambda|$ and $\mu\neq\lambda$.

Proof. Recall that $H_{2\mu}$ is a highest weight vector of weight 2μ and $H_{2\mu}^*$ is a lowest weight vector of weight -2μ. Hence

$$C_\mu \in U_q(\mathfrak{g})\cdot(H_{2\mu}H_{2\mu}^*) = (U^-\cdot H_{2\mu})(U^+\cdot H_{2\mu}^*).$$

It follows from the description of the action of $U_q(\mathfrak{g})$ on $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ (see for example 3.2 and [19], Lemma 5.4) that

$$U_+^-\cdot H_{2\mu} \in \sum_{\substack{(\nu)>(2\mu)\\ |(\nu)|=|(2\mu)|}} (\mathscr{P}_\theta)_\nu \quad\text{and}\quad U_+^+\cdot H_{2\mu}^* \in \sum_{\substack{(\nu)>(2\mu)\\ |(\nu)|=|(2\mu)|}} (\mathscr{D}_\theta)_{-\nu}.$$

Note that $(\mathscr{P}_\theta)_\nu(\mathscr{D}_\theta)_{-\nu'}$ is trivial with respect to the action of U^0 if and only if $\nu=\nu'$. Since C_μ is trivial with respect to the action of elements in U^0 , it follows that (41) is true up to the scalar in front of $H_{2\mu}H_{2\mu}^*$.

To understand the coefficient of $H_{2\mu}H_{2\mu}^*$, recall that C_μ acts as the identity on $U_q(\mathfrak{g})\cdot H_{2\mu}$ via the action of $\mathscr{P}\mathscr{D}_\theta$ on $\mathscr{P}_\theta$. Hence $C_\mu\cdot H_{2\mu}=H_{2\mu}$. Since $H_{2\mu}$ is homogeneous of the same degree as elements in $U_q(\mathfrak{g})\cdot H_{2\mu}^*$, we have $ab\cdot H_{2\mu}=a\langle b,H_{2\mu}\rangle$ for all $a\in\mathscr{P}_\theta$ and $b\in U_q(\mathfrak{g})\cdot H_{2\mu}^*$. In particular, $(H_{2\mu}H_{2\mu}^*)\cdot H_{2\mu}=H_{2\mu}\langle H_{2\mu}^*,H_{2\mu}\rangle$. Hence

$$C_\mu\cdot H_{2\mu} = (\langle H_{2\mu}^*,H_{2\mu}\rangle^{-1}H_{2\mu}H_{2\mu}^*)\cdot H_{2\mu} = H_{2\mu}.$$

On the other hand, by Proposition 5.6, $H_{2\mu}$ is orthogonal to $U_+^+\cdot H_{2\mu}^*$ with respect to the bilinear form $\langle\cdot,\cdot\rangle$. It follows that

$$(U_+^-\cdot H_{2\mu})(U_+^+\cdot H_{2\mu}^*)\cdot H_{2\mu}=0.$$

This forces the coefficient of $H_{2\mu}^*H_{2\mu}$ to be $\langle H_{2\mu}^*,H_{2\mu}\rangle^{-1}$ which establishes (41).

We have already shown the first part of the final assertion. Now suppose that μ satisfies $|\mu|\le|\lambda|$ and $\mu\neq\lambda$. Our assumptions on μ ensures that the degree of $H_{2\mu}$ is at most the degree of $H_{2\lambda}^*$. Hence, arguing as above, Proposition 5.6 ensures that

$$(U_+^-\cdot H_{2\lambda})(U_+^+\cdot H_{2\lambda}^*)\cdot H_{2\mu}=0.$$

Furthermore, Proposition 5.6 also implies that $(H_{2\lambda}H_{2\lambda}^*)\cdot H_{2\mu}=0$ under the assumptions relating μ and λ. Thus $C_\lambda\cdot H_{2\mu}=0$ as desired. □

9.2. **Realization as a polynomial ring.** Define an ordering on tuples of the form (μ, μ') by

$$(\nu, \nu') \preceq (\mu, \mu')$$

provided either $|(\nu, \nu')| < |(\mu, \mu')|$ or both $|(\nu, \nu')| = |(\mu, \mu')|$ and $(\nu, \nu') \geq (\mu, \mu')$. Using this ordering, define the filtration $\mathcal{J}^{\text{fin}}$, which is a refinement of the filtration $\mathcal{J}$, on $\mathscr{P}\mathscr{D}_\theta$ by

$$\begin{aligned}\mathcal{J}^{\text{fin}}_{(\mu,\mu')}(\mathscr{P}\mathscr{D}_\theta) &= \sum_{(\nu,\nu')\preceq(\mu,\mu')} (\mathscr{P}_\theta)_\nu(\mathscr{D}_\theta)_{-\nu'} \\ &= \sum_{\substack{(\nu,\nu'))\geq(\mu,\mu') \\ |(\nu,\nu')|=|(\mu,\mu')|}} (\mathscr{P}_\theta)_\nu(\mathscr{D}_\theta)_{-\nu'} + \sum_{r<|(\mu,\mu')|} \mathcal{J}_r(\mathscr{P}\mathscr{D}_\theta).\end{aligned}$$

Let $\mathcal{G}_{\text{fin}}$ denote the associated graded algebra of $\mathscr{P}\mathscr{D}_\theta$ defined by the filtration $\mathcal{J}^{fin}$. Note that the relations for both $\mathscr{P}_\theta$ and $\mathscr{D}_\theta$ are homogeneous with respect to this filtration and hence we can identify these algebras with subalgebras of $\mathcal{G}_{fin}$. Hence the only relation that is changed in the graded algebra is the twisted tensor product. By Theorem 5.1, we have the following relations in $\mathcal{G}_{\text{fin}}$

$$\bar{d}_{ab}\bar{x}_{ef} - q^{\delta_{af}+\delta_{ae}+\delta_{bf}+\delta_{be}}\bar{x}_{ef}\bar{d}_{ab} \tag{42}$$

for all choices of a, b, e, f where here we are representing the image of an element y in $\mathcal{G}_{\text{fin}}$ as $\bar{y}$. Note that $x_{ef}d_{ab} \in (\mathscr{P}_\theta)_{\epsilon_e+\epsilon_f}(\mathscr{D}_\theta)_{\epsilon_a+\epsilon_b}$. Hence (42) can be rewritten as

$$\bar{d}_{ab}\bar{x}_{ef} - q^{(\epsilon_a+\epsilon_b,\epsilon_e+\epsilon_f)}\bar{x}_{ef}\bar{d}_{ab}.$$

By a straightforward inductive argument, we have that

$$\bar{H}^*_{2\mu}\bar{H}_{2\gamma} = q^{(2\mu,2\gamma)}\bar{H}_{2\gamma}\bar{H}^*_{2\mu} \tag{43}$$

for all $\mu, \gamma \in \Lambda^+_\Sigma$. This means that the subring of $\mathcal{G}_{fin}$ generated by the $\bar{H}_{2\mu}, \bar{H}^*_{2\mu}$ for $\mu \in \Lambda^+_\Sigma$ is q-commutative in the sense that all terms commute up to a power of q. Moreover, we can view this as a q-commutative polynomial ring since by Proposition 4.3, the elements $H_{2\mu}$, $\mu \in \Lambda^+_\Sigma$ form a basis for a polynomial ring in n generators and the same result holds for the $H^*_{2\mu}$, $\mu \in \Lambda^+_\Sigma$. The only additional relations involving both $H^*_{2\mu}$ and a $H_{2\gamma}$ terms takes the form of (43).

Proposition 9.2. *The Capelli operators act semisimply on $\mathscr{P}_\theta$, the invariants $\mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})}$ form a commutative subalgebra of $\mathscr{P}\mathscr{D}_\theta$ isomorphic to a polynomial ring in the n generators $C_{\hat{\eta}_1}, \ldots, C_{\hat{\eta}_n}$, and the degree of $C_{\hat{\eta}_i}$ is $2i$ for each $i = 1, \ldots, n$.*

Proof. The definition as $U_q(\mathfrak{g})$ invariants ensures that the subspace spanned by the Capelli operators forms a subalgebra of $\mathscr{P}\mathscr{D}_\theta$. The semisimple action assertion follows from the fact that the Capelli operators are $U_q(\mathfrak{g})$ invariants and $\mathscr{P}_\theta$ is a multiplicity free $U_q(\mathfrak{g})$- module. Moreover, the eigenspaces for the Capelli operators which are the finite-dimensional simple (left) $U_q(\mathfrak{g})$-modules, are joint eigenspaces for the entire space of Capelli operators. This forces the image of invariant subalgebra spanned by the Capelli operators inside End $\mathscr{P}_\theta$ to be commutative. Since the mapping into the endomorphism ring is faithful (Proposition 5.5), the invariant subalgebra must be commutative as well.

Recall that $H_i = H_{2\hat{\eta}_i}$ for $i = 1, \ldots, n$. By definition, $C_{\hat{\eta}_i}$ is in the left $U_q(\mathfrak{g})$-module generated by $H_iH_i^*$, By Proposition 4.3, H_i has degree i. The same holds for H_i^* since it is the image of H_i using the isomorphism $\mathscr{D}_\theta \cong \mathscr{P}_\theta^{op}$. The fact that $U_q(\mathfrak{g})$ action preserves degree ensures that $C_{\hat{\eta}_i}$ must have degree $2i$ for each choice of i.

Consider $\mu = m_1\hat{\eta}_1 + \cdots + m_n\hat{\eta}_n \in \Lambda_\Sigma^+$. It follows from Lemma 9.1 that the image $\bar{C}_{2\mu}$ of C_μ in $\mathcal{G}_{\text{fin}}$ is $\langle H_{2\mu}^*, H_{2\mu}\rangle^{-1}\bar{H}_{2\mu}\bar{H}_{2\mu}^*$. Hence, using (43), Proposition 4.3 and its analog for the $H_{2\mu}^*$ terms, we see that

$$\bar{C}_{2\mu}\bar{C}_{2\mu'} = c_{\mu,\mu'}\bar{H}_{2\mu}H_{2\mu'}\bar{H}_{2\mu}^*\bar{H}_{2\mu'}^* = e_{\mu,\mu'}\bar{H}_{2\mu+2\mu'}\bar{H}_{2\mu+2\mu'}^* = e_{\mu,\mu'}\bar{C}_{2\mu+2\mu'}.$$

for some nonzero scalars $c_{\mu,\mu'}$ and $e_{\mu,\mu'}$. It follows that

$$\bar{C}_{2\hat{\eta}_1}^{m_1}\cdots\bar{C}_{2\hat{\eta}_n}^{m_n} = \bar{C}_{2(m_1\hat{\eta}_1+\cdots+m_n\hat{\eta}_n)}$$

and hence the image of $\mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})}$ in $\mathcal{G}^{\text{fin}}$ is a generated by the elements $\bar{C}_{2\hat{\eta}_1}, \cdots, \bar{C}_{2\hat{\eta}_n}$. The algebraic independence of these elements follows from the algebraic independence of the terms $H_{2\hat{\eta}_1}, \ldots, H_{2\hat{\eta}_n}$ established in Proposition 4.3 and the analogous result for the $H_{2\hat{\eta}_i}^*$. Thus the image of $\mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})}$ in $\mathcal{G}_{\text{fin}}$ is isomorphic to a polynomial ring in the n generators $\bar{C}_{2\hat{\eta}_1}, \cdots, \bar{C}_{2\hat{\eta}_n}$. This combined with the fact that the Capelli operators are a commuting family of elements ensures that the space spanned by the Capelli operators is a polynomial ring in the n variables $C_{2\hat{\eta}_1}, \cdots, C_{2\hat{\eta}_n}$. □

Note that the key idea in the above proof is that the image of the elements $C_{\hat{\eta}_1}, \ldots, C_{\hat{\eta}_n}$ in the associated graded algebra define by the filtration $\mathcal{J}^{\text{fin}}$ are algebraically independent. Since $\mathcal{J}^{\text{fin}}$ is a filtration that is a refinement of $\mathcal{J}$, the independence property for the associated graded algebra must also hold upon replacing $\mathcal{J}^{\text{fin}}$ by $\mathcal{J}$. Hence the top degree terms of the $C_{\hat{\eta}_1}, \ldots, C_{\hat{\eta}_n}$ with respect to the degree function defining $\mathcal{J}$ are algebraically independent.

9.3. **The Center and Capelli Operators.** Recall the definition of the subring Z generated by the central elements $z_1, \ldots, z_n$ living inside U_2 from Section 8.4. .The next theorem relates this subalgebra Z of the center to the algebra of Capelli operators via the mapping Υ of Corollary 7.4.

Theorem 9.3. *The $U_q(\mathfrak{g})$-module algebra map Υ defines an algebra isomorphism from the polynomial subring Z of $Z(U_2)$ to the algebra generated by the Capelli operators.*

Proof. Recall (Theorem 7.4) is $U_q(\mathfrak{g})$-module algebra map of $\mathcal{F}_2(U_q(\mathfrak{g})) = \mathcal{F}(U_q(\mathfrak{g})) \cap U_2$ to $\mathscr{P}\mathscr{D}_\theta$ where the module structure on the first algebra is given by the adjoint action and on the second algebra via the left action. Note that $z \in Z(U_q(\mathfrak{g}))$ if and only if $az = za$ all $a \in U_q(\mathfrak{g})$ if and only if $(\text{ad } a)z = \epsilon(a)z$ all $a \in U_q(\mathfrak{g})$. Thus

$$Z(U_2) = U_2 \cap U_q(\mathfrak{g})^{\text{ad } U_q(\mathfrak{g})} = \mathcal{F}_2(U_q(\mathfrak{g}))^{\text{ad } U_q(\mathfrak{g})}.$$

Hence

$$\Upsilon(Z(U_2)) = \Upsilon(\mathcal{F}_2(U_q(\mathfrak{g}))^{\text{ad } U_q(\mathfrak{g})}) \subseteq \mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})}.$$

The last term is just the subalgebra of $U_q(\mathfrak{g})$ equal to the subspace spanned by the Capelli operators. Hence $\Upsilon(Z(U_2)) \subseteq \sum_\mu \mathbb{C}(q)C_\mu$. By Corollary 8.4, $\Upsilon(Z)$ is a polynomial ring generated by $\Upsilon(z_1), \cdots, \Upsilon(z_n)$ and $\deg(\Upsilon(z_i)) \leq 2i$ for each $i = 1, \ldots, n$.

Set $c_r = \Upsilon(z_r)$ for $r = 1, \ldots, n$. We argue by induction that

$$\mathbb{C}(q)[c_1, \ldots, c_j] = \mathbb{C}(q)[C_{\hat{\eta}_1}, \ldots, C_{\hat{\eta}_j}] \tag{44}$$

for $j = 0, 1, \ldots, n$ where $j = 0$ is true because it is simply the case where both the left and right side of (44) are just the scalars. Now assume that (44) is true for some i satisfying $i - 1 \geq 0$. Recall that $\deg C_{\hat{\eta}_i} = 2i$ for each i and the top degree terms of the elements $C_{\hat{\eta}_1}, \ldots, C_{\hat{\eta}_n}$ are algebraically independent. It follows that any polynomial expression in these elements that includes $C_{\hat{\eta}_j}$ for $j > i$ has degree strictly greater than $2i$. Since c_i is of degree at most $2i$, we must have

$c_i \in \mathbb{C}(q)[C_{\hat{\eta}_1}, \dots, C_{\hat{\eta}_i}]$. Hence, by induction, we must also have $c_i \notin \mathbb{C}(q)[C_{\hat{\eta}_1}, \dots, C_{\hat{\eta}_{i-1}}]$. Since c_i is of degree at most $2i$ and $C_{\hat{\eta}_i}$ has degree $2i$, we get

$$c_i - aC_{\hat{\eta}_i} \in \mathbb{C}(q)[C_{\hat{\eta}_1}, \dots, C_{\hat{\eta}_{i-1}}] = \mathbb{C}(q)[c_1, \dots, c_{i-1}]$$

for some nonzero scalar a. Assertion (44) now holds for $j = i$ and hence, by induction, is true for $j = n$. This proves the theorem. □

Recall the discussion at the end of Section 9.2 that the $C_{\hat{\eta}_i}$ remain algebraically independent upon passage to the associated graded algebra defined by the degree filtration $\mathcal{J}$ on $\mathscr{P}\mathscr{D}_\theta$. Thus, by the proof of Theorem 9.3, c_i must have the same degree, namely $2i$ as $C_{\hat{\eta}_i}$ for each i. In other words the inequalities in the final assertion of Corollary 8.4 are all equalities.

10. Eigenvalues of Capelli Operators

10.1. **Definition and degree.** Let $C \in \sum_\mu \mathbb{C}(q)C_\mu = \mathscr{P}\mathscr{D}_\theta^{U_q(\mathfrak{g})}$ and $z \in Z$ such that $C = \Upsilon(z)$. Set $\mathcal{E}(C) = \tilde{\varphi}_{HC}(z)$. By Theorem 8.3, $\mathcal{E}(C)$ is an element of $\mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma \bullet}$ and determines the eigenvalues of C with respect to its action on $\mathscr{P}_\theta$. In particular, given $H_\beta \in \mathscr{P}_\theta$, by (36) and the definitions of Υ and $\mathcal{E}(C)$, we have

$$C \cdot H_{2\beta} = z \cdot H_{2\beta} = \tilde{\varphi}_{HC}(z) \cdot H_{2\beta} = \mathcal{E}(C) \cdot H_{2\beta}.$$

It follows from Theorem 8.3, Corollary 8.4, and Theorem 9.3, that $\mathcal{E}$ defines an algebra isomorphism from $\mathscr{P}\mathscr{D}_\theta^{(U_q(\mathfrak{g})}$ to $\mathbb{C}(q)[\mathcal{A}_\geq]^{W_\Sigma \bullet}$.

Given $\lambda \in \Lambda_\Sigma^+$, let $\mathcal{E}_\lambda$ be the polynomial in $\mathbb{C}(q)[x_1, \dots, x_n]$ such that

$$\mathcal{E}_\lambda(K_{2\epsilon_1^\Sigma}, \dots, K_{2\epsilon_n^\Sigma}) = \mathcal{E}(C_\lambda).$$

Note that it makes sense to talk about the degree of $\mathcal{E}_\lambda$ using the standard degree function for $\mathbb{C}(q)[x_1, \dots, x_n]$ where $\deg x_i = 1$ for all $i = 1, \dots, n$. Alternatively, we can view $\mathcal{E}_\lambda(K_{2\epsilon_1^\Sigma}, \dots, K_{2\epsilon_n^\Sigma})$, or more generally, any $\mathcal{E}(C)$ for $C \in \sum_\mu \mathbb{C}(q)C_\mu$, as a polynomial in $\mathbb{C}(q)[\mathcal{A}_\leq]$. From this perspective, we can again talk about degree by insisting that $\deg K_{2\epsilon_i^\Sigma} = 1$ for each i. Note that $\deg K_{2w_0\hat{\eta}_i} = i$ and so the same is true for $m_{2w_0\hat{\eta}_i}$ for each $i = 1, \dots, n$.

Lemma 10.1. *For each $\lambda \in \Lambda_\Sigma^+$, the degree of $\mathcal{E}_\lambda$ is $|\lambda|$.*

Proof. We first argue that the degree of $\mathcal{E}_{\hat{\eta}_i} = |\hat{\eta}_i| = i$ for $i = 1, \dots, n$. Consider first $\mathcal{E}_{\hat{\eta}_1} = \mathcal{E}(C_{\hat{\eta}_1})$. In this case, $C_{\hat{\eta}_1} = a\Upsilon(z_1) + b$ where a and b are scalars and $a \neq 0$. Hence $\mathcal{E}_{\hat{\eta}_1} = a\tilde{\varphi}_{HC}(z_1) + b$. As explained in the proof of Theorem 8.3, $\tilde{\varphi}_{HC}(z_1) = m_{2w_0\hat{\eta}_1}$ (up to a nonzero scalar multiple) and so $\mathcal{E}_{\hat{\eta}_1}$ is a polynomial of degree 1. Now assume $\mathcal{E}_{\hat{\eta}_j}$ is a polynomial of degree j for all $1 \leq j \leq i-1$. We argue that $\mathcal{E}_{\hat{\eta}_i}$ is a polynomial of degree i and so the lemma will then follow by induction.

Set $c_i = \Upsilon(z_i)$ for $i = 2, \dots, n$. It follows from Theorem 9.3 and its proof that for each $i \geq 2$, $C_{\hat{\eta}_i}$ can be expressed as a nonzero scalar multiple of $c_i + p_i(C_{\hat{\eta}_1}, \dots, C_{\hat{\eta}_{i-1}})$ for some polynomial p_i. As explained in Section 9.2, $C_{\hat{\eta}_1}, \dots, C_{\hat{\eta}_{i-1}}$ remain algebraically independent upon passage to the graded algebra defined by the degree filtration $\mathcal{J}$. Since $C_{\hat{\eta}_i}$ and c_i both have the same degree $2i$, it follows that every monomial in p must have degree at most $2i$. In particular, any monomial summand of p such as $C_{\hat{\eta}_1}^{m_1} \cdots C_{\hat{\eta}_{i-1}}^{m_{i-1}}$ must satisfy

$$2m_1 + 4m_2 + \cdots + 2(i-1)m_{i-1} \leq 2i.$$

This inequality combined with the inductive hypothesis yields

$$\begin{aligned}\deg \mathcal{E}(C_{\hat{\eta}_1}^{m_1}\cdots C_{\hat{\eta}_{i-1}}^{m_{i-1}}) &= \deg \mathcal{E}(C_{\hat{\eta}_1}^{m_1})\cdots \mathcal{E}(C_{\hat{\eta}_{i-1}}^{m_{i-1}}) \\ &= \deg \mathcal{E}(C_{\hat{\eta}_1}^{m_1}) + \cdots + \deg \mathcal{E}(C_{\hat{\eta}_{i-1}}^{m_{i-1}}) \\ &= m_1 + 2m_2 + \cdots + (i-1)m_{i-1} \leq i\end{aligned}$$

Hence it suffices to show that $\mathcal{E}(c_i) = \tilde{\varphi}_{HC}(z_i)$ has degree i.

As explained in the proof of Theorem 8.3, $\tilde{\varphi}_{HC}(z_i) = m_{2w_0\hat{\eta}_i}$ (up to a nonzero scalar multiple) in Type AI and the diagonal case. Hence, $\mathcal{E}(c_i)$ has degree i in these two cases. For Type AII, the proof of Theorem 8.3 shows that $m_{2w_0\hat{\eta}_i}$ is a linear combination of elements in the set

$$\{\tilde{\varphi}_{HC}(z_i), m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{i-s}} | 1 \leq s < i\}.$$

Since the terms $m_{2w_0\hat{\eta}_s}, s = 1, \ldots, i$ are algebraically independent, it follows that $\tilde{\varphi}_{HC}(z_i)$ can be expressed as a linear combination of the elements in the set

$$\{m_{2w_0\hat{\eta}_i}, m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{i-s}} | 1 \leq s < i\}. \tag{45}$$

Note that $\deg m_{2w_0\hat{\eta}_i} = i$ and $\deg m_{2w_0\hat{\eta}_s} m_{2w_0\hat{\eta}_{i-s}} = \deg m_{2w_0\hat{\eta}_s} + \deg m_{2w_0\hat{\eta}_{i-s}} = s + i - s = i$ for all $1 \leq s < i$. Hence $\deg \tilde{\varphi}_{HC}(z_i) \leq i$ and equality holds because the terms in (45) are algebraically independent even upon passage to the graded setting. □

10.2. **Vanishing and Non-Vanishing Properties.** Consider $H_{2\mu} \in \mathscr{P}_\theta$ for $\mu \in \Lambda_\Sigma^+$. We expand μ as a sum of the form $\mu = (\mu_1\epsilon_1^\Sigma + \cdots + \mu_n\epsilon_n^\Sigma)$. Note that

$$(K_{2\epsilon_i^\Sigma}) \cdot H_{2\mu} = q^{(2\mu, 2\epsilon_i^\Sigma)} H_{2\mu} = q^{m\mu_i} H_{2\mu}$$

where $m = 4$ in Type AI, and $m = 2$ in Types AII and the diagonal case. Here, we are using the relationship between the inner product for the restricted root system and the inner product $(\cdot, \cdot)$ for the root system of $\mathfrak{g}$ as explained in Section 2.5. Given a polynomial $P \in \mathbb{C}(q)[x_1, \ldots, x_n]$, set $P(q^\mu) = P(q^{\mu_1}, \ldots, q^{\mu_n})$. Note that

$$P(K_{2\epsilon_1^\Sigma}, \ldots, K_{2\epsilon_n^\Sigma}) \cdot H_{2\mu} = P(q^{m\mu}) H_{2\mu}.$$

Hence

$$C_\lambda \cdot H_{2\mu} = \mathcal{E}(C_\lambda) \cdot H_{2\mu} = \mathcal{E}_\lambda(q^{m\mu}) H_{2\mu}$$

for each λ and μ in Λ_Σ^+.

Proposition 10.2. *For each $\lambda, \mu \in \Lambda_\Sigma^+$, the eigenvalue function $\mathcal{E}_\lambda$ satisfy*

(i) $\mathcal{E}_\lambda(q^{m\mu}) = 1$ *for* $\lambda = \mu$.
(ii) $\mathcal{E}_\lambda(q^{m\mu}) = 0$ *for* $\mu \neq \lambda$ *and* $|\mu| \leq |\lambda|$.

where $m = 4$ in Type AI, and $m = 2$ in Types AII and the diagonal case.

Proof. It follows from the discussion preceding the proposition that $\mathcal{E}_\lambda(q^{m\mu}) H_{2\mu} = C_\lambda \cdot H_{2\mu}$ for all choices of λ and μ. Hence (i) and (ii) are equivalent to the analogous assertions with $\mathcal{E}_\lambda(q^{m\mu})$ replaced by $C_\lambda \cdot H_{2\mu}$. The proposition now follows from Lemma 9.1. □

10.3. **Knop-Sahi Interpolation Polynomials.** Let $\mathbb{C}(a,g)[x_1,\dots,x_n]$ be the polynomial in n variables over the field $\mathbb{C}(a,g)$ where a and g are two independent parameters. Knop-Sahi interpolation polynomials introduced in [9] and [26], also called shifted Macdonald polynomials in the later paper [24], are a family of polynomials $P^*_\lambda(x;a,g)$ indexed by partitions λ contained in this polynomial ring that satisfy both an invariance condition and a vanishing condition. In particular, the element $P^*_\lambda(x;a,g)$ in $\mathbb{C}(a,g)[x_1,\dots,x_n]$ is the unique (up to nonzero scalar) polynomial in the $x_1,\dots,x_n$ of degree $|\lambda|$ such that

- $P^*_\lambda(x;a,g)$ is symmetric viewed as a polynomial in the n terms $x_1g^{-1},\dots,x_ng^{-n}$
- $P^*_\lambda(a^\mu;a,g)=0$ for each partition $\mu\neq\lambda$ with $|\mu|\leq|\lambda|$ and $P^*_\lambda(a^\lambda;a,g)\neq 0$ where a^μ is the n-tuple $(a^{\hat{\omega}_1},\dots,a^{\hat{\omega}_n})$.

Here, we are mainly following the Onkoukov's aproach [24]. However, we are using the simpler vanishing condition of Sahi [26] and Knop [9]. (The stronger vanishing property in [24] follows from [9], Section 4). Also, the polynomials in Sahi's paper are symmetric in the $x_1,\dots,x_n$ and so these are recovered from the one's defined here by a change of variables.

The following explicit description of the dotted Weyl group invariance for the three families under consideration will be used to determine the value of g in identifying the Capelli eigenvalues with interpolation polynomials.

Lemma 10.3. *The algebra $\mathbb{C}(q)[\mathcal{A}_{\geq}]^{\mathcal{W}_\theta\bullet}$ is the same as the algebra of symmetric polynomials inside of $\mathbb{C}(q)[q^{-2i}K_{2\tilde{\epsilon}_i},i=1,\dots,n]$ in Type AI and the diagonal case and is equal to the algebra of symmetric polynomials in $\mathbb{C}(q)[q^{-4i}K_{2\tilde{\epsilon}_{2i}},i=1,\dots,n]$ for Type AII.*

Proof. Recall that ρ is the half sum of the positive roots for the root system of $\mathfrak{gl}_N$. Note that the set of roots α such that $(\alpha,\epsilon_i)\neq 0$ can be partitioned into two sets $\{\alpha_j+\cdots+\alpha_{i-1}|j\leq i-1\}$ and $\{\alpha_i+\cdots+\alpha_j|\ j\geq i\}$. For elements α in the first set $(\alpha,\epsilon_i)=-1$ and for elements α in the second set, $(\alpha,\epsilon_i)=1$. Since the first set has $i-1$ elements and the second set has $N-i$ elements, we have that $(\rho,\epsilon_i)=(N-2i-1)/2$. Now returning to the restricted root cases under consideration, we set $N=n$ and for $i=1,\dots,n$, we have

$$q^{(\tilde{\rho},2\tilde{\epsilon}_i)}K_{2\tilde{\epsilon}_i}=q^{(\rho,2\tilde{\epsilon}_i)}K_{2\tilde{\epsilon}_i}=q^{(\rho,2\epsilon_i)}K_{2\tilde{\epsilon}_i}=q^{n-2i-1}K_{2\tilde{\epsilon}_i}.$$

for Type AI,

$$q^{(\tilde{\rho},2\tilde{\epsilon}_i)}K_{2\tilde{\epsilon}_i}=q^{(\rho,2\tilde{\epsilon}_i)}K_{2\tilde{\epsilon}_i}=q^{(\rho_1,\epsilon_i)+(\rho_2,\epsilon_{i+n})}K_{2\tilde{\epsilon}_i}=q^{n-2i-1}K_{2\tilde{\epsilon}_i}$$

in the diagonal case and

$$q^{(\tilde{\rho},2\tilde{\epsilon}_{2i})}K_{2\tilde{\epsilon}_{2i}}=q^{(\rho,2\tilde{\epsilon}_{2i})}K_{2\tilde{\epsilon}_{2i}}=q^{(\rho,\epsilon_{2i-1}+\epsilon_{2i})}K_{2\tilde{\epsilon}_{2i}}=q^{2n-4i}K_{2\tilde{\epsilon}_{2i}}$$

in Type AII. Thus, the dotted restricted Weyl group invariance translates to a simple restricted Weyl group invariance property using $q^{u_i}K_{2\tilde{\epsilon}_i}$ instead of $K_{2\tilde{\epsilon}_i}$ for $u_i=-2i$ in Type AI and the diagonal case and $u_i=4i$ in Type AII. The lemma now follows from the fact that the restricted root system is of type A_{n-1} and so the corresponding Weyl group is the symmetric group. □

Note that in [24], the initial formulation for the polynomials above is so that they are unique *up to nonzero scalar multiple*. Later, a normalization is provided (Section 4 of [24], see equality (4.3)) so that these polynomials can be given precisely without worrying about a scalar factor. The papers [26] and [8] normalize these polynomials in a different way. Namely, they assume that the symmetric polynomial m_λ associated to the partition λ appears in the polynomial at λ with coefficient equal to 1. The normalization used here is again different from each of these earlier references. Here, we have assumed that the eigenvalue functions $\mathcal{E}_\lambda$ arise from elements corresponding to the identity in the

appropriate $U_q(\mathfrak{g})$-module. This choice is equivalent to the condition $\mathcal{E}_\lambda(q^{m\lambda}) = 1$ of Proposition 10.2.

Theorem 10.4. *For each λ, the polynomial $\mathcal{E}_\lambda(x_1, \ldots, x_n)$ evaluated at $x_i = K_{2\epsilon_i^\Sigma}$ is equal to $c_\lambda P^*_\lambda(x; a, g)$ where $c_\lambda = P^*_\lambda(a^\lambda; a, g)^{-1}$ and*

- *$(a, g) = (q^4, q^2)$ in Type AI,*
- *$(a, g) = (q^2, q^4)$ in Type AII, and*
- *$(a, g) = (q^2, q^2)$ in the diagonal type*

Proof. Set $g = q^2$ in Type AI and the diagonal case andset $g = q^4$ in Type AII. By Theorem 8.3 and Theorem 9.3, $\mathcal{E}_\lambda(K_{2\epsilon_1^\Sigma}, \ldots, K_{2\epsilon_n^\Sigma})$ is in the dotted Weyl group invariants of $\mathbb{C}(q)[\mathcal{A}_\geq]$. Hence, by Lemma 10.3, $\mathcal{E}_\lambda(K_{2\epsilon_1^\Sigma}, \ldots, K_{2\epsilon_n^\Sigma})$ is a symmetric polynomial in the variables $g^{-1}K_{2\epsilon_{s1}^\Sigma}, \ldots, g^{-n}K_{2\epsilon_n^\Sigma}$. The same claim is true when we replace each $K_{2\epsilon_i^\Sigma}$ with x_i and so $\mathcal{E}_\lambda(x_1, \ldots, x_n)$ satisfies the same invariance property as $P^*_\lambda(x; a, g) = P^*_\lambda(x_1, \ldots, x_n)$.

Now set $a = q^4$ in Type AI and $a = q^2$ in Type AII and the diagonal case. By Proposition 10.2 (ii), $\mathcal{E}_\lambda(a^\mu) = 0$ for each partition $\mu \neq \lambda$ with $|\mu| \leq \lambda$. Hence $\mathcal{E}_\lambda$ satisfies the same vanishing property as $P^*_\lambda(x; a, g)$. By Lemma 10.1, the degree of $\mathcal{E}_\lambda$ is $|\lambda|$. Thus $\mathcal{E}_\lambda$ must be a nonzero scalar multiple of $P^*_\lambda(x; a, g)$ for each λ. The fact that this nonzero scalar is c_λ follows from the fact that $\mathcal{E}_\lambda(a^\lambda) = 1$ which is just Proposition 10.2 (i). □

Remark 10.5. *Recall that $\mathcal{O}_q(SL_N)$ can be obtained from $\mathcal{O}_q(\mathrm{Mat}_N)$ by modding out by $\det_q - 1$. The space of zonal spherical functions $\mathcal{H} := \mathcal{H}_{\chi(\mathcal{B}_\theta),\mathcal{B}_\theta}$ associated to the pair $\chi(\mathcal{B}_\theta), \mathcal{B}_\theta$ and a Hopf algebra automorphism χ of $U_q(\mathfrak{g})$ are the left $\chi(\mathcal{B}_\theta)$ invariants and right $\mathcal{B}_\theta$ invariants of $\mathcal{O}_q[G]$ where $G = SL_n$ in Type AI, $G = SL_{2n}$ in Type AII, and $G = SL_n \times SL_n$ in the diagonal cae (see* [15]*). By* [15] *Theorem 4.2, there is a map from $\mathcal{O}_q[G]$ to functions on U^0 so that for the correct choice of χ, the image of $\mathcal{H}_{\chi(\mathcal{B}_\theta),\mathcal{B}_\theta}$ is a subring of W_Σ invariants. Moreover, for this choice of χ, a special basis φ_λ, $\lambda \in P^+(\Phi)$ can be identified with a family of Macdonald polynomials $P_\lambda(x; a, g)$ where both a and g are powers of q (see* [16]*, Appendix A). The values of a and g for the three families of this paper are*

- *$(a, g) = (q^4, q^2)$ in Type AI,*
- *$(a, g) = (q^2, q^4)$ in Type AII, and*
- *$(a, g) = (q^2, q^2)$ in the diagonal type.*

These are the same parameters that appear in the identification of the quantum Capelli eigenvalues with interpolation polynomials as in Theorem 10.4. This connection between the realization of zonal spherical functions as Macdonald polynomials and quantum Capelli operators and Knop-Sahi interpolation polynomials mirrors the classical situation. Indeed, these parameters agree because in terms of a particular natural grading, the top degree term of Knop-Sahi interpolation polynomials are precisely Macdonald polynomials with the same parameters (see for example [26]*, Theorem 1.1).*

11. Appendix: Commonly used notation

We list here commonly used symbols and notation along with the first section (post the introduction) in which each item appears.

Section 2.1: ϵ_i, α_i, ω_i, Φ_N, P, Q, P^+, Q^+, Λ_N, Λ_N^+, $\hat{\omega}_i$, w_0, $\mathfrak{gl}_N$, $\mathfrak{sl}_N$, $(\cdot, \cdot)$, $\gamma \oplus \gamma'$

Section 2.2: e_i, f_i, h_{ϵ_i}, $K_{\epsilon_j}, K_i, K_\beta, E_i, F_i$, Δ, ϵ, S, $U^0(\mathfrak{gl}_N)$, $U^0(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$, U^0, $U^+(\mathfrak{gl}_N)$, $U^+(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$, U^+, U^-, $\check{U}_q(\mathfrak{sl}_N)$, $\check{U}^0$, U^+_+, U^-_+, $L(\lambda)$

Section 2.3: (ad a), $\mathcal{F}(M)$, (ad E_i), (ad F_i), (ad$|K$)

Section 2.4: $\mathfrak{g}, \theta, \mathfrak{k}$, $\mathcal{B}_\theta$, J, $R_\mathfrak{g}$, R, B_i, Type AI, Type AII, diagonal case

Section 2.5: π, Φ, $\tilde{\beta}$, Σ, α_i^Σ, ϵ_i^Σ, η_i, $\hat{\eta}_i$, P_Σ, P_Σ^+, Λ_Σ^+, W_Σ, $(\cdot,\cdot)_\Sigma$

Section 3.1: Mat_N, $\mathcal{O}_q(\mathrm{Mat}_N)$, $\mathcal{O}_q(\mathrm{Mat}_N)^{op}$, e_{ij}, r_{kl}^{ij}, R^{t_s}, t_{ij}, ι, $t_{i+N,j+N}$, T_1, T_2, $\mathscr{P}, \mathscr{D}$

Section 3.2: $J_{r,s}$, x_{ij}, d_{ij}, $\mathscr{P}_\theta$, $\mathscr{P}_\theta^{op}$, $\mathscr{D}_\theta$, $\mathcal{J}_r$,$\mathscr{P}_\theta^r$, $\mathscr{D}_\theta^r$

Section 4.1: $\det_q(T)$, $\det_q(T')$, H_n

Section 4.2: $\mathfrak{g}_r$, $U_q(\mathfrak{g}_r)$, $\mathscr{P}(\mathfrak{g}_r)$, $\mathcal{B}_\theta^r$, $\mathscr{P}_\theta(\mathfrak{g}_r)$, $x(r)_{ij}$, $\psi_{r,s}$

Section 4.3: H_i, $H_{2\mu}$

Section 4.4: $H_{2\mu}^*$

Section 5.1: $\mathscr{P}\mathscr{D}_q(\mathrm{Mat}_N)$, $\mathscr{P}\mathscr{D}_\theta$, $\mathcal{J}_r(\mathscr{P}\mathscr{D}_\theta)$, ϕ, ϕ_a

Section 5.2: $\mathcal{L}$, $\langle\cdot,\cdot\rangle$, $(a)_0$

Section 6.3: ψ

Section 7.1: $\check{U}_q(\mathfrak{g})$

Section 7.2: $\mathcal{F}_2(U_q(\mathfrak{gl}_N)$

Section 7.3:$\mathcal{F}_2(U_q(\mathfrak{gl}_N \oplus \mathfrak{gl}_N))$, $U_q^2(\mathfrak{gl}_N)$, $U_q^2(\mathfrak{gl}_N \oplus \mathfrak{gl}_N)$, U_2, Υ

Section 8.1: $z_{2\mu}$

Section 8.2: φ_{HC}, $\mathcal{A}$, $\tilde{\varphi}_{HC}$

Section 8.3: $W_\Sigma\bullet$, $w\bullet$, $m_{2w_0\eta_i}$, $m_{2w_0\hat{\eta}_i}$, $\mathcal{A}_\geq$

Section 8.4: z_i, Z

Section 9.1: C_μ, (μ), (μ,μ'), $|(\mu,\mu')|$

Section 10.1: $\mathcal{E}$, $\mathcal{E}_\lambda$, $P(q^\mu)$

Section 10.3: $P_\lambda^*(x;a,g)$

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