Journal of Algebra 320 (2008) 3187–3198

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Direct limits of Schubert varieties and global sections of line bundles

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ARTICLE INFO

Article history:
Received 11 December 2007
Available online 12 August 2008
Communicated by Vera Serganova

Keywords:
Schubert variety
Borel–Weil–Bott theorem
Demazure module
Classical ind-group

ABSTRACT

We study the space of global sections $\Gamma(X,\mathcal{L})$ of a line bundle $\mathcal{L}$ on a B-stable ind-subvariety X of G/B, where G is a classical simple ind-group and B is an arbitrary Borel subgroup of G. We give a necessary and sufficient condition for projectivity of X, and use it to prove that if X is projective and $\mathcal{L}$ is globally generated, then $\Gamma(X,\mathcal{L})$ is the dual of a B-module $V_X(\mathcal{L})$ with finite-dimensional weight multiplicities. Finally, we describe the structure of these multiplicities in terms of Demazure modules.

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1. Introduction

A marvelous example of interplay between representation theory and algebraic geometry is the celebrated Borel–Weil–Bott theorem, which has been generalized vastly beyond reductive algebraic groups, e.g., to ind-groups and loop groups [DPW,Ku]. In connection with the Borel–Weil theorem, it is natural to study global sections (and higher cohomologies) of *globally generated* line bundles on B-stable subvarieties of a flag variety G/B, which are called Schubert varieties. Standard monomial theory and Demazure's character formula are among the highlights of this subject.

A systematic study of representations of direct limit algebraic groups and their geometric realizations was initiated by Dimitrov, Penkov and Wolf [DP1,DPW]. In particular, in [DP2] the authors show that the infinite-dimensional homogeneous spaces G/B for classical simple ind-groups G are realizable as spaces of generalized flags. Somewhat surprisingly, they show that the ind-variety G/B is rarely projective.

In this work our main purpose is to study B-stable subvarieties of G/B (where G is a classical simple ind-group and B is a Borel subalgebra of G) which we call ind-Schubert varieties (see Section 2.4 for a precise definition). We are interested in the following two natural questions:

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doi:10.1016/j.jalgebra.2008.06.036

Question 1. When is an ind-Schubert variety projective?

Question 2. For a globally generated line bundle $\mathcal{L}$ on a projective ind-Schubert variety X, what is the algebraic structure of the B-module $\Gamma(X,\mathcal{L})$?

Our first result is a necessary and sufficient condition for projectivity of an ind-Schubert variety. Our result shows that although there is a rich class of infinite-dimensional projective ind-Schubert varieties, they are essentially subvarieties of products of projective G/B's.

As for the second question, from general facts about ind-varieties it follows that the B-module $\Gamma(X,\mathcal{L})$ is canonically isomorphic to the dual of a B-module $V_X(\mathcal{L})$ (see Section 2.5). Our second result relates the geometry of X to the algebraic structure of $V_X(\mathcal{L})$. As an application of our projectivity criterion, we prove that when the ind-Schubert variety X is projective, for any globally generated line bundle $\mathcal{L}$ on X the weight spaces of $V_X(\mathcal{L})$ are finite-dimensional. It will be seen that there exists a weight λ such that $V_X(\mathcal{L})_\mu \neq \{0\}$ only if $\lambda-\mu$ is a linear combination of positive roots. We conclude the paper by using Littelmann's results on standard monomial theory [Li] to prove a theorem which provides a method of describing weight multiplicities of $V_X(\mathcal{L})$ in terms of weight multiplicities of certain Demazure modules (see Theorem 2).

This manuscript is organized as follows. In Section 2, we explain some preliminary results about ind-groups and ind-Schubert varieties. In Section 3 we study projectivity of ind-Schubert varieties. Section 4 is devoted to proving our results on weight multiplicities.

2. Preliminaries

2.1. Classical simple ind-groups

Let G be either $\mathrm{GL}(\infty)$ or a classical simple ind-group of type $B(\infty)$, $C(\infty)$ or $D(\infty)$. (For our purposes, there is no difference between the groups $\mathrm{GL}(\infty)$ and $A(\infty)$. It is more convenient to work with $\mathrm{GL}(\infty)$, and we will do so.) For the reader's convenience and future reference, we would like to describe explicit realizations of these ind-groups in more detail. We have $\mathrm{GL}(\infty)=\varinjlim \mathrm{GL}_n(\mathbb{C})$ where the monomorphism $f_n:\mathrm{GL}_n(\mathbb{C})\to\mathrm{GL}_{n+1}(\mathbb{C})$ is given by

$$X\mapsto\begin{bmatrix} X & 0\\ 0 & 1\end{bmatrix}.$$

The group $\mathrm{SO}_{2n}(\mathbb{C})$ (respectively $\mathrm{SO}_{2n+1}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$) is realized as the subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ (respectively $\mathrm{GL}_{2n+1}(\mathbb{C})$ and $\mathrm{GL}_{2n}(\mathbb{C})$) of elements of determinant one which preserve a nondegenerate bilinear form $\langle u,v\rangle$ on $\mathbb{C}^{2n}$ (respectively on $\mathbb{C}^{2n+1}$ and $\mathbb{C}^{2n}$) where

$$\langle u,v\rangle=\begin{cases}\sum_{i=1}^{n} u_{2i-1}v_{2i}+u_{2i}v_{2i-1} & \text{for } \mathrm{SO}_{2n}(\mathbb{C}),\\ u_1v_1+\sum_{i=1}^{n} u_{2i+1}v_{2i}+u_{2i}v_{2i+1} & \text{for } \mathrm{SO}_{2n+1}(\mathbb{C}),\\ \sum_{i=1}^{n} u_{2i-1}v_{2i}-u_{2i}v_{2i-1} & \text{for } \mathrm{Sp}_{2n}(\mathbb{C}).\end{cases}$$

The group $B(\infty)$ (respectively $C(\infty)$ and $D(\infty)$) is defined as $\varinjlim G_n$ where $G_n=\mathrm{SO}_{2n+1}(\mathbb{C})$ (respectively $G_n=\mathrm{Sp}_{2n}(\mathbb{C})$ and $G_n=\mathrm{SO}_{2n}(\mathbb{C})$) and the monomorphism $G_n\to G_{n+1}$ is given by $f_{k+1}\circ f_k$ for $k=2n+1$ (respectively $k=2n$ and $k=2n$). Note that $B(\infty)$ and $D(\infty)$ are isomorphic groups, but the isomorphism does not identify the root systems.

2.2. Root systems and special subgroups

From now on we assume $G=\varinjlim G_n$ and $i_n:G_n\to G_{n+1}$ are as in the standard realizations given above. As a root reductive ind-group [DPW], G has a standard Cartan subgroup $H=\varinjlim H_n$, where H_n is the subgroup of diagonal matrices in G_n. Associated to H there is a root system $\Delta=\varinjlim \Delta_n$ of G, which is a subset of the group of rational multiplicative characters of the ind-group H. We

Table 1

G	Δ	Description of ε_i's
$\mathrm{GL}(\infty)$	$\{\varepsilon_i - \varepsilon_j \mid i, j \in \mathbb{Z}^+, i \neq j\}$	$\varepsilon_k(\mathrm{diag}(t_1, t_2, t_3, \ldots)) = t_k$
$B(\infty)$	$\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \mathbb{Z}^+, i \neq j\}$	$\varepsilon_k(\mathrm{diag}(1, t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots)) = t_k$
$C(\infty)$	$\{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \mathbb{Z}^+, i \neq j\}$	$\varepsilon_k(\mathrm{diag}(t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots)) = t_k$
$D(\infty)$	$\{\pm\varepsilon_i \pm \varepsilon_j \colon\ i, j \in \mathbb{Z}^+\}$	$\varepsilon_k(\mathrm{diag}(t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots)) = t_k$

will use the explicit description of Δ given in Table 1 (note that $-\varepsilon_i(t) = \varepsilon_i(t^{-1})$). Every root $\alpha \in \Delta$ determines a reflection s_α of Δ in a standard way.

A Borel subgroup of G is an ind-subgroup B such that for every n, $B_n = B \cap G_n$ is a Borel subgroup of G_n. It can be seen that there exists a $\sigma \in \mathrm{Aut}(G)$ such that $H \subset \sigma(B)$. From now on we assume that $H \subset B$. In the following proposition, which follows immediately from [DP1, Proposition 3], a $\mathbb{Z}_2$-linear order on $\{\pm\varepsilon_1, \pm\varepsilon_2, \ldots\} \cup \{0\}$ means a linear order such that $x > y$ implies $-y > -x$.

Proposition 1. *There is a one-to-one correspondence between Borel subgroups B of G containing H and certain linearly ordered sets Ω as follows.*

a. $G = \mathrm{GL}(\infty)$: *linear orders on* $\{\varepsilon_1, \varepsilon_2, \ldots\}$.
b. $G = B(\infty)$ *or* $G = C(\infty)$: *$\mathbb{Z}_2$-linear orders on* $\{\pm\varepsilon_1, \pm\varepsilon_2, \ldots\} \cup \{0\}$.
c. $G = D(\infty)$: *$\mathbb{Z}_2$-linear orders on* $\{\pm\varepsilon_1, \pm\varepsilon_2, \ldots\} \cup \{0\}$ *such that a minimal positive element, if exists, belongs to* $\{\varepsilon_1, \varepsilon_2, \ldots\}$.

Let B be a Borel subgroup of G corresponding to an ordered set Ω. Set $\Omega^+ = \Omega$ when $G = \mathrm{GL}(\infty)$ and $\Omega^+ = \{x \mid x \in \Omega \text{ and } x > 0\}$ otherwise. Let $\Omega_n = \Omega^+ \cap \{\pm\varepsilon_1, \ldots, \pm\varepsilon_n\}$. The ordered set Ω_{n+1} is obtained from Ω_n by inserting either ε_{n+1} or $-\varepsilon_{n+1}$ between the kth and $k+1$th largest elements of Ω_n, where $k \in \{0, 1, \ldots, n\}$. (Obviously, $k = 0$ (respectively $k = n$) means that ε_{n+1} becomes the largest (respectively smallest) element of Ω_{n+1}.) The integer k will be called the *insertion location* of Ω_n.

For any subset S of Ω^+, set $H(S) = \bigcap_{\varepsilon_r \notin S} \ker \varepsilon_r$ and for any rational character ν of H written formally as $\nu = \sum_{r \in \mathbb{Z}^+} \nu_r \varepsilon_r$, let $\nu(S) = \nu_{|H(S)} = \sum_{r \in S} \nu_r \varepsilon_r$. Clearly when $S = \emptyset$ we have $H(S) = \{1\}$.

2.3. Weyl groups and Schubert varieties

Suppose we have

$$\Omega_n = \{c_{\sigma(1)}\varepsilon_{\sigma(1)} > \cdots > c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$$

where for any l, $c_l \in \{\pm 1\}$. Let $\mathcal{W}_n$ represent the Weyl group of G_n. When $G = \mathrm{GL}(\infty)$, we think of $\mathcal{W}_n$ as $\mathcal{S}_n$, the group of permutations of $\{\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)}\}$. When $G = B(\infty)$ or $G = C(\infty)$, we think of $\mathcal{W}_n$ as $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$ (the group acting by permutations and sign changes on $\{\pm c_{\sigma(1)}\varepsilon_{\sigma(1)}, \ldots, \pm c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$) and when $G = D(\infty)$, we think of $\mathcal{W}_n$ as the subgroup of $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$ acting on $\{\pm c_{\sigma(1)}\varepsilon_{\sigma(1)}, \ldots, \pm c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$ by permutations and sign changes where the number of negative signs is even (for more details, see [Hu, §12]).

The homomorphism i_n induces a closed immersion $\dot{i}_n : G_n/B_n \to G_{n+1}/B_{n+1}$ and one can consider the ind-variety $G/B = \varinjlim G_n/B_n$. For any $\omega \in \mathcal{W}_n$, the Schubert variety $\overline{B_n \omega B_n} \subseteq G_n/B_n$ will be denoted by X_ω. For any point $p \in G_n/B_n$, $\dot{i}_n(p)$ lies inside a Schubert cell in G_{n+1}/B_{n+1}. Consequently, for any $\omega \in \mathcal{W}_n$, there exists a unique $\omega' \in \mathcal{W}_{n+1}$ such that $\dot{i}_n(X_\omega) \subseteq X_{\omega'}$ and moreover if $\omega'' \in \mathcal{W}_{n+1}$ and $\dot{i}_n(X_\omega) \subseteq X_{\omega''}$ then $\omega' \preccurlyeq \omega''$. (Here as well as in the rest of this paper, $\preccurlyeq$ represents the Bruhat order on the Weyl group with respect to the chosen Borel subgroup.) In fact the map $\Omega_n \to \Omega_{n+1}$ gives rise to a monomorphism $\mathcal{W}_n \to \mathcal{W}_{n+1}$, and ω' is the image of ω under the latter map.

2.4. Ind-varieties and ind-Schubert varieties

A locally irreducible ind-variety X is a topological space with a filtration $X_1 \subseteq X_2 \subseteq \cdots$ such that each X_n is an (irreducible) variety and the maps $X_n \to X_{n+1}$ are closed immersions. (Note that the definition is not independent of the filtration.) The topology of X is the usual direct limit topology and its sheaf of regular functions is $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$. Our main example of a locally irreducible ind-variety is $G/B = \varinjlim G_n/B_n$.

A locally irreducible closed subvariety of X is a closed subset Y of X such that for every n, $Y \cap X_n$ is an (irreducible) variety. A line bundle on X is a sheaf of $\mathcal{O}_X$-modules locally isomorphic to $\mathcal{O}_X$. A line bundle $\mathcal{L}$ on X is called globally generated if $\mathcal{L}_{|X_n}$ is globally generated for every n. It is called very ample if for every n, $\mathcal{L}_{|X_n}$ is very ample and the map $\Gamma(X_{n+1}, \mathcal{L}_{|X_{n+1}}) \to \Gamma(X_n, \mathcal{L}_{|X_n})$ is surjective. An ind-variety X is called projective whenever it has a very ample line bundle.

For any character λ of B_n, let $\mathcal{L}_\lambda \in \mathrm{Pic}(G_n/B_n)$ be the sheaf of sections of

$$G_n \times_{B_n} \mathbb{C}_{-\lambda} = \left\{(g,t) \;\middle|\; g \in G,\ t \in \mathbb{C},\ \text{and } \forall b \in B_n\colon\ (gb,t) = \left(g, \lambda\left(b^{-1}\right)t\right)\right\}$$

considered as a (geometric) line bundle over G_n/B_n. Any $\mathcal{L} \in \mathrm{Pic}(G_n/B_n)$ is of the form $\mathcal{L}_\lambda$ for some λ. Since $G/B = \varinjlim G_n/B_n$, we have $\mathrm{Pic}(G/B) = \varprojlim \mathrm{Pic}(G_n/B_n)$ and it follows that any element of $\mathrm{Pic}(G/B)$ is of the form $\mathcal{L}_\lambda$ for a character λ of B.

It is well known [Br] that a line bundle $\mathcal{L}_\lambda \in \mathrm{Pic}(G_n/B_n)$ is globally generated (respectively, very ample) if and only if λ is dominant (respectively, regular dominant). By Borel–Weil theorem, $\Gamma(G_n/B_n, \mathcal{L}_\lambda) \neq \{0\}$ if and only if $\mathcal{L}_\lambda$ is globally generated, in which case $\Gamma(G_n/B_n, \mathcal{L}_\lambda)$ is isomorphic to an irreducible G_n-module with lowest weight $-\lambda$. Its vector space dual $\Gamma(G_n/B_n, \mathcal{L}_\lambda)^*$ is an irreducible G_n-module with highest weight λ. The highest weight vector is given by the evaluation functional at eB_n where $e \in G_n$ is the identity element.

By an *ind-Schubert variety* in $G/B = \varinjlim G_n/B_n$ we mean a B-stable locally irreducible closed ind-subvariety X of G/B. It follows that $X = \varinjlim X_n$ where each $X_n \subseteq G_n/B_n$ is a Schubert variety.

From now on we assume that $X = \varinjlim X_n$ is an ind-Schubert variety where $X_n = X_{\omega_n} = \overline{B_n \omega_n B_n}$ for some $\omega_n \in \mathcal{W}_n$. It follows from Section 2.3 that $\omega'_n \preccurlyeq \omega_{n+1}$.

2.5. The module $V_X(\mathcal{L})$

For any $\mathcal{L} \in \mathrm{Pic}(G_n/B_n)$, the space $\Gamma(X, \mathcal{L}_{|X_n})$ is a B_n-module. If $\mathcal{L} \in \mathrm{Pic}(X)$, then the natural restriction map $\Gamma(X_n, \mathcal{L}_{|X_n}) \to \Gamma(X_n, \mathcal{L}_{|X_{n-1}})$ is a B_n-module homomorphism. As in [DPW, Proposition 10.3], the Mittag–Leffler property of the sequence

$$\cdots \to \Gamma(X_n, \mathcal{L}_{|X_n}) \to \Gamma(X_{n-1}, \mathcal{L}_{|X_{n-1}}) \to \cdots \to \Gamma(X_1, \mathcal{L}_{|X_1}) \tag{2.5.1}$$

implies that $\Gamma(X, \mathcal{L}) = \varprojlim \Gamma(X_n, \mathcal{L}_{|X_n})$. Therefore $\Gamma(X, \mathcal{L})$ has a natural B-module structure.

Lemma 1. *Let $\mathcal{L} \in \mathrm{Pic}(X)$. If $\mathcal{L}_{|X_n}$ is globally generated for every n, then the restriction map $\Gamma(X_n, \mathcal{L}_{|X_n}) \to \Gamma(X_{n-1}, \mathcal{L}_{|X_{n-1}})$ is surjective. In particular, if $\mathcal{L}_{|X_n}$ is very ample for every n, then $\mathcal{L}$ is a very ample line bundle of X as well.*

Proof. For every n, there exists an $\mathcal{L}'_n \in \mathrm{Pic}(G_n/B_n)$ such that $\mathcal{L}'_n$ is globally generated and $\mathcal{L}'_{n|X_n} = \mathcal{L}_{|X_n}$ (see [Ma] or [Br]). In the commutative diagram

$$\begin{array}{ccc} \Gamma(X_{n-1}, \mathcal{L}_{|X_{n-1}}) & \longrightarrow & \Gamma(X_n, \mathcal{L}_{|X_n})^* \\ \downarrow & & \downarrow \\ \Gamma(G_{n-1}/B_{n-1}, \mathcal{L}'_{n|X_{n-1}}) & \longrightarrow & \Gamma(G_n/B_n, \mathcal{L}'_n)^* \end{array}$$

the lower horizontal map is nonzero and by Borel–Weil theorem, it is one-to-one. By Demazure's theorem, the vertical maps are one-to-one too. Therefore the upper horizontal map has to be one-to-one. □

Let $\mathcal{L} \in \mathrm{Pic}(X)$ be globally generated. From Lemma 1 it follows that $\Gamma(X, \mathcal{L})$ is naturally dual to the B-module $V_X(\mathcal{L}) = \varinjlim \Gamma(X_n, \mathcal{L}_{|X_n})^*$.

Example. Any Schubert variety in G_n/B_n has an open cell isomorphic to an affine space. Unlike Schubert varieties, an ind-Schubert variety may not contain an "open affine cell." Let ω_{4n} be the permutation of $\{1, \ldots, 4n\}$ such that for any $k \in \{0, \ldots, n-1\}$ and $s \in \{1, 2, 3, 4\}$ we have $\omega_{4n}(4k+s) = 4k + \sigma(s)$ where $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$ and $\sigma(4) = 2$. Consider $X_{\omega_{4n}} \subset \mathrm{GL}_{4n}(\mathbb{C})/B_{4n}$ where B_{4n} corresponds to the positive system $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{4n-1} - \varepsilon_{4n}\}$. It is not hard to see that $X_{\omega_{4n}}$ is a subvariety of $X_{\omega_{4(n+1)}}$. However, it is impossible to find an open subset U of $X = \varinjlim X_{\omega_{4n}}$ such that $U \cap X_{\omega_{4k}}$ is an affine space for every k. In fact by the criterion of Lakshmibai and Seshadri [LS] it follows that $X_{\omega_{4n}}$ is inside the singular locus of $X_{\omega_{4(n+1)}}$. Since $U \cap X_{\omega_{4(n+1)}}$ is an affine space, it is smooth and therefore it does not intersect this singular locus; hence $U \cap X_{\omega_{4n}} = \emptyset$, a contradiction.

3. Line bundles and projectivity of X

3.1. A characterization of line bundles on X

The map $i_n : X_n \to X_{n+1}$ induces a group homomorphism $i_n^* : \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ between the Picard groups of these varieties in a standard way, and we have $\mathrm{Pic}(X) = \varprojlim \mathrm{Pic}(X_n)$.

Proposition 2. *Any line bundle on X is the restriction of a line bundle on G/B.*

Proof. By the definition of inverse limit, the maps $\mathrm{Pic}(G/B) \to \mathrm{Pic}(G_n/B_n) \to \mathrm{Pic}(X_n)$ yield a canonical map $\mathrm{Pic}(G/B) \to \mathrm{Pic}(X)$ and we need to show that it is surjective.

For simplicity we assume $G = \mathrm{GL}(\infty)$. The proof for other cases is quite similar. Let $\Omega_n = \{\varepsilon_{\sigma(1)} > \cdots > \varepsilon_{\sigma(n)}\}$. It is well known that $\mathrm{Pic}(G_n/B_n) = \mathbb{Z}^n/\langle(1, \ldots, 1)\rangle$ where (the coset of) any element $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ corresponds to the line bundle on G_n/B_n associated to the character $a_1\varepsilon_{\sigma(1)} + \cdots + a_n\varepsilon_{\sigma(n)}$ of B_n. If the insertion location of Ω_n is k, then the map

$$i_n^* : \mathrm{Pic}(G_{n+1}/B_{n+1}) \to \mathrm{Pic}(G_n/B_n)$$

is given by $i_n^*(a_1, \ldots, a_{n+1}) = (a_1, \ldots, a_k, \widehat{a_{k+1}}, \ldots, a_{n+1})$. Let $j_n : X_n \to G_n/B_n$ be the usual embedding. It is known (see [Ma] or [Br]) that the natural map

$$j_n^* : \mathrm{Pic}(G_n/B_n) \to \mathrm{Pic}(X_n) \tag{3.1.1}$$

is surjective and

$$\ker j_n^* = \big\{(a_1, \ldots, a_{n+1}) \colon\ a_i = a_{i+1} \text{ whenever } s_{\varepsilon_{\sigma(i)} - \varepsilon_{\sigma(i+1)}} \preccurlyeq \omega_n\big\}. \tag{3.1.2}$$

For any positive integers $p > q$, let $i_{p\to q}^*$ denote the map

$$i_q^* \circ i_{q+1}^* \circ \cdots \circ i_{p-2}^* \circ i_{p-1}^* : \mathrm{Pic}(G_p/B_p) \to \mathrm{Pic}(G_q/B_q)$$

and let $K_{p\to q} = i_{p\to q}^*(\ker j_p^*)$. Obviously $K_{q+1\to q} \supseteq K_{q+2\to q} \supseteq K_{q+3\to q} \supseteq \cdots$. Since $\ker j_p^*$ is given by equalities of coordinates, the same is true for any $K_{p\to q}$. This implies that the set $\{K_{q+j\to q} \colon\ j \in \mathbb{Z}^+\}$ is finite. Set $K_{\infty\to q} = \bigcap_{j\in\mathbb{Z}^+} K_{q+j\to q}$.

To complete the proof it suffices to show that for any infinite sequence

$$\cdots \mapsto m_n \mapsto m_{n-1} \mapsto \cdots \mapsto m_2 \mapsto m_1$$

(where $m_n \in \mathrm{Pic}(X_n)$) there exists a sequence

$$\cdots \mapsto m'_n \mapsto m'_{n-1} \mapsto \cdots \mapsto m'_2 \mapsto m'_1$$

where $m'_n \in \mathrm{Pic}(G_n/B_n)$ and $j_n^*(m'_n) = m_n$. Fix $n_\circ \in \mathbb{Z}^+$, and for any $n > n_\circ$ choose an element $m''_n \in (j_n^*)^{-1}(m_n)$. The set of elements of $\mathrm{Pic}(G_n/B_n)$ which map to m_n is equal to the coset $m''_n + \ker j_n^*$. For $n \gg n_\circ$ we have $i_{n\to n_\circ}^*(m''_n + \ker j_n^*) = i_{n\to n_\circ}^*(m''_n) + K_{\infty\to n_\circ}$. Any two cosets of $K_{\infty\to n_\circ}$ are either identical or disjoint. Surjectivity of j_{n+1}^* implies that the cosets $i_{n\to n_\circ}^*(m''_n) + K_{\infty\to n_\circ}$ and $i_{n+1\to n_\circ}^*(m''_{n+1}) + K_{\infty\to n_\circ}$ are not disjoint. Therefore there exists an element $\overline{m}_{n_\circ} \in \mathrm{Pic}(G_{n_\circ}/B_{n_\circ})$ such that for every $n \gg n_\circ$,

$$i_{n\to n_\circ}^*\big(m''_n\big) + K_{\infty\to n_\circ} = \overline{m}_{n_\circ} + K_{\infty\to n_\circ}.$$

It is easily checked that $j_{n_\circ}^*(\overline{m}_{n_\circ} + K_{\infty\to n_\circ}) = m_{n_\circ}$. Moreover,

$$i_{n_\circ+1,n_\circ}^*(\overline{m}_{n_\circ+1} + K_{\infty\to n_\circ+1}) = \overline{m}_{n_\circ} + K_{\infty\to n_\circ}.$$

It is now clear how to obtain the required sequence of m'_n's. □

3.2. A combinatorial criterion for projectivity of X

The main purpose of this section is to give a combinatorial criterion for the projectivity of any ind-Schubert variety X. This criterion will be used in the proof of Theorem 1 below. To state our criterion, we need to introduce a combinatorial invariant of X, which we call the connectivity diagram, that is essentially an equivalence relation $\sim$ on Ω^+. To make our discussion more readable, we first define the connectivity diagram for an ind-Schubert variety of $\mathrm{GL}(\infty)/B$, state its main properties, and then show how its definition can be generalized to the remaining cases.

Suppose $G = \mathrm{GL}(\infty)$. Fix $n \in \mathbb{Z}^+$ and let $\Omega_n = \{\varepsilon_{\sigma(1)} > \cdots > \varepsilon_{\sigma(n)}\}$. We can define the following connectivity diagram for Ω_n: Imagine that the elements of Ω_n are nodes, and connect any two consecutive nodes $\varepsilon_{\sigma(r)} > \varepsilon_{\sigma(r+1)}$ by an edge if and only if $s_\alpha \preccurlyeq \omega_n$ where s_α is the reflection corresponding to the root $\alpha = \varepsilon_{\sigma(r)} - \varepsilon_{\sigma(r+1)}$. This graph is called the connectivity diagram of Ω_n.

The purpose of the connectivity diagram is to store information about the kernel of the map $j_n^* : \mathrm{Pic}(G_n/B_n) \to \mathrm{Pic}(X_n)$ given in (3.1.1). In fact $\varepsilon_{\sigma(m)}$ and $\varepsilon_{\sigma(m+1)}$ are connected by an edge if and only if for any $a = (a_1, \ldots, a_n) \in \ker j_n^*$, we have $a_m = a_{m+1}$.

Using the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(G_{n+1}/B_{n+1}) & \xrightarrow{i_n^*} & \mathrm{Pic}(G_n/B_n) \\ \Big\downarrow{\scriptstyle j_{n+1}^*} & & \Big\downarrow{\scriptstyle j_n^*} \\ \mathrm{Pic}(X_{n+1}) & \xrightarrow{i_n^*} & \mathrm{Pic}(X_n) \end{array}$$

it can be seen that the connectivity diagrams of Ω_n and Ω_{n+1} are compatible. In other words, if ε_i and ε_j belong to the same connected component for the connectivity diagram of Ω_n, then they also belong to the same connected component of the connectivity diagram of Ω_m for any $m > n$. Therefore we have a well-defined connectivity diagram for Ω^+. It is merely the equivalence relation $\sim$ on Ω^+ where $\varepsilon_p \sim \varepsilon_q$ if and only if there exists an n such that ε_p and ε_q are in the same connected component of the connectivity diagram of Ω_n.

When G is $B(\infty)$, $C(\infty)$ or $D(\infty)$, we use the same idea: Suppose

$$\Omega_n = \{c_{\sigma(1)}\varepsilon_{\sigma(1)} > \cdots > c_{\sigma(n)}\varepsilon_{\sigma(n)}\}$$

where the c_r's are in $\{\pm 1\}$. We connect two consecutive nodes $c_{\sigma(m)}\varepsilon_{\sigma(m)} > c_{\sigma(m+1)}\varepsilon_{\sigma(m+1)}$ by an edge whenever $s_\alpha \preccurlyeq \omega_n$, where $\alpha \in \Delta_n$ is a simple positive root of the form $\alpha = c_{\sigma(m)}\varepsilon_{\sigma(m)} \pm c_{\sigma(m+1)}\varepsilon_{\sigma(m+1)}$. (Note that $\alpha = c_{\sigma(m)}\varepsilon_{\sigma(m)} + c_{\sigma(m+1)}\varepsilon_{\sigma(m+1)}$ happens only when $G = D(\infty)$.)

Definition 1. Let $n \in \mathbb{Z}^+$. When $G = B(\infty)$ (respectively $C(\infty)$ and $D(\infty)$), we call the positive simple root $\alpha \in \Delta_n$ special if $\alpha = c_{\sigma(n)}\varepsilon_{\sigma(n)}$ (respectively $\alpha = 2c_{\sigma(n)}\varepsilon_{\sigma(n)}$ and $c_{\sigma(n-1)}\varepsilon_{\sigma(n-1)} + c_{\sigma(n)}\varepsilon_{\sigma(n)}$).

We call a connected component C of the connectivity diagram of Ω_n special if for the special simple root $\alpha \in \Delta_n$, we have $s_\alpha \preccurlyeq \omega_n$ and moreover C contains $c_{\sigma(n)}\varepsilon_{\sigma(n)}$.

Obviously, the connectivity diagram of Ω_n has at most one special connected component. Again one can see that connected components (and special connected components, if they exist) of Ω_n's are compatible. Therefore we obtain an equivalence relation $\sim$ on Ω^+ (and there may be at most one special equivalence class).

Proposition 3. *X is projective if and only if every equivalence class of $\sim$ is isomorphic (as an ordered set) to a subset of $\mathbb{Z}$, and the special equivalence class, if exists, is isomorphic to a subset of $\mathbb{Z}^+$.*

Proof. We give the details of the proof for the case $G = \mathrm{GL}(\infty)$. For other cases the proof is quite similar and is left to the reader. By Proposition 2, we can start with an arbitrary line bundle $\mathcal{L} = \mathcal{L}_\lambda$ (associated to a rational character $\lambda = \sum_{m\in\mathbb{Z}^+} \lambda_m\varepsilon_m$ of B) on G/B to see when $\mathcal{L}_{|X}$ is very ample. (Note that the λ_m's are integers.) Suppose $\mathcal{L}_n = \mathcal{L}_{|X_n}$ is very ample. It is known (see [Ma] or [Br]) that any very ample line bundle on X_n is the restriction of a very ample line bundle on G_n/B_n. In other words, there exists an $\mathcal{L}'_n \in \mathrm{Pic}(G_n/B_n)$ corresponding to $[(a_1,\ldots,a_n)] \in \mathbb{Z}^n/\langle(1,\ldots,1)\rangle$ such that $a_1 > \cdots > a_n$ and $j_n^*(\mathcal{L}'_n) = \mathcal{L}_n$. From the description of $\ker j_n^*$ given in (3.1.2) we can conclude the following statement: for any line bundle $\mathcal{L}''_n \in \mathrm{Pic}(G_n/B_n)$, corresponding to $[(b_1,\ldots,b_n)] \in \mathbb{Z}^n/\langle(1,\ldots,1)\rangle$, if $j_n^*(\mathcal{L}''_n) = \mathcal{L}_n$ then the b_r's corresponding to any connected component of the connectivity diagram of Ω_n form a strictly decreasing sequence. Since this statement holds for every n, it follows that the λ_m's corresponding to every equivalence class of $\sim$ in Ω^+ form a strictly decreasing sequence. It follows that the equivalence class is isomorphic to a subset of $\mathbb{Z}$.

Conversely, if every equivalence class of $\sim$ in Ω^+ is isomorphic to a subset of $\mathbb{Z}$, then one can find integers λ_m such that the λ_m's corresponding to the ε_m's which belong to any equivalence class of $\sim$ form a strictly decreasing sequence. The line bundle on G/B associated to $\sum_{m=1}^{\infty} \lambda_m\varepsilon_m$ restricts to a very ample line bundle on X. □

4. Main results

4.1. Finiteness of H-multiplicities of $V_X(\mathcal{L})$

Before we state and prove our theorems, we need to introduce some notation. Let $\mathcal{I}$ be an index set and $\{C^i\}_{i\in\mathcal{I}}$ be the family of equivalence classes of the connectivity diagram of Ω^+. Fix $n \in \mathbb{Z}^+$ and set $C_{n,i} = \Omega_n \cap C^i$. The partition of Ω_n by $C_{n,i}$'s corresponds to a unique standard parabolic subgroup P_n of G_n as follows: Set

$$\Sigma_{n,i} = \left\{\alpha \in \Delta_n \;\middle|\; \alpha \text{ is a simple positive root, } \alpha \in \mathrm{Span}_{\mathbb{R}}(C_{n,i}) \text{ and } s_\alpha \preccurlyeq \omega_n\right\}$$

and suppose $\langle\Sigma_{n,i}\rangle$ denotes the additive semigroup generated by $\Sigma_{n,i}$. Suppose $U_{n,i}$ is the unipotent subgroup of G_n which corresponds to $\langle\Sigma_{n,i}\rangle \cap \Delta_n$ in the standard way, and set $B_{n,i} = H_{n,i} \ltimes U_{n,i}$ where $H_{n,i} = H(C_{n,i})$. The group generated by $B_{n,i}$ and the reflections s_α for $\alpha \in \Sigma_{n,i}$ is a reductive subgroup of G_n which we will denote by $G_{n,i}$. The parabolic P_n is the group generated by B_n and

the reflections s_α for $\alpha \in \bigcup_{i\in\mathcal{I}} \Sigma_{n,i}$. The standard Levi subgroup of P_n is isomorphic to the direct product $\prod_{i\in\mathcal{I}} G_{n,i}$. We think of $G_{n,i}/B_{n,i}$ as well as $\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}$ as a Schubert variety embedded inside G_n/B_n in the obvious way. Note that for all but finitely many $i \in \mathcal{I}$, $C_{n,i} = \emptyset$ and consequently $G_{n,i}/B_{n,i}$ is just a point. Let $\mathcal{W}_{n,i}$ denote the Weyl group of $G_{n,i}$, i.e., the group generated by s_α where $\alpha \in \Sigma_{n,i}$.

The compatibility of connectivity diagrams for Ω_n's implies that $i_n(P_n) \subset P_{n+1}$, and therefore $P = \varinjlim P_n$ is a standard parabolic subgroup of G. In fact P is the unique standard parabolic subgroup of G corresponding to the partition of Ω^+ by C^i's. The standard Levi factor of P is isomorphic to $\prod_{i\in\mathcal{I}} G^i$ where for every $i \in \mathcal{I}$, $G^i = \varinjlim G_{n,i}$. Set $B^i = B \cap G^i$ and $H^i = H \cap G^i$.

Given any globally generated line bundle $\mathcal{L}$ on X, by Proposition 2 we can assume $\mathcal{L} = \mathcal{L}_{\lambda|X}$ where $\mathcal{L}_\lambda$ is the line bundle on G/B associated to the character $\lambda = \sum_{m\in\mathbb{Z}^+} \lambda_m \varepsilon_m$ of H. Note that for any n and i the weight $\lambda(C_{n,i})$ is dominant. Set $\lambda_{n,i} = \lambda(C_{n,i})$ and let $V(\lambda_{n,i})$ be the irreducible $B_{n,i}$-highest weight module for $G_{n,i}$ with highest weight $\lambda_{n,i}$. (If $C_{n,i} = \emptyset$ we take $V(\lambda_{n,i}) = \mathbb{C}$.) For any $i \in \mathcal{I}$, let $V(\lambda^i)$ be the irreducible B^i-highest weight module of G^i with highest weight $\lambda^i = \lambda(C^i)$.

The tensor products which appear below are all *restricted* in the following sense: Fix an ordering $\mathcal{I} = \{i_1, i_2, \ldots\}$ of $\mathcal{I}$. Suppose $\{M_i\}_{i\in\mathcal{I}}$ is a family of complex vector spaces, each of which possessing a distinguished nonzero vector $m_i \in M_i$. Then $\bigotimes_{i\in\mathcal{I}} M_i$ is defined as $\varinjlim \bigotimes_{i\in\{i_1,\ldots,i_n\}} M_i$, where the map $\bigotimes_{i\in\{i_1,\ldots,i_n\}} M_i \to \bigotimes_{i\in\{i_1,\ldots,i_{n+1}\}} M_i$ is given by $w \mapsto w \otimes m_{i_{n+1}}$. In the modules which will appear below, the distinguished vectors are determined (up to a scalar) as follows: either $M_i = \mathbb{C}$, where we can take any nonzero vector, or M_i is a weight module (for an appropriate subgroup of H) with a unique (up to scaling) highest weight vector. Formally, $\bigotimes_{i\in\mathcal{I}} M_i$ is spanned by $\bigotimes_{i\in\mathcal{I}} v_i$ where for all but finitely many i, $v_i = m_i$.

Theorem 1. *Let $X = \varinjlim X_n$ be a projective ind-Schubert variety in G/B. If $\mathcal{L} = \mathcal{L}_{\lambda|X}$ is a globally generated line bundle on X, then the H-weight spaces of $V_X(\mathcal{L})$ are finite-dimensional.*

Proof. For simplicity, assume $G = \mathrm{GL}(\infty)$. The argument is easily adapted to other cases. Since $\mathcal{L}$ is globally generated, it follows that for any integers $m, m' \in \{1, \ldots, n\}$, if $\varepsilon_m \sim \varepsilon_{m'}$ and in Ω_n we have $\varepsilon_m > \varepsilon_{m'}$, then $\lambda_m \geqslant \lambda_{m'}$. With a slight abuse of notation, let $\mathcal{L}_n$ denote both $\mathcal{L}_{|X_n}$ and $\mathcal{L}_{|\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}}$.

Recall that $X_n = X_{\omega_n}$ where $\omega_n \in \mathcal{W}_n$. It is easily seen that ω_n is expressible as $\omega_n = \prod_{i\in\mathcal{I}} \omega_{n,i}$ where $\omega_{n,i} \in \mathcal{W}_{n,i}$. It follows that X_n is a subvariety of $\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}$ of the form $\prod_{i\in\mathcal{I}} X_{n,i}$ where each $X_{n,i}$ is a Schubert variety inside $G_{n,i}/B_{n,i}$ corresponding to $\omega_{n,i}$. From Demazure's theorem it follows that the natural map

$$\Gamma\left(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n\right) \to \Gamma(X_n, \mathcal{L}_n)$$

is surjective, and therefore its dual map is a monomorphism of B_n-modules. But Borel–Weil theorem implies that as $\prod_{i\in\mathcal{I}} G_{n,i}$-modules

$$\Gamma\left(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n\right)^* \cong \bigotimes_{i\in\mathcal{I}} V(\lambda_{n,i}). \tag{4.1.1}$$

We also have a commutative diagram

$$\begin{array}{ccc} \Gamma(X_n, \mathcal{L}_n)^* & \longrightarrow & \Gamma(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n)^* \\ \downarrow & & \downarrow \\ \Gamma(X_{n+1}, \mathcal{L}_{n+1})^* & \longrightarrow & \Gamma(\prod_{i\in\mathcal{I}} G_{n+1,i}/B_{n+1,i}, \mathcal{L}_{n+1})^* \end{array}$$

in which all of the maps are one-to-one. Therefore the induced map

$$\varinjlim \Gamma(X_n, \mathcal{L}_n)^* \to \varinjlim \Gamma\left(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n\right)^* \tag{4.1.2}$$

is one-to-one.

Choose any H_n-weight vector v_μ in $\Gamma(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n)^*$ of weight

$$\mu = \sum_{r=1}^{n} \mu_r \varepsilon_r.$$

Since the $\prod_{i\in\mathcal{I}} B_{n,i}$-highest weight vector of $\Gamma(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n)^*$ is the functional which evaluates global sections at the base point $\prod_{i\in\mathcal{I}} e_{n,i}B_{n,i}$ (where $e_{n,i}$ is the identity element of $G_{n,i}$), a simple argument shows that the image of v_μ under the map

$$\Gamma\left(\prod_{i\in\mathcal{I}} G_{n,i}/B_{n,i}, \mathcal{L}_n\right)^* \to \Gamma\left(\prod_{i\in\mathcal{I}} G_{n+1,i}/B_{n+1,i}, \mathcal{L}_{n+1}\right)^*$$

has H_{n+1}-weight $\mu + \lambda_{n+1}\varepsilon_{n+1}$. From this statement, (4.1.1), and injectivity of (4.1.2) it follows that $\varinjlim \Gamma(\prod_{i\in\mathcal{I}} X_{n,i}, \mathcal{L}_n)^*$ is a B-submodule of $\bigotimes_{i\in\mathcal{I}} V(\lambda^i)$. Since X is projective, Proposition 3 implies that for every $i \in \mathcal{I}$, the Borel subgroup B^i has a basis (for the definition of a basis, see [DP1, Section 4]). From [DP1, Theorem 5] it follows that H^i-weight spaces of $V(\lambda^i)$ are finite-dimensional. Therefore the H-weight spaces of $V_X(\mathcal{L})$ are finite-dimensional as well. □

Remark. From the proof of Theorem 1 it follows easily that a weight space $V_X(\mathcal{L})_\mu$ is nonzero only if $\lambda - \mu$ is a sum of positive roots in Δ.

4.2. Demazure modules and H-multiplicities

We start with some notation. Let S be a finite subset of C^i. Choose n such that $S \subset \Omega_n$, S lies in a connected component of the connectivity diagram of Ω_n, and if C^i is special then the connected component of Ω_n containing S is special as well. Let $U(S)$ be the unipotent subgroup of G_n which corresponds to $\langle \Sigma_{n,i}\rangle \cap \Delta_n \cap \mathrm{Span}_{\mathbb{R}}(S)$ in the standard way, and set $B(S) = H(S) \ltimes U(S)$. Let $\mathcal{W}(S)$ be the group generated by all reflections s_α where α is a simple positive root of Δ_n such that $\alpha \in \langle \Sigma_{n,i}\rangle \cap \mathrm{Span}_{\mathbb{R}}(S)$, and let $G(S)$ be the subgroup of G_n generated by $B(S)$ and $\mathcal{W}(S)$. It is easy to see that if n is sufficiently large then the notation is independent of n.

Let $\mathcal{L} = \mathcal{L}_{\lambda|X}$ be globally generated. Suppose $\mathcal{E}$ is the infinite-dimensional lattice $\mathrm{Span}_{\mathbb{Z}}(\{\varepsilon_r\}_{r=1}^{\infty})$. A natural question is to find the dimension of the weight spaces $V_X(\mathcal{L})_\mu$ for any $\mu \in \lambda + \mathcal{E}$. (As mentioned in the above remark, the weights μ for which $V_X(\mathcal{L})_\mu \neq \{0\}$ lie in a cone in $\lambda + \mathcal{E}$ that is based at λ.) It is reasonable to expect a connection between the dimensions of $V_X(\mathcal{L})_\mu$, when μ varies within finite-dimensional sublattices $\lambda + \mathcal{E}_n$ of $\lambda + \mathcal{E}$, and characters of finite-dimensional Demazure modules. The main goal of this section is to prove Theorem 2, which establishes such a connection.

Theorem 2. *Let $X = \varinjlim X_n$ be projective and $\mathcal{L} = \mathcal{L}_{\lambda|X}$ be a globally generated line bundle on X. Then there exists a sequence $S_1 \subseteq S_2 \subseteq \cdots$ of finite subsets of Ω^+ such that $\Omega^+ = \bigcup_{m\in\mathbb{Z}^+} S_m$ and*

a. *If $V(m)$ represents the $\prod_{i\in\mathcal{I}} G(S_m \cap C^i)$-highest weight module*

$$\bigotimes_{i\in\mathcal{I}} V\left(\lambda\left(S_m \cap C^i\right)\right)$$

then $V_X(\mathcal{L})$ is isomorphic to a B-submodule of $\varinjlim V(m)$. Here the map $V(m) \to V(m+1)$ is $\bigotimes_{i\in\mathcal{I}} \phi_{m,i}$, where

$$\phi_{m,i} : V\big(\lambda\big(S_m \cap C^i\big)\big) \to V\big(\lambda\big(S_{m+1} \cap C^i\big)\big)$$

identifies a $B(S_m \cap C^i)$-highest weight vector inside $V(\lambda(S_m \cap C^i))$ with a $B(S_{m+1} \cap C^i)$-highest weight vector of $V(\lambda(S_{m+1} \cap C^i))$.

b. *If we identify $V_X(\mathcal{L})$ with its image in $\varinjlim V(m)$, then for any $m \in \mathbb{Z}^+$ there exist elements $\sigma_{m,i} \in \mathcal{W}(S_m \cap C^i)$ such that*

$$V_X(\mathcal{L}) \cap V(m) = \bigotimes_{i\in\mathcal{I}} V_{\sigma_{m,i}}\big(\lambda\big(S_m \cap C^i\big)\big).$$

Here $V_{\sigma_{m,i}}(\lambda(S_m \cap C^i))$ denotes the $B(S_m \cap C^i)$-module generated by the extremal weight vector of $V(\lambda(S_m \cap C^i))$ with weight $\sigma_{m,i} \cdot \lambda(S_m \cap C^i)$.

Remark. From the proof of Theorem 2 it follows that

$$V_X(\mathcal{L}) \cap V(m) = \bigoplus_{\mu\in\lambda+\mathcal{E}_m} V_X(\mathcal{L})_\mu$$

where $\mathcal{E}_m$ is the finite-dimensional sublattice of $\mathcal{E}$ generated by $\{\varepsilon_r \mid r \in S_m\}$.

Proof. For simplicity we give the proof for $G = \mathrm{GL}(\infty)$ and leave the necessary modifications for other cases to the reader. From the proof of Theorem 1 it follows that when $X = \varinjlim X_n$ is projective, we have $X_n = \prod_{i\in\mathcal{I}} X_{n,i}$ where each $X_{n,i}$ is a Schubert variety inside $G_{n,i}/B_{n,i}$. Therefore as $\prod_{i\in\mathcal{I}} B_{n,i}$-modules

$$\Gamma(X_n, \mathcal{L}_n)^* \cong \bigotimes_{i\in\mathcal{I}} V_{\omega_{n,i}}(\lambda_{n,i}) \tag{4.2.1}$$

where $V_{\omega_{n,i}}(\lambda_{n,i})$ denotes the $B_{n,i}$-submodule of $V(\lambda_{n,i})$ generated by the extremal weight vector with weight $\omega_{n,i}\cdot\lambda_{n,i}$. Since X is projective, we have $\Omega = \bigcup_{i\in\mathcal{I}} C^i$ where for any $i \in \mathcal{I}$ there exist a_i, b_i satisfying $a_i < b_i$, $a_i \in \mathbb{Z} \cup \{-\infty\}$, and $b_i \in \mathbb{Z} \cup \{+\infty\}$ such that $C^i = \{\varepsilon_{r_{m,i}} \mid a_i < m < b_i\}$ and for any $a_i < m < m' < b_i$ we have $\varepsilon_{r_{m',i}} > \varepsilon_{r_{m,i}}$.

For any $p \in \mathbb{Z}^+$, set

$$C^{p,i} = \big\{\varepsilon_r \,\big|\, \varepsilon_r \in C^i \text{ and } r = r_{m,i} \text{ where } \max\{-p, a_i\} < m < \min\{b_i, p\}\big\}$$

and set $G^{p,i} = G(C^{p,i})$, $\lambda^{p,i} = \lambda(C^{p,i})$. Let $V(\lambda^{p,i})$ be the $B \cap G^{p,i}$-highest weight module of $G^{p,i}$ with highest weight $\lambda^{p,i}$.

Fix $q \in \mathbb{Z}^+$, choose any $i_1, \ldots, i_q \in \mathcal{I}$, and let $p \in \mathbb{Z}^+$ be large enough such that for any $j \in \{1, \ldots, q\}$, $C^{p,i_j} \neq \emptyset$. set $W = \bigotimes_{m=1}^{q} V(\lambda^{p,i_m})$. Choose $n_\circ \in \mathbb{Z}^+$ large enough such that $\Omega_{n_\circ} \supset C^{p,i_1} \cup \cdots \cup C^{p,i_q}$ and for any $i \in \{i_1, \ldots, i_q\}$, elements of $C^{p,i}$ lie in a connected component of the connectivity diagram of Ω_n.

By mapping the highest weight vector of W (as a $G^{p,i_1} \times \cdots \times G^{p,i_q}$-module) to the highest weight vector of $\bigotimes_{i\in\mathcal{I}} V(\lambda_{n_\circ,i})$, W can be identified with a $G^{p,i_1} \times \cdots \times G^{p,i_q}$-submodule of $\bigotimes_{i\in\mathcal{I}} V(\lambda_{n_\circ,i})$. (The image of W in $\bigotimes_{i\in\mathcal{I}} V(\lambda_{n_\circ,i})$ is spanned by a set of weight vectors of weights of the form $\mu = \sum_{\varepsilon_r\in\Omega_{n_\circ}} \mu_r\varepsilon_r$ where $\mu_r = \lambda_r$ whenever $\varepsilon_r \notin C^{p,i_1} \cup \cdots \cup C^{p,i_q}$.) At the same time, from (4.2.1) it

follows that $\Gamma(X_{n_\circ}, \mathcal{L}_{n_\circ})^*$ can be considered as a subspace of $\bigotimes_{i\in\mathcal{I}} V(\lambda_{n_\circ,i})$ in the obvious way. It is not difficult to see that $\Gamma(X_{n_\circ}, \mathcal{L}_{n_\circ})^* \cap W = \bigotimes_{i\in\mathcal{I}} W_i$ where

$$W_i = \begin{cases} V(\lambda^{p,i}) \cap V_{\omega_{n_\circ,i}}(\lambda_{n_\circ,i}) & \text{if } i \in \{i_1, \ldots, i_q\}, \\ V_e(\lambda_{n_\circ,i}) & \text{otherwise} \end{cases}$$

(here e represents the trivial element of the corresponding Weyl group). Next we prove that for any $1 \leqslant m \leqslant q$, there exists an element $\sigma_{n_\circ,p,i_m}$ in the Weyl group $\mathcal{W}(C^{p,i_m})$ of G^{p,i_m} such that

$$V\left(\lambda^{p,i_m}\right) \cap V_{\omega_{n_\circ,i_m}}(\lambda_{n_\circ,i_m}) = V_{\sigma_{n_\circ,p,i_m}}\left(\lambda^{p,i_m}\right). \tag{4.2.2}$$

Before we prove (4.2.2), we show how the proof of Theorem 2 can be completed. The sequence $\sigma_{n_\circ,p,i_m}$ satisfies $\sigma_{n_\circ,p,i_m} \preccurlyeq \sigma_{n_\circ+1,p,i_m} \preccurlyeq \cdots$ and therefore for $n_\circ$ sufficiently large it will be constant. Let us denote the stable value of the sequence by σ_{p,i_m}. It follows that $V_X(\mathcal{L}) \cap W = \bigotimes_{i\in\mathcal{I}} V_{\sigma_{p,i}}(\lambda^{p,i})$. To complete the proof of Theorem 2, we assume $\mathcal{I} = \{i_1, i_2, \ldots\}$ and for any $q \in \mathbb{Z}^+$ we take $S_q = C^{N_q,i_1} \cup \cdots \cup C^{N_q,i_q}$ where N_q is large enough such that for any $1 \leqslant m \leqslant q$, $C^{N_q,i_m} \neq \emptyset$.

The proof of (4.2.2) follows from Propositions 4 and 5 below, both of which are of independent interest. □

Proposition 4. *Let N be a positive integer and $X^1, X^2 \subseteq G_N/B_N$ be arbitrary Schubert varieties such that $X^1 \cap X^2$ is irreducible. Suppose $\widetilde{\mathcal{L}}$ is a globally generated line bundle on G_N/B_N. Then the intersections of the images of the canonical maps $\Gamma(X^1, \widetilde{\mathcal{L}})^* \to \Gamma(G_N/B_N, \widetilde{\mathcal{L}})^*$ and $\Gamma(X^2, \widetilde{\mathcal{L}})^* \to \Gamma(G_N/B_N, \widetilde{\mathcal{L}})^*$ is equal to the image of the canonical map $\Gamma(X^1 \cap X^2, \widetilde{\mathcal{L}})^* \to \Gamma(G_N/B_N, \widetilde{\mathcal{L}})^*$.*

Proof. Since we are not aware of a reference, we would like to sketch a proof. Let $\eta_1, \eta_2 \in \mathcal{W}_N$ be the elements corresponding to X^1, X^2. The statement of the proposition is equivalent to showing that in the commutative diagram

$$\begin{array}{ccc} \Gamma(G_N/B_N, \widetilde{\mathcal{L}}) & \xrightarrow{\phi_1} & \Gamma(X^1, \widetilde{\mathcal{L}}) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} \\ \Gamma(X^2, \widetilde{\mathcal{L}}) & \xrightarrow{\phi_4} & \Gamma(X^1 \cap X^2, \widetilde{\mathcal{L}}) \end{array}$$

we have $\ker\phi_4 \subseteq \phi_2(\ker\phi_1)$. From now on we use the notation of [Li] without defining it. By [Li, §5, Corollary 2], a basis for the kernel of ϕ_4 is given by p_π's such that $i(\pi) \preccurlyeq \eta_2$ but $i(\pi) \not\preccurlyeq \eta_1$, and [Li, §5, Corollary 2] immediately implies $\phi_1(p_\pi) = 0$. □

Proposition 5. *Let $n \in \mathbb{Z}^+$, $X_{\omega_n} \subset G_n/B_n$ be a Schubert variety, and Q_n be an arbitrary parabolic subgroup of G_n containing B_n. Then $X_{\omega_n} \cap Q_n/B_n$ is an irreducible Schubert variety.*

Proof. It is well known that there exists a one-parameter subgroup $\varphi : \mathbb{C}^\times \to H_n$ such that $Q_n = \{x \in G_n \mid \lim_{t\to 0} \varphi(t)x\varphi(t^{-1}) \text{exists}\}$. Let $\overline{B}_n$ be the opposite Borel and $Y = \overline{B}_n B_n/B_n$ be the open cell. Then Y is H_n-equivariantly isomorphic to the tangent space $T_e(G_n/B_n) = \text{Lie}(G_n)/\text{Lie}(B_n)$ (where e is the identity element of G_n), and therefore it has a natural H_n-module structure. Consequently, we have $Y = Y' \times Y''$ where $Y' = \{x \in Y \mid \forall t \in \mathbb{C}^\times\colon\ \varphi(t)x\varphi(t^{-1}) = x\}$ and $Y'' = \{x \in Y \mid \lim_{t\to\infty} \varphi(t)x\varphi(t^{-1}) = e\}$. Moreover, the map $\pi(x) = \lim_{t\to\infty} \varphi(t)x\varphi(t^{-1})$ is simply the projection morphism $Y \to Y'$.

Obviously $X_{\omega_n} \cap Y$ is irreducible, and therefore $\pi(X_{\omega_n} \cap Y)$ is irreducible. However, $\pi(X_{\omega_n} \cap Y) = X_{\omega_n} \cap Y \cap Q_n/B_n$. Every irreducible component of $X_{\omega_n} \cap Q_n/B_n$ contains eB_n/B_n and therefore intersects Y nontrivially. Therefore $X_{\omega_n} \cap Q_n/B_n$ must be irreducible as well. □

When $G = \mathrm{GL}(\infty)$, for any $i \in \mathcal{I}$ we have

$$C_{n_\circ,i} = \{\varepsilon_{u_1} > \cdots > \varepsilon_{u_x} > \varepsilon_{u_{x+1}} > \cdots > \varepsilon_{u_y} > \varepsilon_{u_{y+1}} > \cdots > \varepsilon_{u_z}\}$$

and $C^{p,i} = \{\varepsilon_{u_{x+1}} > \cdots > \varepsilon_{u_y}\}$. The proof of (4.2.2) follows from Proposition 4 when $\eta_1 = \omega_{n_\circ,i}$ and

$$\eta_2(t) = \begin{cases} t & \text{if } j \leqslant x \text{ or } j > y, \\ y + x + 1 - t & \text{otherwise.} \end{cases}$$

The irreducibility condition of Proposition 4 follows from Proposition 5.

Remark. At least when $G = \mathrm{GL}(\infty)$, it is possible to find a combinatorial description of $\sigma_{n_\circ,p,i_m}$ in terms of $\omega_{n_\circ,i_m}$. The permutation $\sigma_{n_\circ,p,i}$ of $C^{p,i}$ is the unique maximal permutation of $C_{n_\circ,i}$ satisfying $\sigma_{n_\circ,p,i} \preccurlyeq \omega_{n_\circ,i}$ and fixing $\varepsilon_{u_1}, \ldots, \varepsilon_{u_x}$ and $\varepsilon_{u_{y+1}}, \ldots, \varepsilon_{u_z}$. The combinatorial description of this permutation follows by a simple induction from the following lemma, whose proof is left to the reader. (We expect that a similar combinatorial description exists for cases other than $\mathrm{GL}(\infty)$. However, note that to modify the proof for cases other than $\mathrm{GL}(\infty)$, such a combinatorial description is not really necessary, and Proposition 5 would be sufficient.)

Lemma 2. *Let N be a positive integer and σ be an arbitrary element of the symmetric group $\mathcal{S}_N$. Then the permutation σ' for which $\sigma'(N) = N$ and*

$$\sigma'(t) = \begin{cases} \max\{\sigma(t+1), \ldots, \sigma(N)\} & \text{if } t < N \text{ and } \sigma(t) \in \{\sigma'(1), \ldots, \sigma'(t-1)\} \cup \{N\}, \\ \sigma(t) & \text{if } t < N \text{ and } \sigma(t) \notin \{\sigma'(1), \ldots, \sigma'(t-1)\} \cup \{N\} \end{cases}$$

has the following property: $\sigma' \preccurlyeq \sigma$, and for any $\sigma_1 \in \mathcal{S}_N$, $\sigma_1(N) = N$ and $\sigma_1 \preccurlyeq \sigma$ imply $\sigma_1 \preccurlyeq \sigma'$.

Similarly, the permutation σ'' defined by $\sigma''(1) = 1$ and

$$\sigma''(t) = \max\big(\{\sigma(1), \ldots, \sigma(t)\} - \{\sigma''(1), \ldots, \sigma''(t-1)\}\big) \quad \text{if } t > 1$$

has the following property: $\sigma'' \preccurlyeq \sigma$, and for any $\sigma_1 \in \mathcal{S}_N$, $\sigma_1(1) = 1$ and $\sigma_1 \preccurlyeq \sigma$ imply $\sigma_1 \preccurlyeq \sigma''$.

Acknowledgments

This work was completed while the author was a PIMS Postdoctoral Fellow at University of Alberta. The author thanks Ivan Dimitrov for insightful conversations and carefully reading this manuscript, and Jochen Kuttler for discussions which led to Proposition 5.

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