THE REFINED SOLUTION TO THE CAPELLI EIGENVALUE PROBLEM FOR $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ AND $\mathfrak{gl}(m|2n)$

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Abstract. Let $\mathfrak{g}$ be either the Lie superalgebra $\mathfrak{gl}(V) \oplus \mathfrak{gl}(V)$ where $V := \mathbb{C}^{m|n}$ or the Lie superalgebra $\mathfrak{gl}(V)$ where $V := \mathbb{C}^{m|2n}$. Furthermore, let W be the $\mathfrak{g}$-module defined by $W := V \otimes V^*$ in the former case and $W := \mathscr{S}^2(V)$ in the latter case. Associated to $(\mathfrak{g}, W)$ there exists a distinguished basis of *Capelli operators* $\{D^\lambda\}_{\lambda \in \Omega}$, naturally indexed by a set of hook partitions Ω, for the subalgebra of $\mathfrak{g}$-invariants in the superalgebra $\mathscr{P}\mathscr{D}(W)$ of superdifferential operators on W.

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. We compute eigenvalues of the D^λ on the irreducible $\mathfrak{g}$-submodules of $\mathscr{P}(W)$ and obtain them explicitly as the evaluation of the interpolation super Jack polynomials of Sergeev–Veselov at suitable affine functions of the $\mathfrak{b}$-highest weight. While the former case is straightforward, the latter is significantly more complex. This generalizes a result by Sahi, Salmasian and Serganova for these cases, where such formulas were given for a fixed choice of Borel subalgebra.

1. Introduction

Throughout this paper the base field will be $\mathbb{C}$. Let $\mathfrak{g}$ be a Lie superalgebra and let W be a $\mathfrak{g}$-module. Then the superalgebra $\mathscr{P}(W)$ of superpolynomials on W has a canonical $\mathfrak{g}$-module structure, and $\mathscr{P}(W) \cong \mathscr{S}(W^*)$ as $\mathfrak{g}$-modules. Suppose that $\mathscr{P}(W)$ has a multiplicity-free decomposition into irreducible $\mathfrak{g}$-modules, that is

$$\mathscr{P}(W) \cong \bigoplus_{\lambda \in \Omega} W_\lambda, \tag{1}$$

where the W_λ's are pairwise non-isomorphic irreducible $\mathfrak{g}$-modules and Ω is an index set. In addition, assume that the W_λ are modules of type M, that is, $\mathrm{End}_{\mathfrak{g}}(W_\lambda) \cong \mathbb{C}$.

Let $\mathscr{D}(W)$ denote the algebra of constant-coefficient superdifferential operators on W. Then there is a canonical isomorphism $\mathscr{D}(W) \cong \mathscr{S}(W)$. Hence, as explained in Appendix A, we have a $\mathfrak{g}$-module decomposition

$$\mathscr{D}(W) \cong \bigoplus_{\lambda \in \Omega} W_\lambda^*.$$

Let $\mathscr{P}\mathscr{D}(W)$ denote the algebra of superdifferential operators on W with superpolynomial coefficients. We have an isomorphism of $\mathfrak{g}$-modules

$$\mathscr{P}\mathscr{D}(W) \cong \mathscr{P}(W) \otimes \mathscr{D}(W).$$

Taking $\mathfrak{g}$-invariant elements of both sides, it follows that we have vector-space isomorphisms

$$\mathscr{P}\mathscr{D}(W)^{\mathfrak{g}} \cong \bigoplus_{\lambda,\mu \in \Omega} (W_\lambda \otimes W_\mu^*)^{\mathfrak{g}} \cong \bigoplus_{\lambda \in \Omega} \mathbb{C} id_{W_\lambda}. \tag{2}$$

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Definition 1.1. The Capelli operator D^λ for $\lambda \in \Omega$ is the $\mathfrak{g}$-invariant differential operator in $\mathscr{P}\mathscr{D}(W)^{\mathfrak{g}}$ that corresponds to $id_{W_\lambda} \in \mathrm{Hom}_{\mathfrak{g}}(W_\lambda, W_\lambda)$ via the isomorphism (2).

Note that id_{W_λ} corresponds to the tensor $\sum_{i=1}^{d_\lambda} w_i^* \otimes w_i$, where $d_\lambda := \dim W_\lambda$ and the sets of vectors $\{w_i\}_{i=1}^{d_\lambda}$ and $\{w_i^*\}_{i=1}^{d_\lambda}$ are bases of W_λ^* and W_λ that are dual to each other. The above choices give a well-defined normalization of the operator D^λ. Indeed as we explain in Appendix A, if $W_\lambda \subseteq \mathscr{P}^d(W)$ then D^λ acts on W_λ by the scalar $d!$.

Clearly the set $\{D^\lambda \mid \lambda \in \Omega\}$ forms a basis for $\mathscr{P}\mathscr{D}(W)^{\mathfrak{g}}$. By Schur's Lemma, the operators D^μ for $\mu \in \Omega$ act on the irreducible components W_λ of $\mathscr{P}(W)$ by scalars $c_\mu(\lambda)$. Thus we have the following natural problem.

The Capelli Eigenvalue Problem. *Find the eigenvalue $c_\mu(\lambda)$ for each $\lambda, \mu \in \Omega$.*

The Capelli Eigenvalue Problem (henceforth CEP) goes back to the work of Kostant and Sahi [KS91, KS93]. It has been studied extensively and spawned the theory of interpolation Jack polynomials [Sah94, OO97].

In this paper we are interested in a refinement of the CEP when $(\mathfrak{g}, W)$ is either the pair $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V), V \otimes V^*)$ with $V := \mathbb{C}^{m|n}$ or the pair $(\mathfrak{gl}(V), \mathscr{S}^2(V))$ with $V := \mathbb{C}^{m|2n}$. The CEP for these pairs (and several others that are obtained by the TKK construction) was studied in [SSS20]. As explained in *loc. cit.*, these two cases are related to the diagonal symmetric pair $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V), \mathfrak{gl}(V))$ and the symmetric pair $(\mathfrak{gl}(V), \mathfrak{osp}(V))$, respectively. (For further details, as well as the explanation for the special value of the parameter θ used here in terms of deformed root systems, see [SSS20].)

In [SSS20] the authors first parametrize the representations of $\mathfrak{g}$ by hook partitions. Then they fix a Borel subalgebra $\mathfrak{b}$ (coming from the standard Borel subalgebra or its opposite; see [SSS20, Table 4]), and write down a formula for the $\mathfrak{b}$-highest weights of irreducible components of $\mathscr{P}(V)$ in terms of the hook partition. Finally they show that a formula for the eigenvalue $c_\mu(\lambda)$ can be computed as an interpolation super Jack polynomial evaluated at an affine function of the $\mathfrak{b}$-highest weight.

The fixed choice of the Borel subalgebra essentially means that the formula for the eigenvalues of a Capelli operator is really dependent on the parametrization of modules by partitions. Our goal in this paper is to remove this constraint. That is we compute formulas for $c_\mu(\lambda)$ which are polynomial in the $\mathfrak{b}$-highest weights of the W_λ, for *arbitrary* Borel subalgebras $\mathfrak{b}$. In other words, our goal in this paper is to study the following variation of the CEP for the aforementioned pairs $(\mathfrak{g}, W)$.

The Refined Capelli Eigenvalue Problem. *For any Borel subalgebra $\mathfrak{b}$, find the eigenvalue $c_\mu(\lambda)$ as a polynomial function in the $\mathfrak{b}$-highest weight of W_λ.*

Since the Borel subalgebras $\mathfrak{gl}(m|n)$ are not conjugate by the inner automorphism group, the solution to the CEP does *not* carry over to the refined form. Indeed, the work of Section 5 shows that the choice of affine function varies significantly with the type of Borel subalgebra.

Here is an outline of the paper. In Section 2 we recall the definitions and properties of the interpolation super Jack polynomials of Sergeev and Veselov [SV05]. In Section 3 we show that for the diagonal symmetric pair the solution of the refined CEP is obtained in a relatively straightforward way from the results of [SSS20]; this result is stated in Theorem 3.2.

We begin the study of the pair $(\mathfrak{gl}(m|2n), \mathcal{S}^2(\mathbb{C}^{m|2n}))$ in Section 4, where we establish formulae for $\mathfrak{b}$-highest weights, for any decreasing Borel subalgebra $\mathfrak{b}$, and derive necessary

conditions for a solution to the refined CEP. In Section 5, we solve the refined CEP in a series of results of increasing complexity: for *very even* Borel subalgebras in Section 5.1; for *generic* highest weights in Section 5.2; for *relatively even* Borel subalgebras in Section 5.3 and finally in all cases in Section 5.4. In Appendix A, we carefully prove the well-definedness of the CEP and establish the normalization of the Capelli operators.

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Notation and Conventions. Since all Cartan subalgebras of $\mathfrak{gl}(V)$ are conjugate by inner automorphisms, for the refined CEP it suffices to consider all Borel subalgebras that contain a fixed Cartan subalgebra. Throughout this paper $\mathfrak{h}_{m|n} \cong \mathbb{C}^{m|n}$ denotes the standard (*i.e.*, diagonal) Cartan subalgebra of $\mathfrak{gl}(m|n)$; we denote the standard basis of $\mathfrak{h}_{m|n}$, or equivalently of $\mathbb{C}^{m|n}$, indiscriminately by $\{e_i \mid 1 \leq i \leq m+n\}$. We also denote the dual basis for $\mathfrak{h}^*_{m|n}$ by $\{\epsilon_i, \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Henceforth $(\cdot,\cdot)$ is the nondegenerate supersymmetric form on $\mathfrak{h}^*_{m|n}$ that is diagonal with respect to this basis and satisfies $(\epsilon_i, \epsilon_i) = -(\delta_j, \delta_j) = 1$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Let $\mathfrak{b}_{m|n}$ denote the standard upper triangular Borel subalgebra of $\mathfrak{gl}(m|n)$ and $\mathfrak{b}^{\mathrm{op}}_{m|n}$ its opposite. The set of Borel subalgebras has several orbits under the Weyl group. A representative of each orbit (called an *increasing* Borel subalgebra) may be obtained from $\mathfrak{b}_{m|n}$ by performing a sequence of simple odd reflections; such a Borel subalgebra again contains $\mathfrak{h}_{m|n}$. Alternatively, applying simple odd reflections to $\mathfrak{b}^{\mathrm{op}}_{m|n}$ produces another set of representatives, whose elements are the *decreasing* Borel subalgebras.

Let $\mathscr{H}(m|n)$ denote the set of $m|n$ hook partitions, *i.e.*, integer partitions $(\lambda_1, \lambda_2, \ldots)$ satisfying $\lambda_{m+1} \leq n$. Doubling the elements of such a partition yields an element of $\mathscr{H}_2(m|2n)$, which is the subset of $m|2n$ hook partitions whose parts are all even. Each $\lambda \in \mathscr{H}(m|n)$ corresponds to a unique irreducible finite-dimensional representation of $\mathfrak{gl}(m|n)$, denoted V_λ, whose highest weight with respect to the standard Borel subalgebra $\mathfrak{b}_{m|n}$ is given by $\sum_i \lambda_i \epsilon_i + \sum_j \max\{\lambda'_j - m, 0\}\delta_j$. Note that its negative is then the highest weight of V^*_λ with respect to $\mathfrak{b}^{\mathrm{op}}_{m|n}$.

2. Interpolation super Jack Polynomials

Let $m, n \in \mathbb{Z}_{\geq 0}$ and let $R_{m|n} := \mathbb{C}[x_1, \ldots, x_m, y_1 \ldots, y_n]$ be the ring of polynomials in $m+n$ indeterminates with coefficients in $\mathbb{C}$. Let $\theta \in \mathbb{C} \backslash \mathbb{Q}^{\leq 0}$. We say a polynomial $f \in R_{m|n}$ is *separately symmetric* if f is symmetric in $\{x_i\}_{1 \leq i \leq m}$ and in $\{y_j\}_{1 \leq j \leq n}$ separately.

Definition 2.1. We denote by $\Lambda_{m,n,\theta}$ the subalgebra of separately symmetric polynomials f such that

$$f(\ldots, x_i + 1/2, \ldots, \ldots, y_j - 1/2, \ldots) = f(\ldots, x_i - 1/2, \ldots, \ldots, y_j + 1/2, \ldots) \tag{3}$$

on every hyperplane $x_i + \theta y_j = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We call the symmetry property defined in Equation (3) the *monoidal symmetry* of f.

Definition 2.2. Let $u, v \in \mathbb{C}^{m+n}$. We say u is equivalent to v if $f(u) = f(v)$ for all $f \in \Lambda_{m,n,\theta}$, in which case we write $u \sim_E v$. We say that u and v are *monoidally equivalent* if they are related by a sequence of separate symmetries and monoidal symmetries, and write $u \sim v$ in this case.

Evidently if $u \sim v$ then $u \sim_E v$. As we discuss in more detail (and use) in Section 5.4, these two notions of equivalence differ when there exist elements in $\mathbb{C}^{m|n}$ whose orbits under $\sim$ are infinite.

Lemma 2.3. *Let $u = (x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{C}^{m+n}$ and suppose $1 \leq i \leq m$, $1 \leq j \leq n$. Then we have*

$$u \sim \begin{cases} u - e_i + e_{m+j} & \text{whenever } x_i + \theta y_j = \frac{1}{2}(1-\theta), \text{ and} \\ u + e_i - e_{m+j} & \text{whenever } x_i + \theta y_j = -\frac{1}{2}(1-\theta). \end{cases}$$

Proof. For any $v = (x_1', \cdots, x_m', y_1', \cdots, y_n') \in \mathbb{C}^{m+n}$, we have $v + \frac{1}{2}e_i - \frac{1}{2}e_{m+j} \sim v - \frac{1}{2}e_i + \frac{1}{2}e_{m+j}$ whenever $x_i' + \theta y_j' = 0$. If $u = v + \frac{1}{2}e_i - \frac{1}{2}e_{m+j}$ then $x_i' + \theta y_j' = 0$ if and only if $x_i + \theta y_j = (x_i' + \frac{1}{2}) + \theta(y_j' - \frac{1}{2}) = \frac{1}{2}(1-\theta)$, yielding the first equivalence. Alternately, if $u = v - \frac{1}{2}e_i + \frac{1}{2}e_{m+j}$, then $x_i' + \theta y_j' = 0$ if and only if $x_i + \theta y_j = (x_i' - \frac{1}{2}) + \theta(y_j' + \frac{1}{2}) = -\frac{1}{2}(1-\theta)$, and the second equivalence holds. □

We need one extra definition before we can define the interpolation super Jack polynomials. Given an (m,n)-hook partition $\lambda \in \mathscr{H}(m,n)$, the *twisted Frobenius coordinates* $(p(\lambda), q(\lambda)) \coloneqq (p_1(\lambda), \dots, p_m(\lambda), q_1(\lambda), \dots, q_n(\lambda))$ of λ are defined by

$$\begin{aligned} p_i(\lambda) &\coloneqq \lambda_i - \theta\left(i - \frac{1}{2}\right) - \frac{1}{2}(n - \theta m), \text{ and} \\ q_j(\lambda) &\coloneqq \langle \lambda_j' - m \rangle - \theta^{-1}\left(j - \frac{1}{2}\right) + \frac{1}{2}\left(\theta^{-1}n + m\right), \end{aligned} \tag{4}$$

where $1 \leq i \leq m, 1 \leq j \leq n$ and $\langle x \rangle \coloneqq \max\{0, x\}$ for all $x \in \mathbb{R}$.

Definition 2.4. ([SV05, Section 6]) Let $\lambda \in \mathscr{H}(m,n)$ and let $\theta \in \mathbb{C} \backslash \mathbb{Q}^{\leq 0}$. The interpolation super Jack polynomial $P_{\lambda,\theta}$ is the polynomial in $R_{m|n}$ that is uniquely determined by the following properties:

(i) $P_{\lambda,\theta} \in \Lambda_{m,n,\theta}$;
(ii) $\deg(P_{\lambda,\theta}) \leq |\lambda|$ where the degree of $P_{\lambda,\theta}$ means the total degree in both x and y;
(iii) $P_{\lambda,\theta}(p(\lambda), q(\lambda)) = d!$ where $d := |\lambda|$;
(iv) $P_{\lambda,\theta}(p(\mu), q(\mu)) = 0$ for all (m,n)-hook partitions μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.

Remark 2.5. More precisely, the polynomials $SP_\lambda^*(x, y, \theta)$ introduced in [SV05] differ from the $P_{\lambda,\theta}$ by a normalization: for each λ, the value of $SP_\lambda^*(p(\lambda), q(\lambda), \theta)$ is instead given by a hook-type product.

Remark 2.6. The set $\{P_{\lambda,\theta}\}_{\lambda \in \mathscr{H}(m,n)}$ forms a basis for $\Lambda_{m,n,\theta}$. Indeed one can prove that there exists a polynomial $\mathbf{P}_\lambda$ in $m+n$ variables with coefficients in the field of rational functions in a parameter $\bar{\theta}$ such that the specialization of $\mathbf{P}_\lambda$ at $\bar{\theta} := \theta$ yields $P_{\lambda,\theta}$. In particular, the coefficients of $\mathbf{P}_\lambda$ do not have any poles in $\mathbb{C} \setminus \mathbb{Q}^{\leq 0}$. For these facts and further properties of the polynomials $\mathbf{P}_\lambda$ see [SV05].

3. The refined CEP for $(\mathfrak{gl}(V) \oplus \mathfrak{gl}(V), V \otimes V^*)$

In this section we set $\theta := 1$, $V := \mathbb{C}^{m|n}$, $\mathfrak{g} := \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $W := V \otimes V^*$, and $\mathfrak{h} := \mathfrak{h}_{m|n} \oplus \mathfrak{h}_{m|n}$. Then $\mathfrak{b}_0 := \mathfrak{b}_{m|n}^{\mathrm{op}} \oplus \mathfrak{b}_{m|n}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. The

polynomial algebra $\mathscr{P}(W)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module, with decomposition

$$\mathscr{P}(W) = \bigoplus_{\lambda \in \mathscr{H}(m|n)} V_\lambda^* \otimes V_\lambda.$$

Let us denote the highest weight of V_λ with respect to $\mathfrak{b}_{m|n}$ by $\underline{\lambda}_0$; then the $\mathfrak{b}_{m|n}^{\mathrm{op}}$-highest weight of V_λ^* is $-\underline{\lambda}_0$.

We begin by restating the solution to the Capelli Eigenvalue Problem from [SSS20] in terms of the Weyl vector of $\mathfrak{b}_0$. Write $\tilde{x} \in \mathbb{C}^{m|n}$ for the image of $x \in \mathfrak{h}_{m|n}^*$ under the standard isomorphism that sends ϵ_i to e_i and δ_j to e_{m+j}. Let $\tau_0 \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$ denote the affine map

$$\tau_0(x, y) = \tilde{y} + \tilde{\rho}_0$$

where we identify $(x, y) \in \mathfrak{h}^*$ with a pair of vectors $x, y \in \mathbb{C}^{m|n}$ and ρ_0 denotes the Weyl vector corresponding to $\mathfrak{b}_{m|n}$. Observing that

$$\rho_0 = \sum_{i=1}^{m} \frac{m-n+1-2i}{2} \epsilon_i + \sum_{j=1}^{n} \frac{m+n+1-2j}{2} \delta_j,$$

we can rephrase the result of [SSS20, Table 3] pertaining to $\mathfrak{g}$ as follows.

Theorem 3.1. [SSS20, Theorem 1.13] *Let $\mathfrak{g}$ and W be as above. For every $\lambda, \mu \in \mathscr{H}(m|n)$, the eigenvalue of the Capelli operator D^μ on the irreducible summand $V_\lambda^* \otimes V_\lambda$ with $\mathfrak{b}_0$-highest weight $(-\underline{\lambda}_0, \underline{\lambda}_0)$ is equal to*

$$P_{\mu,1} \circ \tau_0 \left(-\underline{\lambda}_0, \underline{\lambda}_0\right),$$

where $P_{\mu,1} \in \Lambda_{m,n,1}$ is the interpolation super Jack polynomial associated to μ for parameter value $\theta = 1$.

Now suppose $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ is an arbitrary Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then $\mathfrak{b}_i$ is a Borel subalgebra of $\mathfrak{gl}(m|n)$ for $i = 1, 2$ and we denote its Weyl vector $\rho_{\mathfrak{b}_i}$. Write $\underline{\lambda}_{\mathfrak{b}_1}$ for the highest weight of V_λ^* with respect to $\mathfrak{b}_1$ and $\underline{\lambda}_{\mathfrak{b}_2}$ for the highest weight of V_λ with respect to $\mathfrak{b}_2$. Then we may define affine maps $\tau_{\mathfrak{b},i} \colon \mathfrak{h}^* = \mathfrak{h}_{m|n}^* \oplus \mathfrak{h}_{m|n}^* \to \mathbb{C}^{m|n}$ via

$$\tau_{\mathfrak{b},1}(x, y) = -\tilde{x} - \tilde{\rho}_{\mathfrak{b}_1}, \quad \text{and} \quad \tau_{\mathfrak{b},2}(x, y) = \tilde{y} + \tilde{\rho}_{\mathfrak{b}_2}.$$

In this section we prove the following theorem.

Theorem 3.2. *Let $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ be an arbitrary Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and let $\lambda \in \mathscr{H}(m|n)$. Then*

$$P_{\mu,1} \circ \tau_{\mathfrak{b},1}(\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_2}) = P_{\mu,1} \circ \tau_{\mathfrak{b},2}(\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_2}) = P_{\mu,1} \circ \tau_0(-\underline{\lambda}_0, \underline{\lambda}_0).$$

It follows that for any $\mathfrak{b}$, the eigenvalue of the Capelli operator D^μ on each irreducible component $V_\lambda^* \otimes V_\lambda$ of $\mathscr{P}(W)$ is given by evaluating a polynomial on the highest weight of this module with respect to $\mathfrak{b}$.

Proof. Since $P_{\mu,1} \in \Lambda_{m,n,1}$, it suffices to show that for $i = 1, 2$ and any $\lambda \in \mathscr{H}(m|n)$,

$$\tau_{\mathfrak{b},i}(\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_2}) \sim \tau_0(-\underline{\lambda}_0, \underline{\lambda}_0). \tag{5}$$

We verify this in stages. First assume that $\mathfrak{b}_2$ is an *increasing* Borel subalgebra of $\mathfrak{gl}(m|n)$, that is, it is obtained from the standard Borel subalgebra $\mathfrak{b}_{m|n}$ by a finite sequence of simple odd reflections. Suppose $\mathfrak{b}_2 = r_\alpha(\mathfrak{b}')$ where $\mathfrak{b}'$ is an increasing Borel subalgebra and

$\alpha = \epsilon_i - \delta_j$ is a simple isotropic root with respect to $\mathfrak{b}'$. By induction, we may suppose that (5) holds for $\mathfrak{b}'$, namely, that

$$\widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}} \sim \widetilde{\underline{\lambda}_0} + \widetilde{\rho}_0.$$

We need to show that $\widetilde{\underline{\lambda}_{\mathfrak{b}_2}} + \widetilde{\rho_{\mathfrak{b}_2}} \sim \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}}$. Recall from [CW13, Lemma 1.40] that the highest weight $\underline{\lambda}_{\mathfrak{b}_2}$ is obtained from $\underline{\lambda}_{\mathfrak{b}'}$ by the formula

$$\underline{\lambda}_{\mathfrak{b}_2} = \begin{cases} \underline{\lambda}_{\mathfrak{b}'} - \alpha & \text{if } (\underline{\lambda}_{\mathfrak{b}'}, \alpha) \neq 0; \\ \underline{\lambda}_{\mathfrak{b}'} & \text{if } (\underline{\lambda}_{\mathfrak{b}'}, \alpha) = 0, \end{cases} \tag{6}$$

while by [CW13, Prop 1.33] we have $(\rho_{\mathfrak{b}'}, \alpha) = 0$ and $\rho_{\mathfrak{b}_2} = \rho_{\mathfrak{b}'} + \alpha$. Therefore if $(\underline{\lambda}_{\mathfrak{b}'}, \alpha) \neq 0$, we have

$$\underline{\lambda}_{\mathfrak{b}_2} + \rho_{\mathfrak{b}_2} = (\underline{\lambda}_{\mathfrak{b}'} - \alpha) + (\rho_{\mathfrak{b}'} + \alpha) = \underline{\lambda}_{\mathfrak{b}'} + \rho_{\mathfrak{b}'},$$

and we are done. Otherwise, we have $(\underline{\lambda}_{\mathfrak{b}'}, \alpha) = 0$, so that $\underline{\lambda}_{\mathfrak{b}_2} = \underline{\lambda}_{\mathfrak{b}'}$ and we instead have

$$\widetilde{\underline{\lambda}_{\mathfrak{b}_2}} + \widetilde{\rho_{\mathfrak{b}_2}} = \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}} + \widetilde{\alpha} = \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}} + e_i - e_{m+j}.$$

Since $(\underline{\lambda}_{\mathfrak{b}'} + \rho_{\mathfrak{b}'}, \alpha) = 0$, it follows that the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n) := \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}}$ satisfies $x_i + y_j = 0$. Thus by Lemma 2.3 with $\theta = 1$ we have

$$\widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}} \sim \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho_{\mathfrak{b}'}} + e_i - e_{m+j} = \widetilde{\underline{\lambda}_{\mathfrak{b}_2}} + \widetilde{\rho_{\mathfrak{b}_2}},$$

as required. Thus by induction, (5) holds for $i = 2$ and all increasing Borel subalgebras $\mathfrak{b}_2$.

Now suppose $\mathfrak{b}_1$ is a decreasing Borel subalgebra, meaning that $\mathfrak{b}_1^{\mathrm{op}}$ is increasing. Then by the symmetry of the definition of $\tau_{\mathfrak{b},i}$ we deduce directly that

$$\begin{aligned} \tau_{\mathfrak{b},1}(\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_2}) &= \tau_{\mathfrak{b}_1 \oplus \mathfrak{b}_1^{\mathrm{op}},1}\ (\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_1^{\mathrm{op}}}) \\ &= \tau_{\mathfrak{b}_1 \oplus \mathfrak{b}_1^{\mathrm{op}},2}\ (\underline{\lambda}_{\mathfrak{b}_1}, \underline{\lambda}_{\mathfrak{b}_1^{\mathrm{op}}}) \\ &\sim \tau_0(-\underline{\lambda}_0, \underline{\lambda}_0), \end{aligned}$$

which implies (5) holds with $i = 1$ whenever $\mathfrak{b}_1$ is decreasing.

Finally, suppose that $\mathfrak{b} = \mathfrak{b}_1' \oplus \mathfrak{b}_2'$ is an arbitrary Borel subalgebra of $\mathfrak{g}$. Then for $i = 1, 2$ there exist permutations $\sigma_i \in S_m$ (permuting $\epsilon_1, \ldots, \epsilon_m$) and $\sigma_i' \in S_n$ (permuting $\delta_1, \ldots, \delta_n$) such that $(\sigma_i \times \sigma_i')(\mathfrak{b}_i') = \mathfrak{b}_i$ is a decreasing (respectively, increasing) Borel subalgebra of $\mathfrak{gl}(m|n)$ when $i = 1$ (respectively, $i = 2$). Then we have for $i = 1, 2$ that $(\sigma_i \times \sigma_i')(\underline{\lambda}_{\mathfrak{b}_i'}) = \underline{\lambda}_{\mathfrak{b}_i}$ and similarly $(\sigma_i \times \sigma_i')(\rho_{\mathfrak{b}_i'}) = \rho_{\mathfrak{b}_i}$. By the separate symmetry of the elements of $\Lambda_{m,n,1}$ these permutations leave polynomials in $\Lambda_{m,n,1}$ invariant, so (5) follows immediately. □

4. Borel subalgebras and highest weights for $\mathfrak{gl}(m|2n)$

From now on, we set $\theta := 1/2$, $V := \mathbb{C}^{m|2n}$, $\mathfrak{g} := \mathfrak{gl}(m|2n)$ and $W := \mathscr{S}^2(V)$. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}_0 := \mathfrak{b}^{\mathrm{op}}$ be the **opposite** standard (that is, lower triangular) Borel subalgebra of $\mathfrak{g}$.

4.1. **The solution to the CEP with respect to $\mathfrak{b}_0$.** The polynomial algebra $\mathscr{P}(W)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module. A key result of [SSS20] is that we can parametrize this decomposition by $m|n$ hook tableaux. Given $\lambda \in \mathscr{H}(m|n)$, let $2\lambda \in \mathscr{H}_2(m|2n)$ be the $(m|2n)$ hook tableau obtained by doubling each row, and let $W_\lambda := (V_{2\lambda})^*$. Then we have the decomposition

$$\mathscr{P}(W) = \bigoplus_{\lambda \in \mathscr{H}(m|n)} W_\lambda. \tag{7}$$

In this case, given $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathscr{H}(m|n)$ with transpose $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, the highest weight $\underline{\lambda}_0$ of W_λ with respect to $\mathfrak{b}_0 = \mathfrak{b}^{\mathrm{op}}_{m|2n}$ is given by

$$\underline{\lambda}_0 = -\sum_{i=1}^{m} 2\lambda_i \epsilon_i - \sum_{j=1}^{n} \mu_j \left(\delta_{2j-1} + \delta_{2j}\right) \tag{8}$$

where $\mu_j := \langle \lambda'_j - m \rangle = \max\{0, \lambda'_j - m\}$. Note that this corresponds precisely to the $m|2n$ hook partitition of 2λ, which is $(2\lambda_1, \ldots, 2\lambda_m | \mu_1, \mu_1, \ldots, \mu_n, \mu_n) \in \mathscr{H}_2(m|2n)$.

Let $\Omega_{m|2n} = \{\underline{\lambda}_0 \mid \lambda \in \mathscr{H}(m|n)\}$ be the set of all $\mathfrak{b}_0$-highest weights of $\mathfrak{g}$-modules appearing in the decomposition (7). The Zariski closure of $\Omega_{m|2n}$ in $\mathfrak{h}^*$ is a subspace we denote $\mathfrak{a}^*$, since it coincides with the subspace of linear functionals on $\mathfrak{h}$ that vanish on $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{k}$. Let $\Gamma \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$ be the linear map defined by

$$\Gamma(a_1, \ldots, a_m | b_1, \ldots, b_{2n}) = \left(\frac{a_1}{2}, \cdots, \frac{a_m}{2} \mid \frac{b_1 + b_2}{2}, \cdots, \frac{b_{2n-1} + b_{2n}}{2} \right).$$

Observe that if ρ_0 denotes the Weyl vector corresponding to $\mathfrak{b}_0$, then we have

$$\Gamma(\rho_0) = -\sum_{i=1}^{m} \frac{m - 2n + 1 - 2i}{4} e_i - \sum_{k=1}^{n} \frac{m + 2n + 2 - 4k}{2} e_{m+k}. \tag{9}$$

Define, for any $x \in \mathfrak{a}^*$, the map $\tau_0^{\mathfrak{a}^*} \colon \mathfrak{a}^* \to \mathbb{C}^{m|n}$ by $\tau_0^{\mathfrak{a}^*}(x) = -\Gamma(x + \rho_0)$. Then it is easy to see that we can rephrase the result of [SSS20, Table 3] pertaining to $\mathfrak{g}$ as follows.

Theorem 4.1. [SSS20, Theorem 1.13.] *Let $\mathfrak{g}$ and W be as above. Then for each $\lambda, \mu \in \mathscr{H}(m|n)$, the eigenvalue of the Capelli operator D^μ on the irreducible summand W_λ with highest weight $\underline{\lambda}_0 \in \mathfrak{a}^*$ with respect to $\mathfrak{b}_0$ is equal to $P_{\mu, \frac{1}{2}} \circ \tau_0^{\mathfrak{a}^*}(\underline{\lambda}_0)$.*

4.2. **Explicit form of $\tau_0(\underline{\lambda}_0)$.** The affine map $\tau_0^{\mathfrak{a}^*}$ is only characterized by the theorem on elements of $\mathfrak{a}^*$. To explicitly describe all extensions of $\tau_0^{\mathfrak{a}^*}$ to affine maps $\tau_0 \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$, let M_0 denote the matrix of the linear transformation $-\Gamma \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$ with respect to our standard bases. Then we have

$$M_0 = \begin{bmatrix} -\frac{1}{2} I_m & 0_{m \times 2n} \\ 0_{n \times m} & D \end{bmatrix} \tag{10}$$

where $D = (d_{ij})$ is a $n \times 2n$ matrix whose entries satisfy

$$d_{ij} = \begin{cases} -\frac{1}{2} & \text{if } j = 2i - 1 \text{ or } 2i \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2. Let $\mathcal{C}$ be the set of all matrices of the form

$$M = M_0 + \begin{bmatrix} 0_{(m+n) \times m} & A \end{bmatrix}$$

where A is any $(m+n) \times (2n)$ matrix whose columns A_j satisfy the relations $A_{2k} = -A_{2k-1}$ for all $1 \leq k \leq n$. We also set

$$X_0 = -\Gamma(\rho_0) = M_0 \rho_0 = \tau_0^{\mathfrak{a}^*}(0). \tag{11}$$

Lemma 4.3. *Every extension of $\tau_0^{\mathfrak{a}^*} \colon \mathfrak{a}^* \to \mathbb{C}^{m|n}$ to an affine map on $\mathfrak{h}^*$ is given on $x \in \mathfrak{h}^*$ by $\tau_0(x) = Mx + X_0$ for some $M \in \mathcal{C}$.*

Proof. It suffices to observe that $\mathfrak{a}^* = \mathrm{span}\{\epsilon_1, \cdots, \epsilon_m, \delta_1 + \delta_2, \cdots, \delta_{2n-1} + \delta_{2n}\}$; thus the set of all linear maps to $\mathbb{C}^{m|n}$ vanishing on $\mathfrak{a}^*$ is given by

$$\mathcal{K} = \left\{ \begin{bmatrix} 0_{(n+m)\times m} & A \end{bmatrix} \mid A_{2k-1} = -A_{2k} \text{ for all } 1 \leq k \leq n, \right\}$$

where A_j denotes the jth column of A. □

Note that since $\rho_0 \notin \mathfrak{a}^*$, the equation $M\rho_0 = X_0$ with $M \in \mathcal{C}$ holds *only* for $M = M_0$. In the following sections, we will see that the single matrix M_0 is not suitable for use with other Borel subalgebras, but that other subsets of $\mathcal{C}$ will be.

4.3. **Borel subalgebras and decreasing $\delta\epsilon$ sequences.** One may parametrize the set $\mathcal{B}$ of Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ by sequences $(\xi_i)_{i=1}^{m+2n}$ where $\{\xi_i | 1 \leq i \leq m+2n\} = \{\epsilon_i, \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq 2n\}$ — called $\delta\epsilon$ sequences — by associating to $\mathfrak{b}$ the sequence for which $\{\xi_i - \xi_{i+1} \mid 1 \leq i < m+2n\}$ is its set of simple roots. In particular, the simple odd (isotropic) roots of $\mathfrak{b}$ correspond to pairs $\epsilon_i\delta_j$ or $\delta_j\epsilon_i$ in the sequence, and if α is such a root, then the $\delta\epsilon$ sequence of $r_\alpha(\mathfrak{b})$ is obtained from that of $\mathfrak{b}$ by swapping ϵ_i and δ_j.

For example, $\mathfrak{b}^{\mathrm{op}}$ corresponds to the sequence $\delta_{2n}\delta_{2n-1}\cdots\delta_1\epsilon_m\epsilon_{m-1}\cdots\epsilon_1$ and it has a single odd simple root $\delta_1 - \epsilon_m$.

If the indices of the ϵ_i and of the δ_j are (separately) decreasing, then we say this is a decreasing $\delta\epsilon$-sequence. These correspond exactly to the decreasing Borel subalgebras defined earlier, since each one may be obtained from $\mathfrak{b}^{\mathrm{op}}$ be a sequence of simple odd reflections. Let $\mathcal{B}^d$ denote the set of decreasing Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$.

Definition 4.4. Let $\mathfrak{b} \in \mathcal{B}^d$, and consider the $\delta\epsilon$ sequence associated to $\mathfrak{b}$.

(1) For each $1 \leq i \leq m$, let $\ell_{i,\mathfrak{b}}$ be the number of δs to the right of ϵ_i.
(2) For each $1 \leq k \leq 2n$, let $j_{k,\mathfrak{b}}$ be the number of ϵs to the left of δ_k.

When $\mathfrak{b}$ is clear from the context, we may write ℓ_i and j_k for $\ell_{i,\mathfrak{b}}$ and $j_{k,\mathfrak{b}}$ respectively. It is easy to see that, for example, $j_k = \#\{i \mid \ell_i \geq k\}$. Thus a decreasing Borel subalgebra (or equivalently, the corresponding decreasing $\delta\epsilon$ sequence) is completely determined by specifying the values $\ell_1 \leq \cdots \leq \ell_m$ (or equivalently, the values $j_1 \geq \cdots \geq j_{2n}$).

Definition 4.5. Let $\mathfrak{b} \in \mathcal{B}^d$. With ℓ_i as defined above, let $\mathcal{G}_\mathfrak{b}$ be the set of roots

$$\begin{aligned} &\delta_1 - \epsilon_m, \delta_2 - \epsilon_m, \cdots, \delta_{\ell_m} - \epsilon_m; \\ &\delta_1 - \epsilon_{m-1}, \delta_2 - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1}} - \epsilon_{m-1}; \\ &\cdots \\ &\delta_1 - \epsilon_1, \delta_2 - \epsilon_1, \cdots, \delta_{\ell_1} - \epsilon_1. \end{aligned} \tag{12}$$

The importance of $\mathcal{G}_\mathfrak{b}$ is as follows. Suppose that we obtain $\mathfrak{b}$ from $\mathfrak{b}^{\mathrm{op}}$ by a sequence of odd simple reflections, that is, such that $\mathfrak{b}_0 = \mathfrak{b}^{\mathrm{op}}$, $\mathfrak{b} = \mathfrak{b}_t$ for some $t > 0$ and for each $1 \leq k \leq t$, $\mathfrak{b}_k = r_{\alpha_k}(\mathfrak{b}_{k-1})$ where α_k is an odd simple (isotropic) root of $\mathfrak{b}_{k-1}$. Then if t is minimal we necessarily have the equality of sets $\{\alpha_i \mid 1 \leq i \leq t\} = \mathcal{G}_\mathfrak{b}$, since the effect of each simple odd reflection $\pm(\epsilon_i - \delta_j)$ is to swap the adjacent terms ϵ_i and δ_j in the corresponding $\delta\epsilon$ sequence. Moreover, the ordered list of roots in (12), (ordered from the top row to the bottom row) is one sequence satisfying these conditions. Set

$$r_\mathfrak{b} := \sum_{\alpha \in \mathcal{G}_\mathfrak{b}} \alpha. \tag{13}$$

The following lemma is immediate.

Lemma 4.6. *For each $\mathfrak{b} \in \mathcal{B}^d$ we have*

$$r_{\mathfrak{b}} = -\sum_{i=1}^{m} \ell_{i,\mathfrak{b}}\epsilon_i + \sum_{k=1}^{2n} j_{k,\mathfrak{b}}\delta_k.$$

4.4. **A formula for highest weights with respect to $\mathfrak{b}$.** Let $\lambda \in \mathscr{H}(m|n)$. For any $\mathfrak{b} \in \mathcal{B}^d$ (respectively, for $\mathfrak{b}^{\mathrm{op}}$) write $\underline{\lambda}_{\mathfrak{b}}$ (respectively, $\underline{\lambda}_0$) for the corresponding highest weight of W_λ.

Suppose $\mathfrak{b}$ is obtained from $\mathfrak{b}^{\mathrm{op}}$ by the sequence of simple odd reflections in the roots (12). For any adjacent pair of Borel subalgebras $\mathfrak{b}'' = r_{\delta_j-\epsilon_i}(\mathfrak{b}')$ of this sequence, we have from (6) that

$$\underline{\lambda}_{\mathfrak{b}''} = \underline{\lambda}_{\mathfrak{b}'} - \delta_j + \epsilon_i$$

unless $(\underline{\lambda}_{\mathfrak{b}'}, \delta_j - \epsilon_i) = 0$. Since all coefficients of $\underline{\lambda}_{\mathfrak{b}'}$ are nonpositive, this latter holds if and only if its ith and $(m+j)$th coefficients are both zero.

It follows that the highest weight of W_λ with respect to $\mathfrak{b}$ takes the form

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\lambda, \mathfrak{b}), \quad \text{where} \quad r(\lambda, \mathfrak{b}) := \sum_{\alpha \in \mathcal{N}_{\mathfrak{b}}} \alpha$$

for some subset $\mathcal{N}_{\mathfrak{b}} \subseteq \mathcal{G}_{\mathfrak{b}}$.

Definition 4.7. If $\mathcal{N}_{\mathfrak{b}} = \mathcal{G}_{\mathfrak{b}}$, then $r(\lambda, \mathfrak{b}) = r_{\mathfrak{b}}$ and we say that $\lambda \in \mathscr{H}(m|n)$ is *generic* for $\mathfrak{b}$. Otherwise, λ is called nongeneric for $\mathfrak{b}$.

Definition 4.8. Let $\mathfrak{b} \in \mathcal{B}^d$ and $\lambda \in \mathscr{H}(m|n)$. If $\ell_{m,\mathfrak{b}} > 2\lambda_m$, then let $I_{\lambda,\mathfrak{b}}$ denote the least index $I \geq 1$ for which $\ell_I > 2\lambda_I$. Further, set

$$\ell(\lambda, i, \mathfrak{b}) := \min\{\ell_{i,\mathfrak{b}}, 2\lambda_i\}$$

for all $1 \leq i \leq m$; then $\ell(\lambda, i, \mathfrak{b}) = \ell_{i,\mathfrak{b}}$ if and only if $i < I_{\lambda,\mathfrak{b}}$.

Proposition 4.9. *Let $\mathfrak{b} \in \mathcal{B}^d$ and $\lambda \in \mathscr{H}(m|n)$. Then $r(\lambda, \mathfrak{b}) = \sum_{\alpha \in \mathcal{N}_{\mathfrak{b}}} \alpha$ where $\mathcal{N}_{\mathfrak{b}}$ is the following set of roots:*

$$\begin{aligned}
&\delta_1 - \epsilon_m, \delta_2 - \epsilon_m, \cdots, \delta_{\ell(\lambda,m,\mathfrak{b})} - \epsilon_m;\\
&\delta_1 - \epsilon_{m-1}, \delta_2 - \epsilon_{m-1}, \cdots, \delta_{\ell(\lambda,m-1,\mathfrak{b})} - \epsilon_{m-1};\\
&\cdots\\
&\delta_1 - \epsilon_1, \delta_2 - \epsilon_1, \cdots, \delta_{\ell(\lambda,1,\mathfrak{b})} - \epsilon_1,
\end{aligned} \tag{14}$$

where we interpret any row corresponding to a value of $\ell(\lambda, i, \mathfrak{b}) = 0$ as an empty row. In particular $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\lambda, \mathfrak{b})$ is the $\mathfrak{b}$-highest weight of W_λ.

Proof. We have already addressed the case that λ is generic for $\mathfrak{b}$, so suppose λ is nongeneric for $\mathfrak{b}$ and let $I := I_{\lambda,\mathfrak{b}}$. It suffices to show that $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\lambda, \mathfrak{b})$. Note that in $\underline{\lambda}_0 = (2\lambda_1, \cdots, 2\lambda_m | \mu_1, \mu_1, \cdots, \mu_n, \mu_n)$, we have $\mu_k = 0$ for all $k > \lambda_m$.

We proceed inductively from m down to 1, following the rows of the list of roots (12), showing that if $\mathfrak{b}_{j+1}$ denotes the Borel subalgebra obtained after completing all reflections up to the end of those in the row corresponding to ϵ_{j+1}, that

$$\underline{\lambda}_{\mathfrak{b}_{j+1}} = \underline{\lambda}_0 - \sum_{i=j+1}^{m} (\delta_1 + \cdots + \delta_{\ell(\lambda,i,\mathfrak{b})} - \ell(\lambda, i, \mathfrak{b})\epsilon_i). \tag{15}$$

The base case corresponds to $j = m$, $\mathfrak{b} = \mathfrak{b}^{\mathrm{op}}$, where there is nothing to show.

Suppose we have shown our inductive hypothesis up to ϵ_{j+1}, as in (15). Let us first determine the coefficients of ϵ_j and of δ_k with $k > 2\lambda_j$.

The coefficient of ϵ_j in $-\underline{\lambda}_{\mathfrak{b}_{j+1}}$ is the same as that of $-\underline{\lambda}_0$; this value is therefore $2\lambda_j$. Let $k > 2\lambda_j \geq 2\lambda_m$. Then the coefficient of δ_k in $\underline{\lambda}_0$ was zero. Moreover, since we have $2\lambda_j \geq 2\lambda_i \geq \ell(\lambda, i, \mathfrak{b})$ for each $j+1 \leq i \leq m$, none of the additional terms in (15) contribute to the coefficient of δ_k. Thus the coefficient of δ_k in $-\underline{\lambda}_{\mathfrak{b}_{j+1}}$ is 0 for $k > 2\lambda_j$.

As we iterate through each of the roots

$$\delta_1 - \epsilon_j, \delta_2 - \epsilon_j, \cdots, \delta_{\ell(\lambda,j,\mathfrak{b})} - \epsilon_j$$

we fall into the first case of (6) because the coefficient of ϵ_j is nonzero at each iteration. Therefore if $\mathfrak{b}'$ denotes the resulting Borel subalgebra, we have

$$\underline{\lambda}_{\mathfrak{b}'} = \underline{\lambda}_{\mathfrak{b}_{j+1}} - (\delta_1 + \cdots + \delta_{\ell(\lambda,j,\mathfrak{b})} - \ell(\lambda, j, \mathfrak{b})\epsilon_j). \tag{16}$$

If $\ell(\lambda, j, \mathfrak{b}) = \ell_{j,\mathfrak{b}}$ (that is, if $j \leq I$) then we are done. Otherwise, $\ell(\lambda, j, \mathfrak{b}) = 2\lambda_j$ and we infer that the coefficient of ϵ_j in $\underline{\lambda}_{\mathfrak{b}'}$ is 0, as is the coefficient of δ_k for any $k > \ell(\lambda, j, \mathfrak{b}) = 2\lambda_j$. Thus for any remaining roots

$$\delta_{\ell(\lambda,j,\mathfrak{b})+1} - \epsilon_j, \cdots, \delta_{\ell_{j,\mathfrak{b}}} - \epsilon_j$$

we fall into the second case of (6), and so $\underline{\lambda}_{\mathfrak{b}_j} = \underline{\lambda}_{\mathfrak{b}'}$. Therefore (15) and (16) together imply

$$\underline{\lambda}_{\mathfrak{b}_j} = \underline{\lambda}_0 - \sum_{i=j}^{m} (\delta_1 + \cdots + \delta_{\ell(\lambda,i,\mathfrak{b})} - \ell(\lambda, i, \mathfrak{b})\epsilon_i),$$

and we may conclude by induction that

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - \sum_{i=1}^{m} (\delta_1 + \cdots + \delta_{\ell(\lambda,i,\mathfrak{b})} - \ell(\lambda, i, \mathfrak{b})\epsilon_i) = \underline{\lambda}_0 - r(\lambda, \mathfrak{b}), \tag{17}$$

as required. □

Remark 4.10. Note in particular that (17) implies that the coefficient of ϵ_i in $\underline{\lambda}_{\mathfrak{b}}$ is given by

$$-(2\lambda_i - \ell(\lambda, i, \mathfrak{b})) = \begin{cases} -(2\lambda_i - \ell_i) < 0 & \text{if } \ell_i < 2\lambda_i \ ; \\ 0 & \text{if } \ell_i \geq 2\lambda_i. \end{cases}$$

The coefficient of ϵ_i is always ≤ 0.

Corollary 4.11. *Let $\mathfrak{b} \in \mathcal{B}^d$ and $\lambda \in \mathscr{H}(m|n)$. Thhe following statements are equivalent.*

(i) *λ is $\mathfrak{b}$-generic.*
(ii) *$2\lambda_m \geq \ell_{m,\mathfrak{b}}$.*
(iii) *$2\lambda_i \geq \ell_{i,\mathfrak{b}}$ for $1 \leq i \leq m$.*

Proof. Equivalence of (ii) and (iii) is an immediate consequence of the fact that the $\ell_{i,\mathfrak{b}}$ are increasing whereas the λ_i are decreasing. The equivalence of (i) and (ii) follows from Proposition 4.9 and the fact that if $2\lambda_i \geq \ell_{i,\mathfrak{b}}$ for all i, then $r(\lambda, \mathfrak{b}) = r_{\mathfrak{b}}$ so λ is generic for $\mathfrak{b}$. □

For ease of reference, we list the roots contributing to $r_{\mathfrak{b}} - r(\lambda, \mathfrak{b})$, that is, the subset $\mathcal{G}_{\mathfrak{b}} \setminus \mathcal{N}_{\mathfrak{b}}$ of the set of roots in (12) which we will *not* subtract from $\underline{\lambda}_0$ when computing the

highest weight $\underline{\lambda}_{\mathfrak{b}}$. Using the index $I = I_{\lambda,\mathfrak{b}}$ for simplicity, it follows from (14) that they are

$$\begin{aligned} &\delta_{2\lambda_m+1} - \epsilon_m, \cdots, \delta_{\ell_{m,\mathfrak{b}}} - \epsilon_m, \\ &\delta_{2\lambda_{m-1}+1} - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1,\mathfrak{b}}} - \epsilon_{m-1}, \\ &\cdots \\ &\delta_{2\lambda_I+1} - \epsilon_I, \cdots, \delta_{\ell_{I,\mathfrak{b}}} - \epsilon_I, \end{aligned} \tag{18}$$

4.5. **Some necessary conditions.** Our goal is to define, for each $\mathfrak{b} \in \mathcal{B}^d$, an affine map $\tau_{\mathfrak{b}} \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$ such that for all $\lambda \in \mathscr{H}(m|n)$ we have the equivalence

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim_E \tau_0(\underline{\lambda}_0). \tag{19}$$

As in Section 3, this would imply that $P_{\lambda,1/2} \circ \tau_{\mathfrak{b}}$ is the interpolation polynomial we seek. Such a map $\tau_{\mathfrak{b}}$ will be given on $x \in \mathfrak{h}^*$ by

$$\tau_{\mathfrak{b}}(x) = M_{\mathfrak{b}} x + X_{\mathfrak{b}},$$

for some $(m+n) \times (m+2n)$ matrix $M_{\mathfrak{b}}$ and vector $X_{\mathfrak{b}} \in \mathbb{C}^{m|n}$. In this section derive some necessary conditions that $\tau_{\mathfrak{b}}$ must satisfy in order for the relation (19) to hold.

Lemma 4.12. *Let $\mathfrak{b} \in \mathcal{B}^d$. If (19) holds, then we must have*

$$X_{\mathfrak{b}} \sim_E X_0.$$

Proof. Since $\underline{\lambda}_0 = \underline{\lambda}_{\mathfrak{b}} = 0$ when λ corresponds to the trivial module, the requirement that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim_E \tau_0(\underline{\lambda}_0)$ for all λ implies in particular that $X_{\mathfrak{b}} \sim_E X_0$. □

Proposition 4.13. *Let $\mathfrak{b} \in \mathcal{B}^d$. If (19) holds, then for all $\lambda \in \mathscr{H}(m|n)$ that are $\mathfrak{b}$-generic we must have the equality*

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_0\underline{\lambda}_0 + X_0. \tag{20}$$

In consequence, we must have

$$M_{\mathfrak{b}} \in \mathcal{C} \quad \textit{and} \quad M_{\mathfrak{b}} r_{\mathfrak{b}} = X_{\mathfrak{b}} - X_0.$$

Proof. By (11), the right hand side of (19) is equal to $M_0(\underline{\lambda}_0 + \rho_0)$. On the other hand, the definition of genericity implies $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r_{\mathfrak{b}}$ so the left hand side is

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_{\mathfrak{b}}\underline{\lambda}_0 + (-M_{\mathfrak{b}} r_{\mathfrak{b}} + X_{\mathfrak{b}}).$$

We first show that the equivalence of the two sides can only be achieved by equality.

Namely, recall that λ is generic for a fixed $\mathfrak{b} \in \mathcal{B}^d$ only if $2\lambda_i \geq \ell_{i,\mathfrak{b}}$ for all i. Therefore we may freely choose generic λ sufficiently large such that the vector $M_0(\underline{\lambda}_0 + \rho_0)$ does not lie on any hyperplane of the form $x_i + \frac{1}{2} y_j = \pm 1/4$, where a symmetry of the form $\sim$ holds (with $\theta = 1/2$) and therefore no symmetry of the form (3) can apply. Similarly, the fact that $X_0 \neq 0$, and that equivalence must hold for the $\mathfrak{b}$-generic weights $s\underline{\lambda}_0$ for every $s \in \mathbb{Z}_{\geq 2}$, excludes the possibility that the equivalence is due to separate symmetry of the variables. Therefore, for all sufficiently large $\mathfrak{b}$-generic λ, the equivalence (19) must occur as an equality, that is, we have

$$M_{\mathfrak{b}}\underline{\lambda}_0 + (-M_{\mathfrak{b}} r_{\mathfrak{b}} + X_{\mathfrak{b}}) = M_0\underline{\lambda}_0 + X_0.$$

Since

$$\operatorname{span}\{\underline{\lambda}_0 \in \Omega_{m|2n} \mid \lambda \text{ is generic for } \mathfrak{b} \text{ and } \underline{\lambda}_0 \gg 0\} = \mathfrak{a}^*,$$

we deduce that this equality is in fact an equality of affine functions on $\mathfrak{a}^*$. Thus we have $X_0 = -M_{\mathfrak{b}} r_{\mathfrak{b}} + X_{\mathfrak{b}}$ and, by Lemma 4.3, we conclude $M_{\mathfrak{b}} \in \mathcal{C}$. □

5. The refined CEP for $\left(\mathfrak{gl}(V), \mathscr{S}^2(V)\right)$

We develop our solution to the refined CEP in stages of increasing complexity of the Borel subalgebra.

5.1. **The very even case.** We begin with the set of Borel subalgebras $\mathfrak{b}$ that have the property that $\underline{\lambda}_\mathfrak{b} \in \mathfrak{a}^*$ for all $\lambda \in \mathscr{H}(m|n)$: the ones for which for all k, the terms $\delta_{2k}\delta_{2k-1}$ are adjacent in the $\delta\epsilon$ sequence.

Definition 5.1. A decreasing Borel subalgebra $\mathfrak{b}$ and its associated $\delta\epsilon$ sequence are called *very even* if $\ell_{i,\mathfrak{b}}$ is even for all $1 \le i \le m$; equivalently, if $j_{2k-1} = j_{2k}$ for all $1 \le k \le n$.

Suppose $\mathfrak{b}$ is very even. For any $\lambda \in \mathscr{H}(m|n)$, the value $\ell(\lambda, i, \mathfrak{b}) = \min\{\ell_{i,\mathfrak{b}}, 2\lambda_i\}$ will also be even. Since $\mathfrak{a}^* = \operatorname{span}\{\delta_{2k-1} + \delta_{2k}, 2\epsilon_i \mid 1 \le k \le n, 1 \le i \le m\}$, it is immediate that the roots comprising $r(\lambda, \mathfrak{b})$ can be partitioned into pairs whose sums lie in this span, and so by Proposition 4.9 we have $\underline{\lambda}_\mathfrak{b} \in \mathfrak{a}^*$. Henceforth we write $\rho_\mathfrak{b}$ for the Weyl vector of $\mathfrak{b}$.

Lemma 5.2. *Suppose $\mathfrak{b}$ is very even and $\tau_\mathfrak{b} \colon \mathfrak{h}^* \to \mathbb{C}^{m|n}$ is an affine map satisfying* (19)*, in the form $\tau_\mathfrak{b}(x) = Mx + X_\mathfrak{b}$ for all $x \in \mathfrak{h}^*$. Then $X_\mathfrak{b} = M_0\rho_\mathfrak{b}$ where M_0 is as in* (10)*, and in fact, $\tau_\mathfrak{b}$ is independent of the choice of $M \in \mathcal{C}$.*

Note that since $\rho_\mathfrak{b} \notin \mathfrak{a}^*$, it is not true that $M\rho_\mathfrak{b} = X_\mathfrak{b}$ for all $M \in \mathcal{C}$.

Proof. Suppose $\mathfrak{b}$ is very even and suppose $\tau_\mathfrak{b}(x) = Mx + X$ is an affine function satisfying (19). By Proposition 4.13, we must have $M \in \mathcal{C}$ and $X = Mr_\mathfrak{b} + X_0$. If $M' \in \mathcal{C}$, then $M - M'$ is zero on $\mathfrak{a}^*$. Thus $M\underline{\lambda}_\mathfrak{b} = M'\underline{\lambda}_\mathfrak{b}$ for every $\lambda \in \mathscr{H}(m|n)$ and $M'r_\mathfrak{b} = Mr_\mathfrak{b}$, so if one choice of $M \in \mathcal{C}$ gives a function satisfying (19), so do all. Further, since by [CW13, Prop 1.33] we have $\rho_\mathfrak{b} = \rho_0 + r_\mathfrak{b}$, we readily compute $M_0\rho_\mathfrak{b} = M_0(\rho_0 + r_\mathfrak{b}) = X_0 + M_0r_\mathfrak{b} = X_\mathfrak{b}$. □

The rest of this subsection is devoted to proving the converse, that is, that for all very even $\mathfrak{b}$, the affine functions $\tau_\mathfrak{b}(x) = Mx + M_0\rho_\mathfrak{b}$ for $M \in \mathcal{C}$ satisfy (19). We first prove our key technical result, which gives a collection of terms that are equivalent to $\tau_0(\underline{\lambda}_0)$. By Lemma 5.2, it suffices to prove the result for $M = M_0$.

Proposition 5.3. *Suppose $\mathfrak{b}$ is very even and that $\underline{\lambda}_0 \in \Omega_{m|2n}$. Then we have*

$$M_0\underline{\lambda}_0 + X_0 \sim M_0(\underline{\lambda}_0 + r_\mathfrak{b} - r(\lambda, \mathfrak{b})) + X_0.$$

For ease of reference in the proof, for all $1 \le i \le m$ and $1 \le j \le n$, let

$$E_i \coloneqq \frac{m + 1 - 2n - 2i}{4} \text{ and } F_k \coloneqq \frac{m + 2 + 2n - 4k}{2}. \tag{21}$$

Then we have $X_0 = M_0\rho_0 = \sum_{i=1}^m E_i e_i + \sum_{k=1}^n F_k e_{m+k}$.

Proof. Suppose $\mathfrak{b}$ is very even and let I be the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$; then we must have $\ell_{I,\mathfrak{b}} \ge 2\lambda_I + 2$. Considering moreover the list of roots occuring in (18), whose sum is $r_\mathfrak{b} - r(\lambda, \mathfrak{b})$, we note that

$$\lambda_m + 1 \le \lambda_{m-1} + 1 \le \cdots \le \lambda_I + 1 \le \frac{1}{2}\ell_{I,\mathfrak{b}} \le \frac{1}{2}\ell_{I+1,\mathfrak{b}} \le \cdots \le \frac{1}{2}\ell_{m,\mathfrak{b}}.$$

That is, the intervals $(\lambda_i + 1, \ell_{i,\mathfrak{b}}/2)$ are nested, as i decreases.

To prove the proposition, we proceed by induction on i from m down to I on the rows of (18). Let

$$r_{\mathfrak{b},\lambda,j} = \sum_{k=2\lambda_j+1}^{\ell_{j,\mathfrak{b}}} \delta_k - (\ell_{j,\mathfrak{b}} - 2\lambda_j)\epsilon_j$$

denote the sum of the roots on the row of (18) corresponding to ϵ_j. Set

$$\Lambda_{i+1} = M_0(\underline{\lambda}_0 + \sum_{j=i+1}^{m} r_{\mathfrak{b},\lambda,j}) + X_0,$$

which is the partial sum obtained after completing the row indexed by $i+1$.

We will show that we can transform $\Lambda_{m+1} := M_0\underline{\lambda}_0 + X_0$, by a sequence of monoidal symmetries as in Lemma 2.3, into the final result

$$\Lambda_I = M_0(\underline{\lambda}_0 + r_{\mathfrak{b}} - r(\lambda, \mathfrak{b})) + X_0.$$

Our inductive hypothesis is that

$$\Lambda_{i+1} \sim \Lambda_i,$$

that the coefficient of e_i in Λ_{i+1} is $\lambda_i + E_i$ and that the coefficients of e_{m+j} in Λ_{i+1} with $\lambda_i + 1 \leq j \leq \frac{1}{2}\ell_{i,\mathfrak{b}}$ are given by $F_j - (m-i)$.

The base case is that $i = m$, so equivalence is a tautology and the statement about the coefficients is true. Suppose it holds for $i+1$. Let us prove the statement for i. Note that

$$\begin{aligned}
\Lambda_i &= \Lambda_{i+1} + M_0 r_{\mathfrak{b},\lambda,i} \\
&= \Lambda_{i+1} + M_0 \left(\left(\sum_{t=2\lambda_i+1}^{\ell_{i,\mathfrak{b}}} \delta_t \right) - (\ell_{i,\mathfrak{b}} - 2\lambda_i)\epsilon_i \right) \\
&= \Lambda_{i+1} + M_0 \left(\sum_{k=\lambda_i+1}^{\ell_{i,\mathfrak{b}}/2} (\delta_{2k-1} + \delta_{2k} - 2\epsilon_i) \right) \\
&= \Lambda_{i+1} + \sum_{k=\lambda_i+1}^{\ell_{i,\mathfrak{b}}/2} (e_i - e_{m+k}).
\end{aligned}$$

Suppose we have shown for some K, $\lambda_i + 1 \leq K < \ell_i/2$, that Λ_{i+1} is monoidally equivalent to

$$\Lambda_{i+1,K-1} = \Lambda_{i+1} + \sum_{k=\lambda_i+1}^{K-1} (e_i - e_{m+k}).$$

We now show that $\Lambda_{i+1,K-1} \sim \Lambda_{i+1,K}$. Then by induction as K runs from $\lambda_i + 1$ to $\ell_{i,\mathfrak{b}}/2$, we will conclude that $\Lambda_i \sim \Lambda_{i+1}$.

Using the outer induction hypothesis, as well as the formula for $\Lambda_{i+1,K-1}$, we compute that the coefficient of e_i in $\Lambda_{i+1,K-1}$ is $\lambda_i + E_i + (K - \lambda_i - 1)$ and the coefficient of e_{m+K} in $\Lambda_{i+1,K-1}$ is $F_K - (m-i)$. Since in this case we have $\theta = 1/2$, Lemma 2.3 says $y \sim y + e_i - e_{m+k}$ if $y_i + \frac{1}{2}y_{m+k} = -\frac{1}{4}$. Using (21), we compute

$$\lambda_i + E_i + (K - \lambda_i - 1) + \frac{1}{2}\left(F_K - (m-i)\right)) = -\frac{1}{4}$$

Therefore monoidal symmetry may be applied and we conclude that

$$\Lambda_{i+1,K-1} \sim \Lambda_{i+1,K-1} + e_i - e_{m+K} = \Lambda_{i+1,K},$$

as required. Therefore we have deduced by induction that

$$\Lambda_{i+1} \sim \Lambda_{i+1,\ell_i/2} = \Lambda_i.$$

Finally, if $i - 1 \geq I$, we compute the coefficients of Λ_i, to complete the outer induction.

The coefficient of e_{i-1} in Λ_i is the same as the coefficient of e_{i-1} in $M_0\underline{\lambda}_0 + X_0$, which is $\lambda_{i-1} + E_{i-1}$, because we have added no roots involving ϵ_{i-1}.

Since $i-1 \geq I$ and $\ell_{i-1,\mathfrak{b}}$ is even, we have $\lambda_i + 1 \leq \lambda_{i-1} + 1 \leq \frac{1}{2}\ell_{i-1,\mathfrak{b}} \leq \frac{1}{2}\ell_{i,\mathfrak{b}}$. Therefore the coefficient of e_{m+j}, for all $\lambda_{i-1} + 1 \leq j \leq \frac{1}{2}\ell_{i-1,\mathfrak{b}}$ was $F_j - (m-i)$ in Λ_{i+1} and is thus $F_j - (m-i) - 1$ in Λ_i, since we subtracted each e_{m+j} exactly once in the course of computing Λ_i. The induction is complete. □

We may now prove our main theorem for very even Borel subalgebras.

Theorem 5.4. *Let $\mathfrak{b}$ be a very even Borel subalgebra in $\mathcal{B}^d$. Set $X_\mathfrak{b} = M_0\rho_\mathfrak{b}$ and let $M_\mathfrak{b} \in \mathcal{C}$. Let $\tau_\mathfrak{b}$ be the affine function defined on $x \in \mathfrak{h}^*$ by*

$$\tau_\mathfrak{b}(x) = M_\mathfrak{b}x + X_\mathfrak{b}.$$

Then for every $\lambda \in \mathscr{H}(m|n)$ we have $\tau_\mathfrak{b}(\underline{\lambda}_\mathfrak{b}) \sim \tau_0(\underline{\lambda}_0)$ so that

$$P_{\mu,\frac{1}{2}} \circ \tau_\mathfrak{b}\left(\underline{\lambda}_\mathfrak{b}\right) = P_{\mu,\frac{1}{2}} \circ \tau_0\left(\underline{\lambda}_0\right).$$

Consequently, the eigenvalue of the Capelli operator D^μ on the irreducible component W_λ is given by the evaluation of a polynomial in the highest weight $\underline{\lambda}_\mathfrak{b}$ with respect to $\mathfrak{b}$.

Proof. It suffices to show $\tau_\mathfrak{b}(\underline{\lambda}_\mathfrak{b}) \sim \tau_0(\underline{\lambda}_0)$ for all $\lambda \in \mathscr{H}(m|n)$. We have that

$$\begin{aligned} M_\mathfrak{b}\underline{\lambda}_\mathfrak{b} + X_\mathfrak{b} &= M_\mathfrak{b}(\underline{\lambda}_0 - r(\lambda,\mathfrak{b})) + X_\mathfrak{b} \\ &= M_\mathfrak{b}(\underline{\lambda}_0 + r_\mathfrak{b} - r(\lambda,\mathfrak{b})) + (-M_\mathfrak{b}r_\mathfrak{b} + X_\mathfrak{b}). \end{aligned}$$

Since $\underline{\lambda}_0 + r_\mathfrak{b} - r(\lambda,\mathfrak{b}) \in \mathfrak{a}^*$, we may replace $M_\mathfrak{b}$ by M_0 in the first term, and as noted in Lemma 5.2 the latter term is equal to X_0. Therefore we have

$$M_\mathfrak{b}\underline{\lambda}_\mathfrak{b} + X_\mathfrak{b} = M_0(\underline{\lambda}_0 + r_\mathfrak{b} - r(\lambda,\mathfrak{b})) + X_0,$$

and thus by applying Proposition 5.3 we conclude that $M_\mathfrak{b}\underline{\lambda}_\mathfrak{b} + X_\mathfrak{b} \sim M_0\underline{\lambda}_0 + X_0$, which completes the proof. □

5.2. **A solution for generic weights.** From now on we suppose that $\mathfrak{b} \in \mathcal{B}^d$ is not very even. We first characterize the non-adjacent pairs $\{\delta_{2k}, \delta_{2k-1}\}$ in the $\delta\epsilon$ sequence corresponding to $\mathfrak{b}$.

Definition 5.5. For each $1 \leq k \leq n$, let

$$\begin{aligned} T_\mathfrak{b} &= \{1 \leq k \leq n \mid j_{2k,\mathfrak{b}} < j_{2k-1,\mathfrak{b}}\} \\ &= \{1 \leq k \leq n \mid \exists 1 \leq i \leq m : \ell_{i,\mathfrak{b}} = 2k-1\}. \end{aligned}$$

One can infer the equality of the given sets from the combinatorial description of $\delta\epsilon$ sequences in Definition 4.4.

Lemma 5.6. *Define $\mathfrak{b}_e \in \mathcal{B}^d$ by the values*

$$\ell_{i,\mathfrak{b}_e} = 2\left\lfloor \frac{1}{2}\ell_{i,\mathfrak{b}} \right\rfloor$$

for all $1 \leq i \leq m$. Then $\mathfrak{b}_e$ is a very even Borel subalgebra and the sequence of odd reflections r_α that take $\mathfrak{b}_e$ to $\mathfrak{b}$ corresponds to the set of roots $\{\delta_{2k-1} - \epsilon_i \mid k \in T_\mathfrak{b}, \ell_{i,\mathfrak{b}} = 2k-1\}$.

Proof. Since $\mathfrak{b}$ is a decreasing Borel subalgebra if and only if the sequence $(\ell_{i,\mathfrak{b}})_{i=1}^m$ is increasing (and bounded by $2n$), it follows that the increasing sequence $(\ell_{i,\mathfrak{b}_e})_{i=1}^m$ uniquely defines a decreasing Borel subalgebra $\mathfrak{b}_e$, and that it is very even. In the $\delta\epsilon$ sequence of $\mathfrak{b}_e$, the terms $\delta_{2k-1}\delta_{2k}$ are adjacent. For each $k \in T_{\mathfrak{b}}$ and i such that $\ell_{i,\mathfrak{b}} = 2k-1$, we have $\ell_{i,\mathfrak{b}_e} = 2k-2$, but for all other i, $\ell_{i,\mathfrak{b}}$ must be even and therefore $\ell_{i,\mathfrak{b}} = \ell_{i,\mathfrak{b}_e}$. Thus to obtain the $\delta\epsilon$ sequence of $\mathfrak{b}$ one must, for each $k \in T_{\mathfrak{b}}$, shift δ_{2k-1} to the right using the reflections corresponding to $\{\delta_{2k-1} - \epsilon_i \mid \ell_{i,\mathfrak{b}} = 2k-1\}$. □

Consequently, if we define for each $k \in T_{\mathfrak{b}}$

$$r_{odd,\mathfrak{b},k} := (j_{2k-1} - j_{2k})\delta_{2k-1} - \sum_{i=m-j_{2k-1}+1}^{m-j_{2k}} \epsilon_i, \tag{22}$$

then we have

$$r_{\mathfrak{b}} = r_{\mathfrak{b}_e} + \sum_{k \in T_{\mathfrak{b}}} r_{odd,\mathfrak{b},k}. \tag{23}$$

Definition 5.7. Let $\mathfrak{b} \in \mathcal{B}^d$. Set

$$\mathcal{C}_{\mathfrak{b}} = \{M \in \mathcal{C} \mid \forall k \in T_{\mathfrak{b}},\ Mr_{odd,\mathfrak{b},k} = 0\}.$$

Lemma 5.8. *The set $\mathcal{C}_{\mathfrak{b}}$ is nonempty.*

Proof. For any $M = M_0 + A' \in \mathcal{C}$ where $A' = [0_{(m+n)\times m}|A]$ for some A with columns satisfying $A_{2k} = -A_{2k-1}$ for all k, one has

$$Mr_{odd,\mathfrak{b},k} = M_0 r_{odd,\mathfrak{b},k} + A(j_{2k-1} - j_{2k})\delta_{2k-1};$$

thus to satisfy the condition $Mr_{odd,\mathfrak{b},k} = 0$, one must set

$$A_{2k-1} := \frac{-1}{j_{2k-1} - j_{2k}} M_0 r_{odd,\mathfrak{b},k}$$

and $A_{2k} := -A_{2k-1}$. These conditions are distinct as k runs over $T_{\mathfrak{b}}$. We conclude that $\mathcal{C}_{\mathfrak{b}}$ is nonempty. □

The set $\mathcal{C}_{\mathfrak{b}}$ contains a unique element if and only if $T_{\mathfrak{b}} = \{1, 2, \cdots, n\}$.

Proposition 5.9. *Set $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$ and suppose $\lambda \in \mathscr{H}(m|n)$ is $\mathfrak{b}$-generic. Then for all $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$ we have*

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_0\underline{\lambda}_0 + X_0.$$

Proof. Suppose λ is $\mathfrak{b}$-generic and suppose $M \in \mathcal{C}_{\mathfrak{b}}$. Then since

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r_{\mathfrak{b}} = \underline{\lambda}_{\mathfrak{b}_e} + \sum_{k \in T_{\mathfrak{b}}} r_{odd,\mathfrak{b},k},$$

we have, by the conditions defining $\mathcal{C}_{\mathfrak{b}}$ and the choice of $X_{\mathfrak{b}}$, that

$$M\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e}.$$

Moreover, if λ is $\mathfrak{b}$-generic, then it is also $\mathfrak{b}_e$-generic since $\ell_{i,\mathfrak{b}_e} \leq \ell_{i,\mathfrak{b}}$ for all $1 \leq i \leq m$. Therefore by Theorem 5.4 and Proposition 4.13, we infer that

$$M\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} = M\underline{\lambda}_0 + X_0,$$

as required. □

5.3. **A solution for the relatively even case.** Suppose now that λ is not $\mathfrak{b}$-generic, but that for each $k \in T_{\mathfrak{b}}$, the set $\mathcal{N}_{\mathfrak{b}}$ of roots defining $r(\lambda, \mathfrak{b})$ include either all, or none, of the roots defining $r_{odd,\mathfrak{b},k}$. We will call such λ *compatible* with $\mathfrak{b}$, but to define compatibility precisely, we must first make some observations about $\delta\epsilon$ sequences.

Suppose that in the $\delta\epsilon$ sequence for $\mathfrak{b}$, there exists at least one ϵ between δ_{2k} and δ_{2k-1}. Since j_{2k-1} is the number of ϵ to the left of δ_{2k-1}, and these are in decreasing order, we deduce that these terms are precisely $\epsilon_m, \cdots, \epsilon_{m-j_{2k-1}+1}$. By the same reasoning, $\epsilon_{m-j_{2k}}$ is the first ϵ to the right of δ_{2k}. Therefore, the $\delta\epsilon$ sequence contains

$$\cdots \delta_{2k}\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1}\delta_{2k-1} \cdots . \tag{24}$$

Now suppose that λ is not $\mathfrak{b}$-generic. Since the sequence of λ_i is decreasing, we have that

$$\lambda_{m-j_{2k}} \leq \lambda_{m-j_{2k}-1} \leq \cdots \leq \lambda_{m-j_{2k-1}+1}. \tag{25}$$

On the other hand, the value of ℓ_i, for each $m - j_{2k} \leq i \leq m - j_{2k-1} + 1$, is exactly $2k - 1$. Therefore, depending on the relation among $2k - 1, 2\lambda_{m-j_{2k}}$ and $2\lambda_{m-j_{2k-1}+1}$, we should subtract all, none or only some of the roots from the set

$$\{\delta_{2k-1} - \epsilon_{m-j_{2k}}, \ldots, \delta_{2k-1} - \epsilon_{m-j_{2k-1}+1}\}$$

to obtain $\underline{\lambda}_{\mathfrak{b}}$ from $\underline{\lambda}_0$.

These observations motivate the following definition.

Definition 5.10. We say that $\lambda \in \mathscr{H}(m|n)$ is *compatible* with $\mathfrak{b}$ if it is either $\mathfrak{b}$-generic or, for each $k \in T_{\mathfrak{b}}$, we have either $2\lambda_{m-j_{2k}} > 2k - 1$ or $2\lambda_{m-j_{2k-1}+1} < 2k - 1$.

Proposition 5.11. *Let $\mathfrak{b} \in \mathcal{B}^d$ and set $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$. Then for any $M \in \mathcal{C}_{\mathfrak{b}}$, the affine function*

$$\tau_{\mathfrak{b}}(x) = M_{\mathfrak{b}}x + X_{\mathfrak{b}}$$

satisfies $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$ for every $\lambda \in \mathscr{H}(m|n)$ that is compatible with $\mathfrak{b}$.

For ease of reference in the proof, for each λ that is compatible with $\mathfrak{b}$, define

$$T_{\mathfrak{b},\lambda} \coloneqq \{k \in T_{\mathfrak{b}} \mid 2\lambda_{m-j_{2k,\mathfrak{b}}} > 2k - 1\}. \tag{26}$$

Proof of Proposition 5.11. Suppose λ is compatible with $\mathfrak{b}$. If it is in fact $\mathfrak{b}$-generic then Proposition 5.9 implies that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = \tau_0(\underline{\lambda}_0)$, so we are done.

Otherwise, recall that the set of roots $\mathcal{G}_{\mathfrak{b}} \setminus \mathcal{N}_{\mathfrak{b}}$ determining $r_{\mathfrak{b}} - r(\lambda, \mathfrak{b})$ is given by (18), where $I = I_{\lambda,\mathfrak{b}}$ denotes the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$. Recall also that for each $k \in T_{\mathfrak{b}}$, $r_{odd,\mathfrak{b},k} = (j_{2k-1} - j_{2k})\delta_{2k-1} - \sum_{i=m-j_{2k-1}+1}^{m-j_{2k}} \epsilon_i$, and that for each index i in this sum, $\ell_i = 2k - 1$. The condition of compatibility implies that either

$$\ell_i = 2k - 1 < 2\lambda_{m-j_{2k}} \leq 2\lambda_{m-j_{2k}-1} \leq \cdots \leq 2\lambda_{m-j_{2k-1}+1}$$

in which case, $I > i$ and none of these roots appear in the expression (18), or

$$2\lambda_{m-j_{2k}} \leq 2\lambda_{m-j_{2k}-1} \leq \cdots \leq 2\lambda_{m-j_{2k-1}+1} < 2k - 1 = \ell_i$$

in which case $I \leq i$ and every root vector that occurs in $r_{odd,\mathfrak{b},k}$ appears in (18).

Therefore, the set $T_{\mathfrak{b},\lambda}$ is exactly the set of $k \in T_{\mathfrak{b}}$ such that none of the roots determining $r_{odd,\mathfrak{b},k}$ occur among the roots in (18); this set may be empty.

Therefore by construction we have

$$r(\lambda, \mathfrak{b}) = r(\lambda, \mathfrak{b}_e) + \sum_{k \in T_{b,\lambda}} r_{odd,\mathfrak{b},k},$$

and so we deduce

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\lambda, \mathfrak{b}) = \underline{\lambda}_0 - r(\lambda, \mathfrak{b}_e) - \sum_{k \in T_{\mathfrak{b},\lambda}} r_{odd,\mathfrak{b},k} = \underline{\lambda}_{\mathfrak{b}_e} - \sum_{k \in T_{\mathfrak{b},\lambda}} r_{odd,\mathfrak{b},k}. \tag{27}$$

Let $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$; since $T_{\mathfrak{b},\lambda} \subset T_{\mathfrak{b}}$, this implies

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} = M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}_e}) - \sum_{k \in T_{b,\lambda}} M_{\mathfrak{b}} r_{odd,\mathfrak{b},k} = M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}_e}).$$

Thus since $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$, we have that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e})$. Finally, as $\mathfrak{b}_e$ is very even and $M_{\mathfrak{b}} \in \mathcal{C}$, we can apply Theorem 5.4 to deduce that

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} \sim M_0\underline{\lambda}_0 + X_0,$$

and the proposition follows. □

We conclude this section by defining the class of Borel subalgebras $\mathfrak{b}$ for which all $\lambda \in \mathscr{H}(m|n)$ are compatible with $\mathfrak{b}$.

Definition 5.12. Let $\mathfrak{b} \in \mathcal{B}^d$. Then $\mathfrak{b}$, or its corresponding $\delta\epsilon$ sequence, is called *relatively even* if for every $1 \leq k \leq n$ we have $j_{2k-1} - j_{2k} \leq 1$, that is, if there is at most one ϵ between $\delta_{j_{2k}}$ and $\delta_{j_{2k-1}}$.

Corollary 5.13. *Suppose $\mathfrak{b} \in \mathcal{B}^d$ is relatively even. Then for any $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$, the affine function $\tau_{\mathfrak{b}}(x) = M_{\mathfrak{b}}x + X_{\mathfrak{b}_e}$ satisfies*

$$P_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}\left(\underline{\lambda}_{\mathfrak{b}}\right) = P_{\mu,\frac{1}{2}} \circ \tau_0\left(\underline{\lambda}_0\right)$$

for every $\lambda \in \mathscr{H}(m|n)$. Consequently, the eigenvalue of the Capelli operator D^{μ} on the irreducible component W_{λ} is given by the evaluation of a polynomial in the highest weight $\underline{\lambda}_{\mathfrak{b}}$ with respect to $\mathfrak{b}$.

Proof. Let $\lambda \in \mathscr{H}(m|n)$. If λ is not compatible with $\mathfrak{b}$, then there exists an index $k \in T_{\mathfrak{b}}$ for which

$$2\lambda_{m-j_{2k}} < 2k - 1 < 2\lambda_{m-j_{2k-1}+1}.$$

This is impossible if $j_{2k} = j_{2k-1}$ (as the λ_i are decreasing) and if $j_{2k-1} - j_{2k} = 1$ (as then $m - j_{2k} = m - j_{2k-1} + 1$). Therefore, if $\mathfrak{b}$ is relatively even it is compatible with every $\lambda \in \mathscr{H}(m|n)$. That the affine function $\tau_{\mathfrak{b}}$ gives a solution to the refined CEP thus follows from Proposition 5.11. □

5.4. **A different approach for $\mathfrak{b}$ that are not relatively even.** Unfortunately, following the approach of the preceding section to establish a formula for $\tau_{\mathfrak{b}}$ when $\mathfrak{b}$ is not relatively even leads to a piecewise affine function, as in [Cao22]. In this section, we follow a different approach.

Recall that the functions in $\Lambda_{m,n,\theta}$ are constant on the orbits $\mathcal{O}$ of elements $\mathbb{C}^{m+n}$ under monoidal symmetry. Since they are polynomial functions, they are constant on the Zariski closure of $\mathcal{O}$, which is a discrete set. Thus, whenever $\mathcal{O}$ is infinite, the polynomials in $\Lambda_{m,n,\theta}$ will not separate orbits, and we may have $P(x) = P(y)$ for x, y in distinct orbits, for all $P \in \Lambda_{m,n,\theta}$.

The question of when infinite orbits exist was answered in [SV17], and in particular this occurs for $\theta = 1/2$. The following is a consequence of their results.

Lemma 5.14. *Suppose $x \in \mathbb{C}^{m|n}$ and that $1 \leq i < i_0 \leq m$, $1 \leq j \leq n$. Let $\mathrm{orb}(x)$ denote the orbit of x under monoidal symmetry. If $x_i - x_{i_0} = 2x_i + x_{m+j} = -\frac{1}{2}$ then*

$$\overline{\mathrm{orb}(x)} \supset \{u \in \mathbb{C}^{m|n} \mid u_i - u_{i_0} = 2u_i + u_{m+j} = \pm\frac{1}{2}, u_s = x_s \forall s \notin \{i, i_0, m+j\}\}, \tag{28}$$

where $\overline{\mathrm{orb}(x)}$ denotes the Zariski closure of the orbit of x.

Proof. In [SV17, §5 (22)], the authors provide a key example for with $\theta = 1/2$. Namely, for any $a \in \mathbb{C}$, it is straightforward to show that

$$\begin{aligned}\mathcal{O}(a) = \{&(\ell + a - \frac{1}{2}, \ell + a, -2(\ell + a) + \frac{1}{2}), (\ell + a + \frac{1}{2}, \ell + a, -2(\ell + a) - \frac{1}{2}),\\ &(\ell + a, \ell + a - \frac{1}{2}, -2(\ell + a) + \frac{1}{2}), (\ell + a, \ell + a + \frac{1}{2}, -2(\ell + a) - \frac{1}{2}) \mid \ell \in \mathbb{Z}\}\end{aligned}$$

is an orbit in $\mathbb{C}^{2|1}$ under monoidal symmetry. The Zariski closure $\overline{\mathcal{O}(a)}$ is thus equal to $\cup_{a\in\mathbb{C}}\mathcal{O}(a)$, which is the union of the two affine lines:

$$u_1 - u_2 = 2u_1 + u_3 = \pm\frac{1}{2}, \quad u_2 - u_1 = 2u_2 + u_3 = \pm\frac{1}{2}.$$

Applying this to the triple $(i, i_0|j)$ of indices in place of $(1, 2|3)$ gives the result. □

We now proceed to the definition of our affine function, which we will denote $\tau_{\mathfrak{b}}^{\mathrm{full}}$ to distinguish it from the function $\tau_{\mathfrak{b}}$ defined on relatively even Borel subalgebras in the previous section.

Definition 5.15. Let $\mathfrak{b} \in \mathcal{B}^d$. Define

$$\mathcal{C}_{\mathfrak{b}}^{\mathrm{full}} = \{M \in \mathcal{C} \mid \forall k \in T_{\mathfrak{b}}, M\delta_{2k-1} = \frac{1}{2}e_{m-j_{2k,\mathfrak{b}}} - e_{m+k}\}.$$

Given any $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}^{\mathrm{full}}$, we set $X_{\mathfrak{b}} = M_{\mathfrak{b}}r_{\mathfrak{b}} + X_0$ and $\tau_{\mathfrak{b}}^{\mathrm{full}}(x) = M_{\mathfrak{b}}x + X_{\mathfrak{b}}$ for $x \in \mathfrak{h}^*$.

Note that the defining condition of $\mathcal{C}_{\mathfrak{b}}^{\mathrm{full}}$ basically describes the columns A_{2k-1} of $M_{\mathfrak{b}}$ for $k \in T_{\mathfrak{b}}$, hence $\mathcal{C}_{\mathfrak{b}}^{\mathrm{full}}$ is nonempty.

Remark 5.16. Indeed the value of $M_{\mathfrak{b}}r_{\mathfrak{b}}$ is independent of the choice of $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}^{\mathrm{full}}$. This can be seen as follows. By (23) we have $r_{\mathfrak{b}} = r_{\mathfrak{b}_e} + \sum_{k\in T_{\mathfrak{b}}} r_{odd,\mathfrak{b},k}$. Since $M_{\mathfrak{b}} \in \mathcal{C}$, the value of $M_{\mathfrak{b}}r_{\mathfrak{b}_e}$ is independent of the choice of $M_{\mathfrak{b}}$. Also, by the definition of $M_{\mathfrak{b}}$ the value of $M_{\mathfrak{b}}r_{odd,\mathfrak{b},k}$ is independent of the choice of $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}^{\mathrm{full}}$.

Theorem 5.17. *For every $\lambda \in \mathscr{H}(m|n)$ we have*

$$P_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}^{\mathrm{full}}(\underline{\lambda}_{\mathfrak{b}}) = P_{\mu,\frac{1}{2}} \circ \tau_0(\underline{\lambda}_0).$$

Consequently, the eigenvalue of the Capelli operator D^μ on the irreducible component W_λ is given by the evaluation of a polynomial in the highest weight $\underline{\lambda}_{\mathfrak{b}}$ with respect to $\mathfrak{b}$.

To prove this, we require several lemmas. As in Lemma 5.6, let $\mathfrak{b}_e$ be the closest very even Borel subalgebra to $\mathfrak{b}$; we will relate $\tau_{\mathfrak{b}}^{\mathrm{full}}$ to the function $\tau_{\mathfrak{b}_e}$ that was defined in Theorem 5.4. For ease of notation, set $\ell_i = \ell_{i,\mathfrak{b}}$ and $j_t = j_{t,\mathfrak{b}}$ in what follows.

For each $k \in T_{\mathfrak{b}}$, the $\delta\epsilon$ sequence for $\mathfrak{b}$ contains

$$\cdots \delta_{2k}\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1}\delta_{2k-1} \cdots$$

whereas that for $\mathfrak{b}_e$ contains

$$\cdots \delta_{2k}\delta_{2k-1}\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1} \cdots . \tag{29}$$

Thus $\ell_i = 2k-1$ and $\ell_{i,\mathfrak{b}_e} = 2k-2$ for $m-j_{2k-1}+1 \le i \le m-j_{2k}$; similarly, $j_{2k-1,\mathfrak{b}_e} = j_{2k,\mathfrak{b}_e}$ whereas by hypothesis $j_{2k-1} - j_{2k} \ge 1$.

We have

$$r_{\mathfrak{b}} = r_{\mathfrak{b}_e} + \sum_{k \in T_{\mathfrak{b}}} r_{odd,\mathfrak{b},k}$$

as in (23), where

$$r_{odd,\mathfrak{b},k} = (\delta_{2k-1} - \epsilon_{m-j_{2k}}) + (\delta_{2k-1} - \epsilon_{m-j_{2k}-1}) + \cdots + (\delta_{2k-1} - \epsilon_{m-j_{2k-1}+1}).$$

To obtain $\mathfrak{b}$ from $\mathfrak{b}_e$, we must apply the reflections in the roots $\delta_{2k-1} - \epsilon_i$ in decreasing lexicographic order, that is, for k decreasing in $T_{\mathfrak{b}}$ and then i decreasing from $m - j_{2k}$ until $m - j_{2k-1} + 1$.

Definition 5.18. Write $(k,i) \succeq (k',i')$ if we have $k, k' \in T_{\mathfrak{b}}$, $m - j_{2k-1} + 1 \le i \le m - j_{2k}$, $m - j_{2k'-1} + 1 \le i' \le m - j_{2k'}$ and $k \ge k'$, $i \ge i'$. Write $(k,i) \succ (k',i')$ if $(k,i) \succeq (k',i')$ and $(k,i) \ne (k',i')$.

Note that we retain the index k in the notation only for convenience. For two such pairs, the statement $(k,i) \succ (k',i')$ is equivalent to stating $i > i'$, as in each pair the first term is fully determined by the second. In particular, we may write $(k,i) \succ (k',i')$ for any $1 \le i' \le m$, not only those corresponding to $k' \in T_{\mathfrak{b}}$ and $m - j_{2k'+1} \le i' \le m - j_{2k'}$, without ambiguity. We require two technical lemmas.

Lemma 5.19. *Fix $M_{\mathfrak{b}} \in \mathcal{C}^{\mathrm{full}}_{\mathfrak{b}}$ and let $k \in T_{\mathfrak{b}}$. Suppose the coefficient of $\epsilon_{m-j_{2k}}$ in $\underline{\lambda}_{\mathfrak{b}_e}$ is zero. Then we have*

$$\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{k'>k} r_{odd,\mathfrak{b},k'} \sim \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{k'>k} r_{odd,\mathfrak{b},k'} + (e_{m-j_{2k}} - e_{m+k}) \tag{30}$$

where $\sim$ denotes monoidal symmetry.

Proof. Recall that $\tau_{\mathfrak{b}_e}(x) = M_{\mathfrak{b}_e} x + X_{\mathfrak{b}_e} = M_0 x + M_0 r_{\mathfrak{b}_e} + X_0$. As in the proof of Proposition 5.3, by a direct calculation using (18) we have

$$M_0(r_{\mathfrak{b}_e} - r(\lambda, \mathfrak{b}_e)) = \sum_{i=I_{\lambda,\mathfrak{b}_e}}^{m} \sum_{j=\lambda_i+1}^{\frac{1}{2}\ell_{i,\mathfrak{b}_e}} (e_i - e_{m+j}).$$

Hence

$$\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) = M_{\mathfrak{b}_e}\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} = M_0\underline{\lambda}_0 + X_0 + \sum_{i=I_{\lambda,\mathfrak{b}_e}}^{m} \sum_{j=\lambda_i+1}^{\frac{1}{2}\ell_{i,\mathfrak{b}_e}} (e_i - e_{m+j}). \tag{31}$$

The hypothesis implies $2\lambda_{m-j_{2k}} \le \ell_{m-j_{2k},\mathfrak{b}_e} = 2k-2$. Thus for $i > m - j_{2k}$ we have

$$2\lambda_i \le 2\lambda_{m-j_{2k}} \le \ell_{m-j_{2k},\mathfrak{b}_e} \le \ell_{i,\mathfrak{b}_e} - 2.$$

This implies that $I_{\lambda,\mathfrak{b}_e} \le m - j_{2k} + 1$.

Recall the formula for the sums $r_{odd,\mathfrak{b},k}$ defined in (22). First note that the coefficients of $e_{m-j_{2k}}$ and e_{m+k} in $M_{\mathfrak{b}} \sum_{k'>k} r_{odd,\mathfrak{b},k'}$ are zero. Namely, by the definition of $M_{\mathfrak{b}}$ we have that $M_{\mathfrak{b}}(\epsilon_i) = -\frac{1}{2}e_i$ and $M_{\mathfrak{b}}(\delta_{2k'-1}) = \frac{1}{2}e_{m-j_{2k'}} - e_{m+k'}$. The only pairs (k',i) occuring in this sum have $k' > k$ and $i > m - j_{2k}$.

Next we compute the coefficients of $e_{m-j_{2k}}$ and e_{m+k} in $\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) = \sum_i x_i e_i$.

Let us begin with $x_{m-j_{2k}}$. The coefficient of $e_{m-j_{2k}}$ in $M_0\underline{\lambda}_0$ is $\lambda_{m-j_{2k}}$ and in X_0 is $E_{m-j_{2k}}$. If it happens that $I_{\lambda,\mathfrak{b}_e} \le m - j_{2k}$, then the index $i = m - j_{2k}$ occurs in the sum (31) whence

the coefficient of $e_{m-j_{2k}}$ in this sum is $\frac{1}{2}\ell_{m-j_{2k},\mathfrak{b}_e} - \lambda_{m-j_{2k}}$. Since $\frac{1}{2}\ell_{m-j_{2k},\mathfrak{b}_e} = k-1$, we deduce that $x_{m-j_{2k}} = E_{m-j_{2k}} + k - 1$. If indeed $I_{\lambda,\mathfrak{b}_e} = m - j_{2k} + 1$, then we must have $2\lambda_{m-j_{2k}} = 2k-2$ and the index $i = m - j_{2k}$ does not occur in (31), hence again we have

$$x_{m-j_{2k}} = \lambda_{m-j_{2k}} + E_{m-j_{2k}} = E_{m-j_{2k}} + k - 1.$$

Next we compute x_{m+k}. Since $2\lambda_m \leq 2\lambda_{m-j_{2k}} \leq \ell_{m-j_{2k},\mathfrak{b}_e} = 2k-2$, we deduce from the structure of the hook partition that $\mu_{2k-1} = \mu_{2k} = 0$, so the coefficient of e_{m+k} in $M_0\underline{\lambda}_0$ is zero. The coefficient of e_{m+k} in X_0 is F_k. The coefficient of e_{m+k} in the sum (31) is the number of indices i such that $\lambda_i < k \leq \ell_{i,\mathfrak{b}_e}/2$. The inequality $k \leq \ell_{i,\mathfrak{b}_e}$ implies ϵ_i is to the left of δ_{2k} in $\mathfrak{b}_e$ in (29) and in particular $i > m - j_{2k}$; but then $\lambda_i \leq \lambda_{m-j_{2k}} \leq k-1$ holds. Thus the coefficient is $-(m-(m-j_{2k})) = -j_{2k}$, whence $x_{m+k} = F_k - j_{2k}$.

Now note that $E_{m-i} + \frac{1}{2}F_j = \frac{3}{4} + \frac{1}{2}i - j$, so that

$$x_{m-j_{2k}} + \frac{1}{2}x_{m+k} = \frac{3}{4} + \frac{1}{2}j_{2k} - k + k - 1 - j_{2k} = -\frac{1}{4},$$

which implies that monoidal symmetry may be applied, yielding the result. □

Lemma 5.20. *Fix $M_\mathfrak{b} \in \mathcal{C}_\mathfrak{b}^{\mathrm{full}}$. Let $k \in T_\mathfrak{b}$ and suppose $m - j_{2k-1} + 1 \leq i < m - j_{2k}$. Suppose the coefficient of ϵ_i in $\underline{\lambda}_{\mathfrak{b}_e}$ is zero. Then we have*

$$\begin{aligned}(32)\qquad &\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_\mathfrak{b} \sum_{(k',i')\succ(k,i)} (\delta_{2k'-1} - \epsilon_{i'}) \\ &\sim_E \quad \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_\mathfrak{b} \sum_{(k',i')\succ(k,i)} (\delta_{2k'-1} - \epsilon_{i'}) + \left(\frac{1}{2}e_{m-j_{2k}} + \frac{1}{2}e_i - e_{m+k}\right)\end{aligned}$$

although they are not related by monoidal symmetry.

Proof. We set $i_0 = m - j_{2k}$ and show that with $x = \sum_t x_t e_t$ equal to the left hand side above, we may apply Lemma 5.14 to the triple (i, i_0, k). Since $\frac{1}{2}e_{m-j_{2k}} + \frac{1}{2}e_i - e_{m+k}$ has coefficients $(\frac{1}{2}, \frac{1}{2}, -1)$, it then follows that the left side of the relation lies in $\overline{\mathrm{orb}(x)}$ as well, as required.

As noted in the previous proof, the coefficients of $\{e_i, e_{i_0}, e_{m+k}\}$ in x coincide with those of

$$(33)\qquad \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_\mathfrak{b} \sum_{i<i'\leq m-j_{2k}} (\delta_{2k-1} - \epsilon_{i'})$$

as the terms with $k' > k$ cannot contribute.

First we compute the coefficients of $\{e_i, e_{i_0}, e_{m+k}\}$ in the right hand side of (31). Since the coefficient of ϵ_i in $\underline{\lambda}_{\mathfrak{b}_e}$ is zero, we have $\lambda_{m-j_{2k}} \leq \lambda_i \leq \ell_{i,\mathfrak{b}_e}/2 = k-1$. It follows that for $i' > m - j_{2k}$ we have

$$\lambda_{i'} \leq \lambda_{m-j_{2k}} \leq k-1 < \ell_{i',\mathfrak{b}_e}/2$$

and in particular, $I_{\lambda,\mathfrak{b}_e} \leq m - j_{2k} + 1$.

Now consider any $s \leq m$. The coefficient of e_s in $M_0\lambda_0 + X_0$ is $\lambda_s + E_s$. If $s \geq I_{\lambda,\mathfrak{b}_e}$, then the coefficient of e_s in the sum on the right hand side of (31) is $\frac{1}{2}\ell_{s,\mathfrak{b}_e} - \lambda_s = k-1-\lambda_s$. Hence the coefficient of $e-s$ on the right hand side of (31) is $\lambda_s + E_s + (k-1-\lambda_s) = E_s + k - 1$. If $i \leq s < I_{\lambda,\mathfrak{b}_e}$, then e_s does not occur in the sum on the right hand side of (31) but we have

$$k - 1 = \ell_{i,\mathfrak{b}_e} \leq \ell_{s,\mathfrak{b}_e} \leq \lambda_s \leq \lambda_i \leq k-1,$$

from which it follows that $\lambda_s = k-1$. This means that in this case also the coefficient of e_s on the right hand side of (31) is equal to $\lambda_s + E_s = E_s + k - 1$. As a consequence, we obtain that the coefficient of e_s for $s = i, i_0$ in (31) is $E_s + k - 1$.

On the other hand, from $\lambda_m \leq k-1$ it follows that $\mu_{2k-1} = \mu_{2k} = 0$ so the coefficient of e_{m+k} in $M_0\lambda_0 + X_0$ is F_k. As in the proof of Lemma 5.19, the coefficient of e_{m+k} in the sum on the right hand side of (31) is $-j_{2k}$.

We now compute the contributions of the second term in (33). First note that

$$\sum_{i<i'\leq m-j_{2k}} (\delta_{2k-1} - \epsilon_{i'}) = (m - j_{2k} - i)\delta_{2k-1} - \sum_{i'=i+1}^{m-j_{2k}} \epsilon_{i'}.$$

Therefore from the definition of $M_{\mathfrak{b}}$ we have

$$\begin{aligned} M_{\mathfrak{b}} \sum_{i<i'\leq m-j_{2k}} (\delta_{2k-1} - \epsilon_{i'}) &= (m - j_{2k} - i)M_{\mathfrak{b}}\delta_{2k-1} - \sum_{i'=i+1}^{m-j_{2k}} M_{\mathfrak{b}}\epsilon_{i'} \\ &= (m - j_{2k} - i)(\frac{1}{2}e_{m-j_{2k}} - e_{m+k}) - \sum_{i'=i+1}^{m-j_{2k}} (-\frac{1}{2}e_{i'}). \end{aligned}$$

Therefore, altogether we have

$$\begin{aligned} x_i &= E_i + k - 1 \\ x_{m-j_{2k}} &= E_{m-j_{2k}} + k - 1 + \frac{1}{2}(m - j_{2k} - i) + \frac{1}{2} \\ x_{m+k} &= F_k - j_{2k} - (m - j_{2k} - i) = F_k - m + i. \end{aligned}$$

We compute

$$\begin{aligned} x_i - x_{m-j_{2k}} &= E_i - E_{m-j_{2k}} - \frac{1}{2}(m - j_{2k} - i + 1) \\ &= \frac{1}{4}(-2i + 2(m - j_{2k})) - \frac{1}{2}(m - j_{2k} - i + 1) = -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} 2x_i + x_{m+k} &= 2(E_i + k - 1) + (F_k - m + i) \\ &= \frac{1}{2}(m - 2n + 1 - 2i) + 2k - 2 + \frac{1}{2}(m + 2n + 2 - 4k) - m + i \\ &= \frac{1}{2} - 2 + 1 = -\frac{1}{2}. \end{aligned}$$

It follows from Lemma 5.14 that the orbit of x is infinite and that $x + \frac{1}{2}e_{m-j_{2k}} + \frac{1}{2}e_i - e_{m+k}$ lies in its Zariski closure. □

Proof of Theorem 5.17. We begin by relating $\tau_{\mathfrak{b}}^{\text{full}}$ to $\tau_{\mathfrak{b}_e}$. Let $I = I_{\lambda,\mathfrak{b}}$ be the least index i for which $2\lambda_i < \ell_{i,\mathfrak{b}}$. Then $2\lambda_I \leq \ell_{I,\mathfrak{b}_e}$ and so by Remark 4.10, for every $j \geq I$, the coefficient of ϵ_j in $\underline{\lambda}_{\mathfrak{b}_e}$ is zero.

Set $k_\lambda = \lceil (1+\ell_{I,\mathfrak{b}})\rceil/2$; if $\ell_{I,\mathfrak{b}}$ is odd, this is the index such that $m - j_{2k_\lambda} + 1 \leq I \leq m - j_{2k_\lambda}$, and otherwise it need not be a relevant index. Then

$$r(\lambda, \mathfrak{b}) = r(\lambda, \mathfrak{b}_e) + \sum_{(k,i)\prec(k_\lambda, I)} (\delta_{2k-1} - \epsilon_i).$$

Since $M_{\mathfrak{b}} \in \mathcal{C}$ and $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_{\mathfrak{b}_e} - (r(\lambda, \mathfrak{b}) - r(\lambda, \mathfrak{b}_e))$, for any λ we have

$$\begin{aligned} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} &= M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}_e} - (r(\lambda, \mathfrak{b}) - r(\lambda, \mathfrak{b}_e))) + (M_{\mathfrak{b}} r_{\mathfrak{b}} + X_0) \\ &= M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} + M_{\mathfrak{b}}((r_{\mathfrak{b}} - r(\lambda, \mathfrak{b})) - (r_{\mathfrak{b}_e} - r(\lambda, \mathfrak{b}_e))) \\ &= \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{(k,i) \succeq (k_\lambda, I)} (\delta_{2k-1} - \epsilon_i). \end{aligned}$$

We now proceed by induction from $J = m+1$ down to $J = I$, showing that (with $K_J = \lceil (1 + \ell_{J,\mathfrak{b}})\rceil/2$) if

$$\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{(k,i) \succeq (K_J, J)} (\delta_{2k-1} - \epsilon_i) \sim_E \tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) \tag{34}$$

for all λ such that the coefficient of ϵ_J in $\underline{\lambda}_{\mathfrak{b}_e}$ is 0, then this same relation also holds when we replace J with $J-1$ and apply it to all λ such that the coefficient of ϵ_{J-1} in $\underline{\lambda}_{\mathfrak{b}_e}$ is 0. Note that if the coefficient of ϵ_i in $\overline{\lambda}_{\mathfrak{b}_e}$ is zero, then so is the coefficient of ϵ_j for any $j > i$; thus the induction hypothesis will apply.

First suppose that $J = m+1$ (which corresponds, for example, to the generic case). Then the sum over $(k,i) \succeq (K_J, J)$ is empty, so there is nothing to show.

Next suppose the result for J. Let λ be such that the coefficient of ϵ_{J-1} in $\underline{\lambda}_{\mathfrak{b}_e}$ is 0. If $\ell_{J-1,\mathfrak{b}} = \ell_{J-1,\mathfrak{b}_e}$, then $J-1$ does not occur as the second index of any pair (k,i) in (34), so we have equality of left hand sides, and are done. Otherwise, let k be the index such that $m - j_{2k-1} + 1 \leq J - 1 \leq m - j_{2k}$.

If $J - 1 = m - j_{2k}$ then $k = K_{J-1} < K_J$. Thus the left hand side of (34) is equal to the left hand side of (30). By Lemma 5.19, with $k = K_{J-1}$, this is monoidally equivalent to

$$\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{k' > k} r_{odd,\mathfrak{b},k'} + (e_{m-j_{2k}} - e_{m+k}).$$

Since $M_{\mathfrak{b}}(\delta_{2k-1} - \epsilon_{m-j_{2k}}) = e_{m-j_{2k}} - e_{m+k}$, the latter formula is (34) for $J-1$ and we are done by the induction hypothesis.

Finally, if $m - j_{2k-1} + 1 \leq J - 1 < m - j_{2k}$ then $K_J = K_{J-1} = k$. Thus the left hand side of (34) is equal to the left hand side of (32) with $i = J - 1$. By Lemma 5.20, this is equivalent (in the sense of $\sim_E$) to

$$\tau_{\mathfrak{b}_e}(\underline{\lambda}_{\mathfrak{b}_e}) + M_{\mathfrak{b}} \sum_{(k',i') \succ (k,i)} (\delta_{2k'-1} - \epsilon_{i'}) + \left(\frac{1}{2} e_{m-j_{2k}} + \frac{1}{2} e_i - e_{m+k} \right).$$

Since $M_{\mathfrak{b}}(\delta_{2k-1} - \epsilon_i) = \frac{1}{2} e_{m-j_{2k}} + \frac{1}{2} e_i - e_{m+k}$, the latter formula is also (34) for $J-1$, and again we are done.

Since the relation $\sim_E$ ensures that composing with $P_{\mu,\frac{1}{2}}$, for any $\mu \in \mathscr{H}(m|n)$, gives the same value, this completes the proof of the theorem. □

Remark 5.21. Note that if $\mathfrak{b}$ is very even, then $\tau_{\mathfrak{b}}^{\text{full}} = \tau_{\mathfrak{b}}$ and thus Theorem 5.4 is a special case of Theorem 5.17.

However, when $\mathfrak{b}$ is relatively even, then for each k such that $j_{2k-1} - j_{2k} = 1$, Definition 5.7 sets $M(\delta_{2k-1} - \epsilon_{m-j_{2k}}) = 0$ for each $M \in \mathcal{C}_{\mathfrak{b}}$, whereas Definition 5.15 sets $M(\delta_{2k-1} - \epsilon_{m-j_{2k}}) = e_{m-j_{2k}} - e_{m+k}$ for each $M \in \mathcal{C}_{\mathfrak{b}}^{\text{full}}$. Thus $\tau_{\mathfrak{b}}^{\text{full}} \neq \tau_{\mathfrak{b}}$ but (as can be deduced from the proof of Lemma 5.19), they are monoidally equivalent on all highest weights $\underline{\lambda}_{\mathfrak{b}}$. Interestingly, as Example 5.22 below shows, the choice of $M\delta_{2k-1}$ made in Definition 5.7 cannot be extended to define a valid $\tau_{\mathfrak{b}}$ in the non-relatively even case, and so Corollary 5.13

is a separate solution to the refined Capelli eigenvalue problem that is neither a consequence nor a special case of Theorem 5.17.

Finally, note that the key step (32) of Lemma 5.20, which is invoked precisely for the non-relatively even Borel subalgebras, was not one of monoidal equivalence or supersymmetry, but rather of the weaker equivalence relation $\sim_E$, thus showing that monoidal equivalence alone does not suffice. Nevertheless, a key result of [Cao22] is that for any $\mathfrak{b}' \in \mathcal{B}^d$, one can define *piecewise* affine functions τ' that satisfy $\tau'(\underline{\lambda}_{\mathfrak{b}'}) \sim \tau_0(\underline{\lambda}_0)$ for all $\lambda \in \Omega_{m|n}$.

It is natural to ask if the solution in Theorem 5.17 is unique. Here we give an example to illustrate this can be true.

Example 5.22. Let $m = 2$ and $n = 1$, so that $\mathfrak{g} = \mathfrak{gl}(2|2)$. The Weyl vector corresponding to the opposite standard Borel subalgebra is $\rho_0 = (1/2, 3/2, -3/2, -1/2)$ and thus $X_0 = M_0\rho_0 = (-1/4, -3/4, 1)$, where M_0 is given in (10). Consider the Borel subalgebra $\mathfrak{b}$ corresponding to the $\delta\epsilon$ sequence $\delta_2\epsilon_2\epsilon_1\delta_1$. This is the smallest example with the property that $j_{2k-1,\mathfrak{b}} - j_{2k,\mathfrak{b}} \geq 2$ for some $1 \leq k \leq n$. Since $\mathfrak{b}_0$ corresponds to $\delta_2\delta_1\epsilon_2\epsilon_1$, we know $\mathcal{G}_{\mathfrak{b}} = \{\delta_1 - \epsilon_2, \delta_2 - \epsilon_1\}$. By Proposition 4.9, we compute the highest weights $\underline{\lambda}_{\mathfrak{b}}$ for $\lambda \in \mathscr{H}(2|1)$ as in Table 1.

λ	$\underline{\lambda}_0$	$\underline{\lambda}_{\mathfrak{b}}$	(condition)
$(r, s\|t)$	$-(2r, 2s, t, t)$	$-(2r-1, 2s-1, t+2, t)$	$r \geq s > 0$, $t \geq 0$, $r, s, t \in \mathbb{Z}$
(r)	$-(2r, 0, 0, 0)$	$-(2r-1, 0, 1, 0)$	$r > 0$, $r \in \mathbb{Z}$
$\emptyset$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	

TABLE 1. The $\mathfrak{b}_0$- and $\mathfrak{b}$-highest weights of W_λ for all $\lambda \in \mathscr{H}(2|1)$.

Suppose now that $\tau'(x) = M'x + X'$ is an affine map such that

$$P_{\mu,\frac{1}{2}}(\tau'(\underline{\lambda}_{\mathfrak{b}})) = P_{\mu,\frac{1}{2}}(\tau_0(\underline{\lambda}_0)) \quad \text{for all } \lambda \in \mathscr{H}(2|1). \tag{35}$$

Note that the orbits of $\underline{\lambda}_0$ under monoidal symmetry, for λ $\mathfrak{b}$-generic, are finite, so (35) holds if and only if the elements are equivalent under monoidal symmetry, and as noted in Proposition 4.13, we must have $M' \in \mathcal{C}$, so that it takes the form

$$M' = \begin{bmatrix} -1/2 & 0 & a & -a \\ 0 & -1/2 & b & -b \\ 0 & 0 & -1/2 + c & -1/2 - c \end{bmatrix} \tag{36}$$

for some $a, b, c \in \mathbb{C}$. Since in this case $r_{\mathfrak{b}} = -\epsilon_1 - \epsilon_2 + 2\delta_1$, the proposition also yields $X' - X_0 = M_{\mathfrak{b}} r_{\mathfrak{b}} = (\frac{1}{2} + 2a, \frac{1}{2} + 2b, -1 + 2c) \in \mathbb{C}^{2|1}$, so that

$$X' = (1/4 + 2a, -1/4 + 2b, 2c), \tag{37}$$

with a, b, c as in (36).

Next consider the highest weights of the form $\underline{\lambda}_0 = (2r, 0, 0, 0)$ with $r > 0$. Then we have $M_0\underline{\lambda}_0 + X_0 = (r - 1/4, -3/4, 1)$. Applying Lemma 2.3, and separate symmetry in the first two coordinates, we deduce that for $r \geq 1$ this vector is monoidally equivalent only to the four vectors

$$(r - \frac{1}{4}, -\frac{3}{4}, 1), \quad (r - \frac{1}{4}, \frac{1}{4}, 0), \quad (-\frac{3}{4}, r - \frac{1}{4}, 1), \quad \text{and} \quad (\frac{1}{4}, r - \frac{1}{4}, 1).$$

Since this orbit is finite, it again coincides with its equivalence class under $\sim_E$. Therefore to satisfy (35) for each $r \geq 1$, the vector $M'\underline{\lambda}_{\mathfrak{b}} + X'$ must occur on this list. Applying (36) and (37), we compute

$$M'\underline{\lambda}_{\mathfrak{b}} + X' = M'(-2r+1, 0, -1, 0) + X' = (r - 1/4 + a, -1/4 + b, 1/2 + c),$$

whence we must have $a = 0$ and either $c = -b = 1/2$ or $c = -b = -1/2$. These two choices (which correspond to Definitions 5.7 and 5.15, respectively), give

$$X' = (1/4, -5/4, 1) \quad \text{and} \quad X' = (1/4, 3/4, -1),$$

respectively.

Finally, to satisfy (35) when $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 = 0$, we apply Lemma 5.14 to deduce that we should have $X' \in \cup_{d \in \mathbb{C}} \mathcal{O}(d)$, which yields the unique choice $X' = (\frac{1}{4}, \frac{3}{4}, -1)$.

We conclude that τ' is uniquely determined by (35) in this case, and is given by $X' = (\frac{1}{4}, \frac{3}{4}, -1)$ and

$$M' = \begin{bmatrix} -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

which corresponds with the map $\tau_{\mathfrak{b}}^{\text{full}}$ of Definition 5.15.

Appendix A. A normalization for Capelli operators

Let W be a $\mathbb{Z}_2$-graded vector space. Then there exists a natural pairing

$$\langle \cdot, \cdot \rangle : W^{\otimes d} \times (W^*)^{\otimes d} \to \mathbb{C}\ ,\ \langle v_1 \otimes \cdots \otimes v_d, v_1^* \otimes \cdots \otimes v_d^* \rangle := \prod_{i=1}^{d} \langle v_{d-i+1}, v_i^* \rangle,$$

where $\langle v, v^* \rangle := v^*(v)$. This gives an isomorphism

$$W^{\otimes d} \xrightarrow{\cong} \left((W^*)^{\otimes d}\right)^*. \tag{38}$$

Now recall the canonical isomorphisms

$$\mathscr{S}^d(W) \cong (W^{\otimes d})^{S_d} \quad \text{and} \quad \mathscr{P}^d(W) \cong \mathscr{S}^d(W^*) \cong \left((W^*)^{\otimes d}\right)^{S_d}. \tag{39}$$

From the maps (38) and (39) we obtain a linear map

$$\boldsymbol{\varphi} : \mathscr{S}^d(W) \to \mathscr{P}^d(W)^* \tag{40}$$

as the composition of the maps

$$\mathscr{S}^d(W) \xrightarrow{\boldsymbol{i}} W^{\otimes d} \xrightarrow{\cong} \left((W^{\otimes d})^*\right)^* \xrightarrow{\boldsymbol{p}} \mathscr{P}^d(W)^*, \tag{41}$$

where $\boldsymbol{i}$ is induced by the inclusion $(W^{\otimes d})^{S_d} \subseteq W^{\otimes d}$ and $\boldsymbol{p}$ is induced by the projection that is dual to the inclusion $((W^*)^{\otimes d})^{S_d} \to (W^*)^{\otimes d}$. It is straightforward to verify that the map $\boldsymbol{\varphi}$ is indeed an isomorphism of $\mathfrak{gl}(W)$-modules. In particular, given $v \in W_{\bar{0}}$ and $v^* \in (W^*)_{\bar{0}}$, for the d-tensors $v^{\otimes d} \in \mathscr{S}^d(W)$ and $(v^*)^{\otimes d} \in \mathscr{P}^d(W)$ we have

$$\langle v^{\otimes d}, (v^*)^{\otimes d} \rangle = \langle v, v^* \rangle^d.$$

We can also embed $\mathscr{S}(W)$ in $\mathrm{End}(\mathscr{P}(W))$ as follows. First for any homogeneous $v \in W \cong \mathscr{S}^1(W)$ we define a superderivation ∂_v on $\mathscr{P}(W)$ by setting

$$\partial_v(v^*) := \langle v, v^* \rangle \quad \text{for } v^* \in W^* \cong \mathscr{P}^1(W),$$

and then extending ∂_v to all of $\mathscr{P}(W)$ by the Leibniz rule

$$\partial_v(ab) := \partial_v(a)b + (-1)^{|v|\cdot|a|}a\partial_v(b).$$

The resulting derivations ∂_v supercommute, and therefore we can extend the embedding $\partial : W \to \mathrm{End}(\mathscr{P}(W))$ to all of $\mathscr{S}(W)$ by the universal property of $\mathscr{S}(W)$, *i.e.*, by setting

$$\partial_{v_1\cdots v_d} := \partial_{v_1}\cdots\partial_{v_d} \text{ for } v_1,\ldots,v_d \in W.$$

This map identifies $\mathscr{S}(W)$ with the algebra $\mathscr{D}(W)$ of constant-coefficient superdifferential operators on $\mathscr{P}(W)$.

Clearly $\partial_u(\mathscr{P}^d(W)) \subseteq \mathscr{P}^{d-d'}(W)$ for $u \in \mathscr{S}^{d'}(W)$. In particular, when $d' = d$ we have $\mathscr{P}^0(W) \cong \mathbb{C}$, hence we obtain a linear map

$$\boldsymbol{\varphi}' : \mathscr{S}^d(W) \mapsto \mathscr{P}^d(W)^* \quad , \quad u \mapsto \big[p \mapsto \partial_u p\big].$$

Lemma A.1. $\boldsymbol{\varphi}' = d!\boldsymbol{\varphi}$.

Proof. The proof of Lemma A.1 is an exercise in super linear algebra, and we omit the details. Here we only outline one proof strategy. First one proves that both $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}'$ are $\mathfrak{gl}(W)$-module isomorphisms and $\mathscr{S}^d(W)$ is an irreducible $\mathfrak{gl}(W)$-module. Thus, it suffices to compare the values of $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}'$ on one nonzero vector. For instance, if $W_{\bar{0}} \neq 0$ then for nonzero $v \in W_{\bar{0}}$ and $v^* \in W_{\bar{0}}^*$ we have

$$\partial_v((v^*)^{\otimes d}) = d\langle v, v^*\rangle (v^*)^{\otimes(d-1)},$$

hence by induction we obtain $\partial_{v^{\otimes d}}((v^*)^{\otimes d}) = d!\langle v, v^*\rangle^d$, which completes the proof. □

Lemma A.1 can be used to fix a normalization for the Capelli operator D^λ in Definition 1.1 as follows. Recall the $\mathfrak{g}$-module decomposition

$$\mathscr{P}(W) \cong \bigoplus_{\lambda\in\Omega} W_\lambda.$$

The map $\boldsymbol{\varphi}$ results in an isomorphism $\mathscr{S}(W) \cong \bigoplus_{\lambda\in\Omega} W_\lambda^*$. For $\lambda \in \Omega$, the operator Capelli D^λ is constructed as follows. We choose a basis $w_1^*,\ldots,w_{d_\lambda}^*$ for W_λ, and a dual basis $w_1,\ldots,w_{d_\lambda}$ for W_λ^*. From the definition of D^λ it is straightforward to check that $D^\lambda = \sum_{i=1}^{d_\lambda} w_i^*\partial_{w_i}$. In particular, if $W_\lambda \subseteq \mathscr{P}^d(W)$ then for $1 \leq j \leq d_\lambda$ we have

$$D^\lambda w_j^* = \sum_{i=1}^{d_\lambda} w_i^*\partial_{w_i} w_j^* = d! \sum_{i=1}^{d_\lambda} w_i^*\langle w_i, w_j^*\rangle = d!w_j^*.$$

Consequently, D^λ acts on W_λ by the scalar $d!$.

References

[Cao22] Mengyuan Cao. *The Refined Solution to the Capelli Eigenvalue Problem for* $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ *and* $\mathfrak{gl}(m|2n)$. PhD thesis, University of Ottawa, 2022.

[CW13] Shun-Jen Cheng and Weiqiang Wang. *Dualities and representations of Lie superalgebras*. American Mathematical Society, 2013.

[KS91] Bertram Kostant and Siddhartha Sahi. The Capelli identity, tube domains, and the generalized Laplace transform. *Advances in Mathematics*, 87(1):7192, 1991.

[KS93] Bertram Kostant and Siddhartha Sahi. Jordan algebras and Capelli identities. *Inventiones Mathematicae*, 112(1):657664, 1993.

[OO97] Andrei Okounkov and Grigori Olshanski. Shifted Jack polynomials, binomial formula, and applications. *Mathematical Research Letters*, 4(1):6778, 1997.

[Sah94] Siddhartha Sahi. The spectrum of certain invariant differential operators associated to a hermitian symmetric space. *Lie Theory and Geometry*, page 569576, 1994.

[SSS20] Siddhartha Sahi, Hadi Salmasian, and Vera Serganova. The Capelli Eigenvalue Problem for Lie superalgebras. *Mathematische Zeitschrift*, 2020.

[SV05] Alexander Sergeev and Alexander Veselov. Generalised discriminants, deformed CalogeroMoser-Sutherland operators and super-Jack polynomials. *Advances in Mathematics*, 192:341–375, 2005.

[SV17] Alexander Sergeev and Alexander Veselov. Orbits and invariants of super Weyl groupoid. *Int. Math. Res. Not. IMRN* 2017, no. 20, 6149–6167.

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