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POSITIVE DEFINITE SUPERFUNCTIONS AND UNITARY REPRESENTATIONS OF LIE SUPERGROUPS

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Abstract. For a broad class of Fréchet–Lie supergroups $\mathcal{G}$, we prove that there exists a correspondence between positive definite smooth (resp., analytic) superfunctions on $\mathcal{G}$ and matrix coefficients of smooth (resp., analytic) unitary representations of the Harish–Chandra pair $(G, \mathfrak{g})$ associated to $\mathcal{G}$.

As an application, we prove that a smooth positive definite superfunction on $\mathcal{G}$ is analytic if and only if it restricts to an analytic function on the underlying manifold of $\mathcal{G}$.

When the underlying manifold of $\mathcal{G}$ is 1-connected we obtain a necessary and sufficient condition for a linear functional on the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ to correspond to a matrix coefficient of a unitary representation of $(G, \mathfrak{g})$.

The class of Lie supergroups for which the aforementioned results hold is characterised by a condition on the convergence of the Trotter product formula. This condition is strictly weaker than assuming that the underlying Lie group of $\mathcal{G}$ is a locally exponential Fréchet–Lie group. In particular, our results apply to examples of interest in representation theory such as mapping supergroups and diffeomorphism supergroups.

1. Introduction

The study of unitarizable modules of infinite-dimensional Lie superalgebras has a long history. Both physicists and mathematicians have obtained several interesting examples and classification results for these Lie superalgebras. Examples include the $N = 1$ and $N = 2$ super Virasoro algebras [BFK86], [FQS85], [GKO86], [Io10, Io08], superconformal current algebras [KaTo85], and affine Lie superalgebras [JaKa89], [JaZh88]. (For finite-dimensional Lie superalgebras, see the references cited in [MeNeSa12] and [Sa10].)

In [CCTV06] the authors initiate harmonic analysis on Lie supergroups by laying a precise mathematical foundation to study unitary representations of finite-

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dimensional Lie supergroups, and use it to classify irreducible unitary representations of translation (and in particular, Poincaré) Lie supergroups. Their main idea is to use the equivalence between the category of Lie supergroups and the category of *Harish–Chandra pairs* [K77], [Ko83]. A Harish–Chandra pair is a pair $(G, \mathfrak{g})$ where G is a Lie group, $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a Lie superalgebra, $\mathfrak{g}_{\overline{0}} = \mathrm{Lie}(G)$, and there is an *adjoint* action of G on $\mathfrak{g}$ (see Definition 4.6.3). To justify the robustness of the category of representations of Harish–Chandra pairs, one needs a nontrivial *stability* result which, when $\mathfrak{g}$ is a finite-dimensional Lie superalgebra, is proved in [CCTV06, Prop. 2].

At this point it is worth mentioning that in [DoPe85a] and [DoPe85b] the authors study unitarizable representations of Lie supergroups such as $SU(2, 2|n)$ by embedding them in smoothly induced representations, whereas our focus is on the Hilbert space completions of these representations.

Several technical issues arise in the extension of the stability result of [CCTV06] to the infinite-dimensional case. These technical issues are resolved in [MeNeSa12] when $\mathfrak{g}$ is a Banach–Lie superalgebra. Nevertheless, many infinite-dimensional Lie supergroups which are interesting from the point of view of representation theory, such as mapping supergroups and diffeomorphism supergroups, are not Banach-Lie groups. In [NeSa13] we succeeded in extending the stability theorem to Harish–Chandra pairs $(G, \mathfrak{g})$ where $\mathfrak{g}$ is a Fréchet–Lie superalgebra and G has the Trotter property, that is, for every $x, y \in \mathrm{Lie}(G)$,

$$\exp(t(x+y)) = \lim_{n\to\infty} \big(\exp\big(\frac{t}{n}x\big)\exp\big(\frac{t}{n}y\big)\big)^n$$

holds in the sense of uniform convergence on compact subsets of $\mathbb{R}$. The latter class of Harish–Chandra pairs is broad enough to include the examples of interest in representation theory (see Example 6.7.4).

Lie supergroups such as mapping and diffeomorphism supergroups are infinite-dimensional supermanifolds modeled on Fréchet spaces. In [DeMo99, Rem. 2.6], it is pointed out that the Berezin–Kostant–Leites theory, which defines a supermanifold as a locally ringed space, is not suitable in the infinite-dimensional context. It turns out that an alternative approach to finite-dimensional supermanifolds, which has a categorical flavor, has the capacity to be extended to the infinite-dimensional case. Apparently, the main ideas behind the latter approach originate from the papers [Sh84] and [Vo84]. The definition and basic properties of the pertinent category of supermanifolds were outlined in a preprint by Molotkov. Later on, this category was studied more extensively in Sachse's thesis [Sac09] (see [AlLa12] as well). The idea behind the definition of the latter category is the functor of points approach adapted to the framework of the DeWitt topology. In this approach, a supermanifold is uniquely determined by its Λ_n-points, where Λ_n denotes the Graßmann algebra with n generators. Therefore a supermanifold can be thought of as a functor $\mathcal{F} : \mathsf{Gr} \to \mathsf{Man}$ together with an atlas which is induced by a Grothendieck (pre-)topology on the category $\mathsf{Man}^{\mathsf{Gr}}$. Here Gr is the category of finite-dimensional Graßmann algebras and Man is the category of smooth or analytic manifolds modeled on locally convex spaces. Infinite-dimensional Lie supergroups are group-objects of this category.

1.1. Our main results

In this article we investigate the relationship between smooth (and analytic) positive definite superfunctions on a (possibly infinite-dimensional) Lie supergroup $\mathcal{G}$, unitary representations of the Harish–Chandra pair $(G, \mathfrak{g})$ associated to $\mathcal{G}$, and positive linear functionals on the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$.

Our first main result (Theorem 5.5.2) identifies the $\mathbb{C}$-superalgebra of smooth superfunctions on $\mathcal{G}$ with a natural subalgebra of $\mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^{\infty}(G, \mathbb{C}))$. This is a well known result for Berezin–Kostant–Leites Lie supergroups [Ko83] but its proof in the infinite-dimensional setting requires new ideas because the supermanifold structure is not given by a sheaf of superalgebras anymore. Another issue in infinite-dimensions is the lack of standard charts obtained by the exponential map. In order to prove Theorem 5.5.2, we need to use several basic facts about the structure of infinite-dimensional Lie supergroups and their left invariant differential operators. We were unable to find a reference for these facts and therefore we have included detailed proofs. The reader who is familiar with Harish–Chandra pairs but not interested in the technical details of the functorial approach to Lie supergroups can continue reading the paper from Section 6.

Our second main result (Theorem 6.7.5) is that a smooth (resp., analytic) positive definite superfunction on $\mathcal{G}$ is the matrix coefficient of a cyclic smooth (resp., analytic) unitary representation of $(G, \mathfrak{g})$ and vice versa. This result is an extension of the well known Gelfand–Naimark–Segal (GNS) construction to Lie supergroups. In a sense it means that to describe unitary representations of Lie supergroups it is sufficient to study unitary representations of their Harish–Chandra pairs. From the GNS construction we also obtain the following interesting corollary (see Corollary 6.7.6): if f is a smooth positive definite superfunction on $\mathcal{G}$ which restricts to an analytic function on the underlying Lie group $\mathcal{G}_{\Lambda_0}$, then f restricts to an analytic function on the Lie group $\mathcal{G}_{\Lambda}$ for every Λ.

Our method to prove Theorem 6.7.5 is similar in spirit to the classical GNS construction, but several technical issues arise. For instance, unlike the classical GNS construction, in our framework one has to work with unbounded representations of semigroups, such as the "semidirect product" $G \ltimes U(\mathfrak{g}_{\mathbb{C}})$, so that we are actually dealing with structures similar to crossed product algebras. The stability result of [NeSa13] plays a crucial role in our argument.

Our third main result (Theorem 7.2.2) is about an extension of the noncommutative moment problem to Lie supergroups. For an elaborate discussion of the history of the noncommutative moment problem see [Ne11, Sec. 1]. If $(\pi, \rho^{\pi}, \mathscr{H})$ is an analytic unitary representation of $(G, \mathfrak{g})$ and $v \in \mathscr{H}^{\omega}$, then one can construct a $\mathbb{C}$-linear map $\lambda_v : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ defined by $\lambda_v(D) := \langle \rho^{\pi}(D)v, v\rangle$. The noncommutative moment problem is to characterize the $\mathbb{C}$-linear maps $\lambda_v : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ which are obtained from unitary representations by the above construction. We obtain a necessary and sufficient condition when $\mathfrak{g}$ is a Fréchet–Lie superalgebra and G is a 1-connected Fréchet BCH–Lie group. This is achieved by modifying the method of [Ne11] using the ideas that were developed in [MeNeSa12].

In future work, which will rely on this article and [NeSa13], we will study global realizations of unitarizable super Virasoro algebras ([Io10]), superconformal current algebras ([KaTo85]), and mapping superalgebras ([JaZh88]).

1.2. Structure of this article

Section 2 will review the background material concerning calculus on locally convex spaces. Section 3 will review the definition of the category of supermanifolds. In Section 4 we show how one can associate a Harish–Chandra pair to a supermanifold. In Section 5 we study left invariant differential operators on Lie supergroups and prove Theorem 5.5.2. Section 6 is devoted to the GNS construction. The necessary and sufficient condition for integrability of functionals is proved in Section 7.

2. Calculus on locally convex spaces

Unless stated otherwise, all vector spaces will be over $\mathbb{R}$. If E and F are vector spaces and $m \geq 1$ then $\mathrm{Alt}_m(E,F)$ will denote the vector space of alternating m-linear maps $f : E^m \to F$. For convenience we set $\mathrm{Alt}_0(E,F) := F$. If $E = E_{\overline{0}} \oplus E_{\overline{1}}$ is a $\mathbb{Z}_2$-graded vector space, then the parity of a homogeneous element $v \in E$ will be denoted by $|v| \in \{\overline{0}, \overline{1}\}$.

2.1. Smooth maps between locally convex spaces

Throughout this paper, by a *locally convex space* we mean a Hausdorff locally convex topological vector space.

We quickly review basic definitions and properties of differentiable maps between locally convex spaces. For further details see [Ha82], [Mil84], and [Ne06].

Definition 2.1.1. Let E and F be locally convex spaces and $U \subseteq E$ be an open set. A map $h : U \to F$ is called *differentiable* at $p \in U$ if the directional derivatives

$$\mathsf{d}h(p)(v) := \mathsf{d}_v h(p) := \lim_{t \to 0} \frac{1}{t}(h(p+tv) - h(p))$$

exist for all $v \in E$. The map h is called C^1 if it is continuous, differentiable at every $p \in U$, and the map

$$\mathsf{d}h : U \times E \to F \ , \ (p,v) \mapsto \mathsf{d}_v h(p)$$

is continuous. If $k > 1$ is an integer then a continuous map $h : U \to F$ is called C^k if the limit

$$\mathsf{d}^j h(p)(v_1, \ldots, v_j) := \lim_{t \to 0} \frac{1}{t} \left(\mathsf{d}^{j-1} h(p+tv_j)(v_1, \ldots, v_{j-1}) - \mathsf{d}^{j-1} h(p)(v_1, \ldots, v_{j-1}) \right)$$

exists for every $1 \leq j \leq k$ and every $(u, v_1, \ldots, v_j) \in U \times E^j$, and the maps

$$\mathsf{d}^j h : U \times E^j \to F \ , \ (p, v_1, \ldots, v_j) \mapsto \mathsf{d}^j h(p)(v_1, \ldots, v_j)$$

are continuous. We call a map $h : U \to F$ *smooth* when it is C^k for every $k \geq 1$.

Remark 2.1.2. The above notion of smooth maps naturally leads to smooth manifolds, Lie groups, etc. For more details see [Gl02].

It is known that if $h : U \to F$ is C^k then for every $p \in U$ the map

$$E^k \to F \ , \ (v_1, \dots, v_k) \mapsto \mathsf{d}^k h(p)(v_1, \dots, v_k)$$

is a continuous symmetric k-linear map. The Chain Rule holds in the following form: if E, F, G are locally convex spaces, $U \subseteq E$ and $V \subseteq F$ are open, and $f : U \to F$ and $g : V \to G$ are C^1 maps such that $f(U) \subseteq V$, then $g \circ f : U \to G$ is also C^1 and

$$\mathsf{d}_v(g \circ f)(p) = \mathsf{d}g\big(f(p)\big)\big(\mathsf{d}_v f(p)\big) \ \text{ for every } p \in U \text{ and every } v \in E.$$

The following lemma is sometimes called the Faà di Bruno formula. Its proof is by induction on n. We omit the proof because the argument in the locally convex setting is the same as the one for finite-dimensional spaces.

Lemma 2.1.3. *Let E and F be locally convex spaces, $U \subseteq E$ be open, and $V \subseteq \mathbb{R}^n$ be an open 0-neighborhood. Let $\mathscr{P}_n$ denote the collection of partitions of the set $\{1, \dots, n\}$. If $f : U \to F$ and $g : V \to U$ are smooth maps then*

$$\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} f \circ g(t_1, \dots, t_n)\Big|_{t_1 = \cdots = t_n = 0} = \sum_{\{A_1, \dots, A_k\} \in \mathscr{P}_n} \mathsf{d}^k f\big(g(0, \dots, 0)\big)(v_{A_1}, \dots, v_{A_k})$$

where

$$v_A := \frac{\partial}{\partial t_{a_1}} \cdots \frac{\partial}{\partial t_{a_\ell}} g(t_1, \dots, t_n)\Big|_{t_1 = \cdots = t_n = 0}$$

for every $A := \{a_1, \dots, a_\ell\} \subseteq \{1, \dots, n\}$.

We briefly mention the definition of analytic maps between locally convex spaces. For a discussion of different notions of analyticity as well as pertinent references, see [Ne11, Sec. 2].

Definition 2.1.4. Let E and F be locally convex spaces and $U \subseteq E$ be an open set. A continuous map $h : U \to F$ is called *analytic* if for every $p \in U$ there exists an open 0-neighborhood V_p in $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ and continuous homogeneous polynomials $h_n : E \to F$ of degree n such that $h(p + v) = \sum_{n=0}^{\infty} h_n(v)$ for every $v \in V_p \cap E$.

2.2. Left invariant differential operators on Lie groups

In this paper we assume that all Lie groups are smooth manifolds modeled on locally convex spaces. The exponential map of a Lie group, if it exists, will be denoted by $x \mapsto e^x$.

Let H be a Lie group and $\mathfrak{h} := \mathrm{Lie}(H)$ be its Lie algebra. Assume that the exponential map

$$\mathfrak{h} \to H \ , \ x \mapsto e^x$$

is smooth. Let $U \subseteq H$ be open, F be a locally convex space, and $h : U \to F$ be a smooth map. For every $x \in \mathfrak{h}$ set

$$L_x h : U \to F \ , \ L_x h(g) := \lim_{t \to 0} \frac{1}{t}\big(h(ge^{tx}) - h(g)\big).$$

Lemma 2.2.1. *Let H be a Lie group with a smooth exponential map, F be a locally convex space, $h : H \to F$ be a smooth map, and $v_1, \ldots, v_n \in \mathrm{Lie}(H)$. For every $g \in H$ consider the map*

$$u_g : \mathbb{R}^n \to F \ , \ u_g(t_1, \ldots, t_n) := h(ge^{t_1 v_1 + \cdots + t_n v_n}).$$

Then

$$\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} u_g(t_1, \ldots, t_n)\Big|_{t_1 = \cdots = t_n = 0} = \frac{1}{n!} \sum_{\sigma \in S_n} L_{v_{\sigma(1)}} \cdots L_{v_{\sigma(n)}} h(g).$$

Proof. Fix $x_1, \ldots, x_n \in \mathbb{R}$ and set $y := \sum_{i=1}^n x_i v_i$. Consider the map

$$\gamma : \mathbb{R} \to H \ , \ \gamma(s) := ge^{sy}.$$

Then $h \circ \gamma(s) = u_g(x_1 s, \ldots, x_n s)$ and therefore for every $s_\circ \in \mathbb{R}$ we can write

$$\begin{aligned} \sum_{i=1}^n x_i L_{v_i} h(ge^{s_\circ y}) = L_y h(ge^{s_\circ y}) &= \frac{\partial}{\partial s}(h \circ \gamma)\Big|_{s = s_\circ} \\ &= \sum_{i=1}^n x_i \Big(\frac{\partial}{\partial t_i} u_g(t_1, \ldots, t_n)\Big|_{t_1 = x_1 s_\circ, \ldots, t_n = x_n s_\circ} \Big). \end{aligned}$$

To finish the proof we use the above relation repeatedly to compute $(L_y)^n h(ge^{s_\circ y})$, set $s_\circ = 0$, and compare the coefficient of $x_1 \cdots x_n$ on both sides. □

3. Supermanifolds as functors

A locally convex space E is called $\mathbb{Z}_2$-*graded* if $E = E_{\overline{0}} \oplus E_{\overline{1}}$ where $E_{\overline{0}}$ and $E_{\overline{1}}$ are locally convex spaces and the direct sum decomposition of E is topological.

Throughout this section $E = E_{\overline{0}} \oplus E_{\overline{1}}$ and $F = F_{\overline{0}} \oplus F_{\overline{1}}$ will denote $\mathbb{Z}_2$-graded locally convex spaces.

3.1. The category Gr

Let Gr denote the category of finite-dimensional real Graßmann algebras, i.e., unital associative $\mathbb{R}$-algebras Λ_n, $n \geq 0$, generated by elements $\lambda_1, \ldots, \lambda_n$ which satisfy the relations $\lambda_i \lambda_j + \lambda_j \lambda_i = 0$ for every $1 \leq i, j \leq n$. Every $\Lambda \in \mathsf{Gr}$ has a canonical $\mathbb{Z}_2$-grading $\Lambda := \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$, i.e., it is an associative superalgebra. The identity element of any $\Lambda \in \mathsf{Gr}$ will be denoted by 1_Λ. Given an integer $n \geq 0$ and a set $I = \{i_1, \ldots, i_\ell\} \subseteq \mathbb{N}$ where $1 \leq i_1 < \cdots < i_\ell \leq n$, we define $\lambda_I \in \Lambda_n$ by $\lambda_I := \lambda_{i_1} \cdots \lambda_{i_\ell}$.

The morphisms between objects of Gr are homomorphisms of $\mathbb{Z}_2$-graded unital algebras. For every $m, n \geq 0$ the set of morphisms from Λ_m into Λ_n will be denoted by $\mathsf{Mor}_{\mathsf{Gr}}(\Lambda_m, \Lambda_n)$.

Observe that $\Lambda_0 \simeq \mathbb{R}$ and therefore for every $\Lambda \in \mathsf{Gr}$ there exist unique morphisms $\varepsilon_\Lambda \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda_0)$ and $\iota_\Lambda \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_0, \Lambda)$. The kernel of ε_Λ is called the *augmentation ideal* of Λ and will be denoted by Λ^+. We set $\Lambda_{\overline{0}}^+ := \Lambda^+ \cap \Lambda_{\overline{0}}$.

For every $m \geq n \geq 0$ let $\varepsilon_{m,n} \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_m, \Lambda_n)$ and $\iota_{n,m} \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_n, \Lambda_m)$ be the homomorphisms uniquely identified by

$$\varepsilon_{m,n}(\lambda_k) := \begin{cases} \lambda_k & \text{if } k \leq n, \\ 0 & \text{otherwise} \end{cases}$$

and $\iota_{n,m}(\lambda_k) := \lambda_k$ for all $1 \leq k \leq n$. In particular $\varepsilon_{m,0} = \varepsilon_{\Lambda_m}$ and $\iota_{0,m} = \iota_{\Lambda_m}$.

3.2. The category $\mathsf{Man}^{\mathsf{Gr}}$

Let Man denote the category of smooth manifolds modeled on locally convex spaces (see Remark 3.6.6). For every two functors $\mathcal{F}, \mathcal{G} : \mathsf{Gr} \to \mathsf{Man}$, the set of natural transformations from $\mathcal{F}$ to $\mathcal{G}$ will be denoted by $\mathsf{Nat}(\mathcal{F}, \mathcal{G})$. The category whose objects are functors $\mathcal{F} : \mathsf{Gr} \to \mathsf{Man}$ and whose morphisms are natural transformations will be denoted by $\mathsf{Man}^{\mathsf{Gr}}$. If $\mathcal{F} \in \mathsf{Man}^{\mathsf{Gr}}$ and $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$ then the morphism in Man from $\mathcal{F}_\Lambda$ to $\mathcal{F}_{\Lambda'}$ that is induced by ϱ will be denoted by $\mathcal{F}_\varrho : \mathcal{F}_\Lambda \to \mathcal{F}_{\Lambda'}$.

Remark 3.2.1. Let $\mathcal{G} \in \mathsf{Man}^{\mathsf{Gr}}$. For every $m \geq n \geq 0$ the map $\mathcal{G}_{\iota_{n,m}} : \mathcal{G}_{\Lambda_n} \to \mathcal{G}_{\Lambda_m}$ is injective and identifies $\mathcal{G}_{\Lambda_n}$ with a subset of $\mathcal{G}_{\Lambda_m}$. We will use this natural identification to simplify our notation. For instance if $f \in \mathsf{Nat}(\mathcal{G}, \mathcal{F})$ for some $\mathcal{F} \in \mathsf{Man}^{\mathsf{Gr}}$, then for every $p \in \mathcal{G}_{\Lambda_n}$ we write $f_{\Lambda_m}(p)$ instead of $f_{\Lambda_m}(\mathcal{G}_{\iota_{n,m}}(p))$.

Let $\mathcal{F}, \mathcal{G} \in \mathsf{Man}^{\mathsf{Gr}}$ and $f \in \mathsf{Nat}(\mathcal{G}, \mathcal{F})$. We write $\mathcal{G} \sqsubseteq_f \mathcal{F}$ if for every $\Lambda \in \mathsf{Gr}$, the map $f_\Lambda : \mathcal{G}_\Lambda \to \mathcal{F}_\Lambda$ is a diffeomorphism onto an open subset of $\mathcal{F}_\Lambda$. We write $\mathcal{G} \sqsubseteq \mathcal{F}$ if for every $\Lambda \in \mathsf{Gr}$ the set $\mathcal{G}_\Lambda$ is an open subset of $\mathcal{F}_\Lambda$ and $f_\Lambda : \mathcal{G}_\Lambda \to \mathcal{F}_\Lambda$ is the canonical injection. If $\mathcal{H} \sqsubseteq \mathcal{G}$ and $f \in \mathsf{Nat}(\mathcal{G}, \mathcal{F})$ then we define $f\Big|_{\mathcal{H}} \in \mathsf{Nat}(\mathcal{H}, \mathcal{F})$ by

$$\left(f\Big|_{\mathcal{H}}\right)_\Lambda := f_\Lambda\Big|_{\mathcal{H}_\Lambda}.$$

3.3. Superdomains and supermanifolds

Let $\mathcal{E}^E \in \mathsf{Man}^{\mathsf{Gr}}$ be defined by

$$\mathcal{E}^E_\Lambda := (E \otimes \Lambda)_{\overline{0}} \text{ for every } \Lambda \in \mathsf{Gr}.$$

The zero vector in $\mathcal{E}^E_\Lambda$ will be denoted by $\mathbf{0}_\Lambda$. Every $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$ induces a map

$$\mathcal{E}^E_\varrho : \mathcal{E}^E_\Lambda \to \mathcal{E}^E_{\Lambda'} \ , \ v \otimes \lambda \mapsto v \otimes \varrho(\lambda).$$

Definition 3.3.1. A functor $\mathcal{U} \sqsubseteq \mathcal{E}^E$ is called a *superdomain*.

The next proposition characterizes superdomains.

Proposition 3.3.2. *If $\mathcal{U} \sqsubseteq \mathcal{E}^E$ then there exists an open set $U \subseteq E_{\overline{0}}$ such that*

$$\mathcal{U}_\Lambda = U + (E \otimes \Lambda^+)_{\overline{0}}$$

for every $\Lambda \in \mathsf{Gr}$.

Proof. Set $U := \mathcal{U}_{\Lambda_0}$. Clearly

$$\mathcal{U}_\Lambda \subseteq \{p \in \mathcal{E}^E_\Lambda \,:\, \mathcal{E}^E_{\varepsilon_\Lambda}(p) \in U\} = U + (E \otimes \Lambda^+)_{\overline{0}}$$

for every $\Lambda \in \mathsf{Gr}$. Next we prove that $\mathcal{U}_\Lambda \supseteq U + (E \otimes \Lambda^+)_{\overline{0}}$. Let $\Lambda := \Lambda_n$. Fix $u \in U$. Then $\mathcal{U}_{\iota_\Lambda}(u) \in \mathcal{U}_\Lambda$, i.e., $u \otimes 1_\Lambda \in \mathcal{U}_\Lambda$. Since $\mathcal{U}_\Lambda \subseteq \mathcal{E}^E_\Lambda$ is open, there exists an open neighborhood V of $\mathbf{0}_\Lambda \in \mathcal{E}^E_\Lambda$ such that $(u \otimes 1_\Lambda) + V \subseteq \mathcal{U}_\Lambda$.

Set $V^+ := V \cap (E \otimes \Lambda^+)_{\overline{0}}$. For every $s > 1$ let $\varrho_s \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda)$ be the homomorphism uniquely defined by $\varrho_s(\lambda_i) := s\lambda_i$ for $1 \leq i \leq n$. Then

$$\mathcal{U}_\Lambda \supseteq \mathcal{U}_{\varrho_s}\left((u \otimes 1_\Lambda) + V^+\right) = (u \otimes 1_\Lambda) + \mathcal{U}_{\varrho_s}(V^+)$$

and $\bigcup_{s>1}\left((u \otimes 1_\Lambda) + \mathcal{U}_{\varrho_s}(V^+)\right) = u \otimes 1_\Lambda + (E \otimes \Lambda^+)_{\overline{0}}$. □

Definition 3.3.3. Let $\mathcal{U} \sqsubseteq \mathcal{E}^E$, $\mathcal{V} \sqsubseteq \mathcal{E}^F$, and $f \in \mathsf{Nat}(\mathcal{U}, \mathcal{V})$. We call f a *smooth morphism* from $\mathcal{U}$ to $\mathcal{V}$ if for every $\Lambda \in \mathsf{Gr}$ and every $p \in \mathcal{U}_\Lambda$ the differential map

$$\mathcal{E}^E_\Lambda \to \mathcal{E}^F_\Lambda \ , \ v \mapsto \mathsf{d}_v f_\Lambda(p)$$

is $\Lambda_{\overline{0}}$-linear.

Definition 3.3.4. Let $\mathcal{U}, \mathcal{V}, \mathcal{M} \in \mathsf{Man}^{\mathsf{Gr}}$, $f \in \mathsf{Nat}(\mathcal{U}, \mathcal{M})$, and $g \in \mathsf{Nat}(\mathcal{V}, \mathcal{M})$. Assume that $\mathcal{V} \sqsubseteq_g \mathcal{M}$. The *fiber product* of $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{M}$ is the functor $\mathcal{U} \times_{\mathcal{M}} \mathcal{V} \sqsubseteq \mathcal{U}$ defined by

$$(\mathcal{U} \times_{\mathcal{M}} \mathcal{V})_\Lambda := f_\Lambda^{-1}(g_\Lambda(\mathcal{V}_\Lambda)) \text{ for every } \Lambda \in \mathsf{Gr}.$$

We set $(\mathcal{U} \times_{\mathcal{M}} \mathcal{V})_\varrho := \mathcal{U}_\varrho\Big|_{(\mathcal{U} \times_{\mathcal{M}} \mathcal{V})_\Lambda}$ for every $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$.

Remark 3.3.5. Let $\mathcal{U} \sqsubseteq \mathcal{E}^E$, $\mathcal{V} \sqsubseteq \mathcal{E}^F$, $f \in \mathsf{Nat}(\mathcal{U}, \mathcal{M})$ and $\mathcal{V} \sqsubseteq_g \mathcal{M}$. From the definition of fiber product it is easily seen that $\mathcal{U} \times_{\mathcal{M}} \mathcal{V} \sqsubseteq \mathcal{E}^E$. However, the canonical projection $p^{\mathcal{V}} \in \mathsf{Nat}(\mathcal{U} \times_{\mathcal{M}} \mathcal{V}, \mathcal{V})$ given by

$$p^{\mathcal{V}}_\Lambda := g_\Lambda^{-1} \circ f_\Lambda\Big|_{(\mathcal{U} \times_{\mathcal{M}} \mathcal{V})_\Lambda} \quad \text{for every } \Lambda \in \mathsf{Gr}$$

might not necessarily be a smooth morphism.

Definition 3.3.6. A *supermanifold* modeled on a $\mathbb{Z}_2$-graded locally convex space $E = E_{\overline{0}} \oplus E_{\overline{1}}$ is a pair $(\mathcal{M}, \mathscr{A})$ where $\mathcal{M} \in \mathsf{Man}^{\mathsf{Gr}}$ and $\mathscr{A}$ is a set of pairs $(\mathcal{U}, f)$ satisfying the following properties.

(i) If $(\mathcal{U}, f) \in \mathscr{A}$ then $\mathcal{U} \sqsubseteq \mathcal{E}^E$, $f \in \mathsf{Nat}(\mathcal{U}, \mathcal{M})$, and $\mathcal{U} \sqsubseteq_f \mathcal{M}$.

(ii) $\mathscr{A}$ is an open covering of $\mathcal{M}$, i.e.,

$$\mathcal{M}_\Lambda = \bigcup_{(\mathcal{U}, f) \in \mathscr{A}} f_\Lambda(\mathcal{U}_\Lambda) \text{ for every } \Lambda \in \mathsf{Gr}.$$

(iii) For every two elements $(\mathcal{U}, f)$ and $(\mathcal{V}, g)$ of $\mathscr{A}$ the canonical projection

$$p^{\mathcal{V}} : \mathcal{U} \times_{\mathcal{M}} \mathcal{V} \to \mathcal{V}$$

is a smooth morphism.

The set $\mathscr{A}$ is called an *atlas* of $\mathcal{M}$. An element of $\mathscr{A}$ is called an *open chart* of $\mathcal{M}$.

Remark 3.3.7. In the rest of this article, when there is no ambiguity about the atlas of a supermanifold $(\mathcal{M}, \mathscr{A})$, we write $\mathcal{M}$ instead of $(\mathcal{M}, \mathscr{A})$.

Lemma 3.3.8. *Let $\mathcal{M}$ be a supermanifold and $\mathcal{U} \sqsubseteq_f \mathcal{M}$. Then*

$$f_\Lambda(\mathcal{U}_\Lambda) = \mathcal{M}_{\varepsilon_\Lambda}^{-1}(f_{\Lambda_0}(\mathcal{U}_{\Lambda_0})) := \{p \in \mathcal{M}_\Lambda \ : \ \mathcal{M}_{\varepsilon_\Lambda}(p) \in f_{\Lambda_0}(\mathcal{U}_{\Lambda_0})\}$$

for every $\Lambda \in \mathsf{Gr}$.

Proof. Follows from Proposition 3.3.2. The proof is straightforward and left to the reader. □

The notion of a smooth morphism between two supermanifolds can now be defined using open charts. Let $(\mathcal{M}, \mathscr{A})$ and $(\mathcal{N}, \mathscr{B})$ be two supermanifolds and $h \in \mathsf{Nat}(\mathcal{M}, \mathcal{N})$. For every $(\mathcal{U}, f) \in \mathscr{A}$ and every $(\mathcal{V}, g) \in \mathscr{B}$ the natural transformations $h \circ f \in \mathsf{Nat}(\mathcal{U}, \mathcal{N})$ and $g \in \mathsf{Nat}(\mathcal{V}, \mathcal{N})$ define a fiber product $\mathcal{U} \times_{\mathcal{N}} \mathcal{V}$. We say h is a *smooth morphism* from $\mathcal{M}$ to $\mathcal{N}$ if for every $(\mathcal{U}, f) \in \mathscr{A}$ and every $(\mathcal{V}, g) \in \mathscr{B}$ the canonical projection $p^{\mathcal{V}} : \mathcal{U} \times_{\mathcal{N}} \mathcal{V} \to \mathcal{V}$ is a smooth morphism.

3.4. Λ-smooth maps

The next definition will simplify our presentation.

Definition 3.4.1. Let $E = E_{\overline{0}} \oplus E_{\overline{1}}$ and $F = E_{\overline{0}} \oplus F_{\overline{1}}$ be $\mathbb{Z}_2$-graded locally convex spaces. Let $\mathcal{M}$ be a supermanifold modeled on E and $\mathcal{N}$ be a supermanifold modeled on F. Fix $\Lambda \in \mathsf{Gr}$ and let $p \in \mathcal{M}_\Lambda$. A smooth map $h_\Lambda : \mathcal{M}_\Lambda \to \mathcal{N}_\Lambda$ is called Λ-*smooth* at p if for every open chart $(\mathcal{U}, f)$ of $\mathcal{M}$ where $p \in f_\Lambda(\mathcal{U}_\Lambda)$, and every open chart $(\mathcal{V}, g)$ of $\mathcal{N}$ where $h_\Lambda(p) \in g_\Lambda(\mathcal{V}_\Lambda)$, the map

$$\mathcal{E}_\Lambda^E \to \mathcal{E}_\Lambda^F \ , \ v \mapsto \mathsf{d}_v \left(g_\Lambda^{-1} \circ h_\Lambda \circ f_\Lambda\right)(p)$$

is $\Lambda_{\overline{0}}$-linear. If h_Λ is Λ-smooth at every $p \in \mathcal{M}_\Lambda$ then h_Λ will simply be called Λ-smooth.

Remark 3.4.2. Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds and $h \in \mathsf{Nat}(\mathcal{M}, \mathcal{N})$. Then h is a smooth morphism if and only if $h_\Lambda : \mathcal{M}_\Lambda \to \mathcal{N}_\Lambda$ is Λ-smooth for every $\Lambda \in \mathsf{Gr}$.

3.5. The category SMan

The category of supermanifolds and their smooth morphisms will be denoted by SMan and the set of smooth morphisms from $\mathcal{M}$ to $\mathcal{N}$ will be denoted by $\mathsf{Mor}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{N})$. It is proved in [AlLa12, Thm 3.36] that the category of finite-dimensional supermanifolds (in the sense of Berezin, Kostant, and Leites) is equivalent to a full subcategory of SMan.

3.6. The superalgebra $C^\infty(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}})$

As usual, we denote the $\mathbb{Z}_2$-graded vector space $\mathbb{C} \oplus \mathbb{C}$ by $\mathbb{C}^{1|1}$. For every supermanifold $\mathcal{M}$ set

$$C^\infty(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}}) := \mathsf{Mor}_{\mathsf{SMan}}(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}}).$$

The next proposition follows from [AlLa12, Prop. 3.4].

Proposition 3.6.1. *Let $\mathcal{U} \sqsubseteq \mathcal{E}^E$ and set $U := \mathcal{U}_{\Lambda_0}$. There exists a bijection between smooth morphisms $h \in C^\infty(\mathcal{U}, \mathcal{E}^{\mathbb{C}^{1|1}})$ and families of maps $\{h_m : m \geq 0\}$ satisfying the following properties:*

(i) *$h_m : U \to \mathrm{Alt}_m(E_{\overline{1}}, \mathbb{C})$ for every $m \geq 0$, and the map*

$$U \times E_{\overline{1}}^m \to F\ ,\ (u, v_1, \ldots, v_m) \mapsto h_m(u)(v_1, \ldots, v_m)$$

is smooth.

(ii) *For every $\Lambda \in \mathsf{Gr}$ we have*

$$h_\Lambda(u + v_{\overline{0}} + v_{\overline{1}}) = \sum_{k,m \geq 0} \frac{1}{k!m!} \mathsf{d}^k h_m(u)(\underbrace{v_{\overline{0}}, \ldots, v_{\overline{0}}}_{k \text{ times}})(\underbrace{v_{\overline{1}}, \ldots, v_{\overline{1}}}_{m \text{ times}}) \qquad (1)$$

where $u \in U$, $v_{\overline{0}} \in E_{\overline{0}} \otimes \Lambda_{\overline{0}}^+$, and $v_{\overline{1}} \in E_{\overline{1}} \otimes \Lambda_{\overline{1}}$.

In (1) the multilinear functions $\mathsf{d}^k h_m(u)$ are extended by linearity to Graßmann variables, i.e.,

$$\mathsf{d}^k h_m(u)(x_1\lambda_{I_1}, \ldots, x_k\lambda_{I_k})(y_1\lambda_{J_1}, \ldots, y_m\lambda_{J_m}) := \mathsf{d}^k h_m(u)(x_1, \ldots, x_k)(y_1, \ldots, y_m) \prod_{i=1}^k \lambda_{I_i} \prod_{j=1}^m \lambda_{J_j}.$$

Remark 3.6.2. The superspace $\mathcal{E}^{\mathbb{C}^{1|1}}$ is a ring object in $\mathsf{Man}^{\mathsf{Gr}}$ and induces a multiplication on $C^\infty(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}})$. Therefore $C^\infty(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}})$ is the superalgebra of smooth superfunctions on the supermanifold $\mathcal{M}$. In this article, we will not need this multiplication.

Definition 3.6.3. Let $\mathcal{M}$ be a supermanifold and $h \in C^\infty(\mathcal{M}, \mathcal{E}^{\mathbb{C}^{1|1}})$. The family of maps $\{h_m : m \geq 0\}$ which satisfies (1) is called the *skeleton* of h.

Lemma 3.6.4. *Let $\mathcal{U} \sqsubseteq \mathcal{E}^E$, $h \in C^\infty(\mathcal{U}, \mathcal{E}^{\mathbb{C}^{1|1}})$, and $\{h_m : m \geq 0\}$ be the skeleton of h. If $x_1, \ldots, x_n \in E_{\overline{1}}$ then*

$$\mathsf{d}^n h_{\Lambda_n}(u)(x_1\lambda_1, \ldots, x_n\lambda_n) = h_n(u)(x_1, \ldots, x_n) \cdot \lambda_1 \cdots \lambda_n \text{ for every } u \in \mathcal{U}_{\Lambda_0}.$$

Proof. If we set $v_{\overline{0}} := \mathbf{0}_{\Lambda_n}$ and $v_{\overline{1}} := t_1x_1\lambda_1 + \cdots + t_nx_n\lambda_n$ in (1) then it follows that $h_{\Lambda_n}(u + t_1x_1\lambda_1 + \cdots + t_nx_n\lambda_n)$ is a vector-valued polynomial in $t_1, \ldots, t_n$ with coefficients in $\mathcal{E}_{\Lambda_n}^{\mathbb{C}^{1|1}}$. The leading term of this polynomial is

$$h_n(u)(x_1, \ldots, x_n) \cdot \lambda_1 \cdots \lambda_n \cdot t_1 \cdots t_n.$$

It follows that

$$\begin{aligned} &\mathsf{d}^n h_{\Lambda_n}(u)(x_1\lambda_1, \ldots, x_n\lambda_n) \\ &\quad = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} h_{\Lambda_n}(u + t_1x_1\lambda_1 + \cdots + t_nx_n\lambda_n)\Big|_{t_1 = \cdots = t_n = 0} \\ &\quad = h_n(u)(x_1, \ldots, x_n) \cdot \lambda_1 \cdots \lambda_n \end{aligned}$$

which completes the proof. □

Lemma 3.6.5. *Let $\mathcal{U} \sqsubseteq \mathcal{E}^E$, $h \in \mathsf{Mor}_{\mathsf{SMan}}(\mathcal{U}, \mathcal{E}^F)$, $\Lambda \in \mathsf{Gr}$, and $p \in \mathcal{U}_\Lambda$.*

(i) *If $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$ for some $\Lambda' \in \mathsf{Gr}$ then*

$$\mathsf{d}^n h_{\Lambda'}\big(\mathcal{U}_\varrho(p)\big)\big(\mathcal{E}^E_\varrho(v_1), \ldots, \mathcal{E}^E_\varrho(v_n)\big) = \mathcal{E}^F_\varrho\Big(\mathsf{d}^n h_\Lambda(p)(v_1, \ldots, v_n)\Big)$$

for every $v_1, \ldots, v_n \in \mathcal{E}^E_\Lambda$.

(ii) *The map*

$$\mathcal{E}^E_\Lambda \times \cdots \times \mathcal{E}^E_\Lambda \to \mathcal{E}^F_\Lambda \ , \ (v_1, \cdots, v_n) \mapsto \mathsf{d}^n h_\Lambda(p)(v_1, \ldots, v_n)$$

is $\Lambda_{\overline{0}}$-linear in $v_1, \ldots, v_n$.

Proof. (i) The case $n = 1$ follows from differentiating the relation $h_{\Lambda'} \circ \mathcal{U}_\varrho = \mathcal{E}^F_\varrho \circ h_\Lambda$ and using the fact that $\mathsf{d}_v \mathcal{U}_\varrho(\mathbf{0}_\Lambda) = \mathcal{E}^E_\varrho(v)$ for every $v \in \mathcal{E}^E_\Lambda$. The case $n > 1$ follows by induction on n.

(ii) The case $n = 1$ is an immediate consequence of Λ-smoothness of h_Λ and the case $n > 1$ follows by induction on n. □

Remark 3.6.6. If Man denotes the category of analytic manifolds, then after minor modifications the definitions and results of this section will remain valid and therefore lead to the category of analytic supermanifolds.

4. Lie Supergroups

Throughout this section $E = E_{\overline{0}} \oplus E_{\overline{1}}$ will be a Mackey complete $\mathbb{Z}_2$-graded locally convex space. For more information on the notion of Mackey completeness see [KrMi97, Sec. 2]. Note that every sequentially complete locally convex space is Mackey complete.

4.1. Lie supergroups as group objects in SMan

In view of our categorical approach to supergeometry, a Lie supergroup will be a group object in the category SMan. For background on group objects in categories see [GeMa88, Sec. II.3.10].

Definition 4.1.1. A supermanifold $\mathcal{G} \in \mathsf{SMan}$ is called a *Lie supergroup* if it is a group object in SMan and the Lie group $\mathcal{G}_{\Lambda_0}$ has a smooth exponential map.

In the rest of Section 4 we assume that $\mathcal{G}$ is a Lie supergroup modeled on $E = E_{\overline{0}} \oplus E_{\overline{1}}$. If $\Lambda \in \mathsf{Gr}$ then $\mathcal{G}_\Lambda$ is a Lie group modeled on the locally convex space $(E \otimes \Lambda)_{\overline{0}}$. For every $\Lambda \in \mathsf{Gr}$, the identity element of $\mathcal{G}_\Lambda$ will be denoted by $\mathbf{1}_\Lambda$.

Let $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the multiplication morphism of $\mathcal{G}$. For every $\Lambda \in \mathsf{Gr}$, if $g \in \mathcal{G}_\Lambda$ then we define the left and right translation maps $(l_g)_\Lambda : \mathcal{G}_\Lambda \to \mathcal{G}_\Lambda$ and $(r_g)_\Lambda : \mathcal{G}_\Lambda \to \mathcal{G}_\Lambda$ by

$$(l_g)_\Lambda(x) := gx := \mu_\Lambda(g, x) \text{ and } (r_g)_\Lambda(x) := xg := \mu_\Lambda(x, g).$$

Note that $(l_g)_\Lambda$ and $(r_g)_\Lambda$ are Λ-smooth. If $g \in \mathcal{G}_{\Lambda_0}$ then based on Remark 3.2.1 we obtain smooth morphisms $l_g, r_g \in \mathsf{Mor}_{\mathsf{SMan}}(\mathcal{G}, \mathcal{G})$.

4.2. A local Lie supergroup

Fix an open chart $(\mathcal{V}, f)$ of $\mathcal{G}$ such that $\mathcal{V} \sqsubseteq \mathcal{E}^E$. After a suitable translation we can assume that $\mathbf{0}_{\Lambda_0} \in \mathcal{V}_{\Lambda_0}$, and $f_{\Lambda_0}(\mathbf{0}_{\Lambda_0}) = \mathbf{1}_{\Lambda_0}$. It follows immediately that $\mathbf{0}_\Lambda \in \mathcal{V}_\Lambda$ and $f_\Lambda(\mathbf{0}_\Lambda) = \mathbf{1}_\Lambda$.

The multiplication and inversion of $\mathcal{G}$ can be pulled back to $\mathcal{V}$ as follows. Fix $\Lambda \in \mathsf{Gr}$. For every $v_1, v_2 \in \mathcal{V}_\Lambda$ if $\mu_\Lambda(f_\Lambda(v_1), f_\Lambda(v_2)) \in f_\Lambda(\mathcal{V}_\Lambda)$ then we set

$$v_1 \bullet v_2 := f_\Lambda^{-1}\big(\mu_\Lambda(f_\Lambda(v_1), f_\Lambda(v_2))\big). \tag{2}$$

Similarly, for every $v \in \mathcal{V}_\Lambda$, if $f_\Lambda(v)^{-1} \in f_\Lambda(\mathcal{V}_\Lambda)$ then we set

$$v^{-1} := f_\Lambda^{-1}\big(f_\Lambda(v)^{-1}\big). \tag{3}$$

Note that the product $v_1 \bullet v_2$ and the inverse v^{-1} are not necessarily defined for all choices of $v_1, v_2, v \in \mathcal{V}_\Lambda$. Nevertheless, the following statement holds.

Lemma 4.2.1. *There exists a $\mathcal{U} \sqsubseteq \mathcal{V}$ which satisfies the following properties:*

(i) $\mathbf{0}_\Lambda \in \mathcal{U}_\Lambda$ *and* $\mathbf{0}_\Lambda \bullet u = u \bullet \mathbf{0}_\Lambda = u$ *for every* $u \in \mathcal{U}_\Lambda$.
(ii) *If* $u \in \mathcal{U}_\Lambda$ *for some* $\Lambda \in \mathsf{Gr}$ *then* u^{-1} *is defined and belongs to* $\mathcal{U}_\Lambda$. *Moreover,* $u \bullet u^{-1} = u^{-1} \bullet u = \mathbf{0}_\Lambda$.
(iii) *The product* $u_1 \bullet u_2 \bullet u_3$ *is defined for every* $\Lambda \in \mathsf{Gr}$ *and for every* $u_1, u_2, u_3 \in \mathcal{U}_\Lambda$.

Proof. Since $\mathcal{G}_{\Lambda_0}$ is a Lie group, it is possible to choose $\mathcal{U}_{\Lambda_0} \subseteq \mathcal{V}_{\Lambda_0}$ suitably such that properties (i)-(iii) hold in the special case $\Lambda = \Lambda_0$. For every $\Lambda \in \mathsf{Gr}$ set

$$\mathcal{U}_\Lambda := \mathcal{V}_{\varepsilon_\Lambda}^{-1}(\mathcal{U}_{\Lambda_0}) := \{\, p \in \mathcal{V}_\Lambda \;:\; \mathcal{V}_{\varepsilon_\Lambda}(p) \in \mathcal{U}_{\Lambda_0} \,\}.$$

Properties (i)-(iii) for general Λ follow from naturality of f and Lemma 3.3.8. $\square$

4.3. The exponential map for $\mathcal{G}_\Lambda$

Let $(\mathcal{U}, f)$ denote the open chart of $\mathcal{G}$ obtained by Lemma 4.2.1. Lemma 3.3.8 implies that the set

$$f_{\Lambda_{n+1}}^{-1}\big(\ker(\mathcal{G}_{\varepsilon_{n+1,n}})\big) = \big(E \otimes (\Lambda_n \cdot \lambda_{n+1})\big)_{\overline{0}} \subseteq \mathcal{U}_{\Lambda_{n+1}}$$

is closed under the multiplication and inversion defined in (2) and (3). Thus, with the latter operations, $\big(E \otimes (\Lambda_n \cdot \lambda_{n+1})\big)_{\overline{0}}$ is a Lie group. The next lemma explicitly describes the multiplication of $\big(E \otimes (\Lambda_n \cdot \lambda_{n+1})\big)_{\overline{0}}$.

Lemma 4.3.1. *If* $x, y \in \big(E \otimes (\Lambda_n \cdot \lambda_{n+1})\big)_{\overline{0}}$, *then* $x \bullet y = x + y$.

Proof. Write $x = x_1 \cdot \lambda_{n+1}$ and $y = y_1 \cdot \lambda_{n+1}$ where $x_1, y_1 \in (E \otimes \Lambda_n)_{\overline{1}}$. Set $\varrho := \varepsilon_{n+2,n+1}$, $x' := \mathcal{U}_{\iota_{n+1,n+2}}(x) \in \mathcal{U}_{\Lambda_{n+2}}$, and $y' := y_1 \cdot \lambda_{n+2} \in \mathcal{U}_{\Lambda_{n+2}}$. Then

$$\mathcal{U}_\varrho(x' \bullet y') = \mathcal{U}_\varrho(x') \bullet \mathcal{U}_\varrho(y') = \mathcal{U}_\varrho(x') \bullet \mathbf{0}_{\Lambda_{n+1}} = \mathcal{U}_\varrho(x')$$

and therefore $x' \bullet y' = x' + w \cdot \lambda_{n+2} + w' \cdot \lambda_{n+1}\lambda_{n+2}$ where $w \in (E \otimes \Lambda_n)_{\overline{1}}$ and $w' \in (E \otimes \Lambda_n)_{\overline{0}}$.

Similarly if we take $\varrho' \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_{n+2}, \Lambda_{n+1})$ defined by

$$\varrho'(\lambda_i) := \begin{cases} \lambda_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i = n+1, \\ \lambda_{n+1} & \text{if } i = n+2 \end{cases}$$

then $w \cdot \lambda_{n+1} = \mathcal{U}_{\varrho'}(x' \bullet y') = \mathcal{U}(x') \bullet \mathcal{U}(y') = \mathbf{0}_{\Lambda_{n+1}} \bullet \mathcal{U}_{\varrho'}(y') = y_1 \cdot \lambda_{n+1}$. It follows that $w \cdot \lambda_{n+2} = y'$ and therefore $x' \bullet y' = x' + y' + w' \cdot \lambda_{n+1}\lambda_{n+2}$. Finally, consider $\varrho'' \in \mathsf{Mor}(\Lambda_{n+2}, \Lambda_{n+1})$ defined by

$$\varrho''(\lambda_i) := \begin{cases} \lambda_i & \text{if } 1 \leq i \leq n, \\ \lambda_{n+1} & \text{if } n+1 \leq i \leq n+2. \end{cases}$$

Then

$$x \bullet y = \mathcal{U}_{\varrho''}(x') \bullet \mathcal{U}_{\varrho''}(y') = \mathcal{U}_{\varrho''}(x' \bullet y') = \mathcal{U}_{\varrho''}(x' + y' + w' \cdot \lambda_{n+1}\lambda_{n+2}) = x + y$$

which completes the proof of the lemma. □

Proposition 4.3.2. *For every $n \geq 0$ the Lie group $\mathcal{G}_{\Lambda_n}$ has a smooth exponential map.*

Proof. We use induction on n. The case $n = 0$ follows from Definition 4.1.1. Next assume $n > 0$. Since E is Mackey complete, Lemma 4.3.1 and [Ne06, Prop. II.5.6] imply that $\ker(\mathcal{G}_{\varepsilon_{n+1,n}})$ is regular in the sense of [Mil84, Def. 7.6]. See [Ne06, Def. II.5.2] for a more detailed discussion. If $1 \to A \to B \to C \to 1$ is a smooth extension of Lie groups such that A is regular and C has a smooth exponential map, then B also has a smooth exponential map. We do not give a proof of the latter statement because the argument is very similar to [KrMi97, Thm 38.6]. See also [GlNe]. To complete the proof of the proposition we use the above statement with the sequence $1 \longrightarrow \ker(\mathcal{G}_{\varepsilon_{n+1,n}}) \longrightarrow \mathcal{G}_{\Lambda_{n+1}} \xrightarrow{\varepsilon_{n+1,n}} \mathcal{G}_{\Lambda_n} \longrightarrow 1$. □

4.4. The exponential map of $(\mathcal{U}, f)$

Let $(\mathcal{U}, f)$ be the open chart of $\mathcal{G}$ obtained by Lemma 4.2.1. Next we show how to pull back the exponential map of $\mathcal{G}$ to $\mathcal{U}$. For every $\Lambda \in \mathsf{Gr}$, the map

$$\mathcal{E}^E_\Lambda \to \mathrm{Lie}(\mathcal{G}_\Lambda)\ ,\ v \mapsto \mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)$$

is a continuous invertible linear transformation with a continuous inverse. For every $v \in \mathcal{E}^E_\Lambda$ we set

$$e^v := f_\Lambda^{-1}(e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)})$$

whenever $e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)} \in f_\Lambda(\mathcal{U}_\Lambda)$. The next lemma implies that the exponential map is well-defined on some $\mathcal{W} \sqsubseteq \mathcal{E}^E$.

Lemma 4.4.1. *There exists a $\mathcal{W} \sqsubseteq \mathcal{E}^E$ which satisfies the following properties.*

(i) $\mathbf{0}_{\Lambda_0} \in \mathcal{W}_{\Lambda_0}$.
(ii) $e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)} \in f_\Lambda(\mathcal{U}_\Lambda)$ *for every* $\Lambda \in \mathsf{Gr}$ *and every* $v \in \mathcal{W}_\Lambda$.

Proof. Since $\mathcal{G}_{\Lambda_0}$ has a smooth exponential map, it is possible to choose an open 0-neighborhood $W \subseteq \mathrm{Lie}(\mathcal{G}_{\Lambda_0})$ such that $\{e^w \ : \ w \in W\} \subseteq f_{\Lambda_0}(\mathcal{U}_{\Lambda_0})$. Set

$$\mathcal{W}_{\Lambda_0} := \{\, v \in \mathcal{E}^E_{\Lambda_0} \ : \ \mathsf{d}_v f_{\Lambda_0}(\mathbf{0}_{\Lambda_0}) \in W \,\}$$

and $\mathcal{W}_\Lambda := \big(\mathcal{E}^E_{\varepsilon_\Lambda}\big)^{-1}(\mathcal{W}_{\Lambda_0}) := \{\, v \in \mathcal{E}^E_\Lambda \ : \ \mathcal{E}^E_{\varepsilon_\Lambda}(v) \in \mathcal{W}_{\Lambda_0} \,\}$ for every $\Lambda \in \mathsf{Gr}$.

Next we prove that if $v \in \mathcal{W}_\Lambda$ then $e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)} \in f_\Lambda(\mathcal{U}_\Lambda)$. By Lemma 3.3.8 it suffices to prove that if $v \in \mathcal{W}_\Lambda$ then $\mathcal{G}_{\varepsilon_\Lambda}\big(e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)}\big) \in f_{\Lambda_0}(\mathcal{U}_{\Lambda_0})$. To prove the latter statement note that if $v \in \mathcal{W}_\Lambda$ then $\mathcal{E}^E_{\varepsilon_\Lambda}(v) \in \mathcal{W}_{\Lambda_0}$ and therefore

$$\mathcal{G}_{\varepsilon_\Lambda}\big(e^{\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)}\big) = e^{\mathsf{d}\mathcal{G}_{\varepsilon_\Lambda}(\mathbf{1}_\Lambda)(\mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda))} = e^{\mathsf{d}f_{\Lambda_0}(\mathbf{0}_{\Lambda_0})\left(\mathcal{E}^E_{\varepsilon_\Lambda}(v)\right)} \in f_{\Lambda_0}(\mathcal{U}_{\Lambda_0})$$

which completes the proof. □

Lemma 4.4.1 implies that for every $\Lambda \in \mathsf{Gr}$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}_\Lambda & \xrightarrow{\quad f_\Lambda \quad} & \mathcal{G}_\Lambda \\ {\scriptstyle v\mapsto e^v}\uparrow & & \uparrow{\scriptstyle x\mapsto e^x} \\ \mathcal{W}_\Lambda & \xrightarrow[v\mapsto \mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda)]{} & \mathrm{Lie}(\mathcal{G}_\Lambda) \end{array}$$

Our next goal is to prove that the induced exponential map is in $\mathsf{Nat}(\mathcal{W},\mathcal{U})$.

Lemma 4.4.2. *Let* $\Lambda, \Lambda' \in \mathsf{Gr}$ *and* $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$. *Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_\Lambda & \xrightarrow{\quad \mathcal{U}_\varrho \quad} & \mathcal{U}_{\Lambda'} \\ {\scriptstyle v\mapsto e^v}\uparrow & & \uparrow{\scriptstyle v\mapsto e^v} \\ \mathcal{W}_\Lambda & \xrightarrow[\mathcal{W}_\varrho]{} & \mathcal{W}_{\Lambda'} \end{array}$$

Proof. Let $w \in \mathcal{W}_\Lambda$. Then

$$\begin{aligned} f_{\Lambda'}\big(e^{\mathcal{W}_\varrho(w)}\big) &= f_{\Lambda'}\big(e^{\mathcal{E}^E_\varrho(w)}\big) = e^{\mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})(\mathcal{E}^E_\varrho(w))} = e^{\mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)(\mathsf{d}_w f_\Lambda(\mathbf{0}_\Lambda))} \\ &= \mathcal{G}_\varrho\big(e^{\mathsf{d}_w f_\Lambda(\mathbf{0}_\Lambda)}\big) = \mathcal{G}_\varrho\big(f_\Lambda(e^w)\big) = f_{\Lambda'}(\mathcal{U}_\varrho(e^w)). \end{aligned}$$

Since $f_{\Lambda'} : \mathcal{U}_{\Lambda'} \to \mathcal{G}_{\Lambda'}$ is an injection, it follows that $e^{\mathcal{W}_\varrho(w)} = \mathcal{U}_\varrho(e^w)$. □

Fix $\Lambda \in \mathsf{Gr}$. The map

$$S_\Lambda : \mathcal{E}^E_\Lambda \to \mathrm{Lie}(\mathcal{G}_\Lambda) \ , \ v \mapsto \mathsf{d}_v f_\Lambda(\mathbf{0}_\Lambda) \tag{4}$$

is a continuous invertible linear transformation with a continuous inverse. Therefore the Lie bracket of $\mathrm{Lie}(\mathcal{G}_\Lambda)$ can be pulled back to $\mathcal{E}^E_\Lambda$, that is, we can define a Lie bracket

$$\mathcal{E}^E_\Lambda \times \mathcal{E}^E_\Lambda \to \mathcal{E}^E_\Lambda \ , \ [x,y]_\Lambda := S^{-1}_\Lambda\left([S_\Lambda(x), S_\Lambda(y)]_{\mathrm{Lie}(\mathcal{G}_\Lambda)}\right) \quad \text{for every } x, y \in \mathcal{E}^E_\Lambda,$$

where $[\,\cdot,\cdot\,]_{\mathrm{Lie}(\mathcal{G}_\Lambda)}$ denotes the Lie bracket of $\mathrm{Lie}(\mathcal{G}_\Lambda)$. Observe that $e^{sx}, e^{ty} \in \mathcal{W}_\Lambda$ for sufficiently small $s, t \in \mathbb{R}$, and therefore

$$[x,y]_\Lambda = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\big(e^{sx} \bullet e^{ty} \bullet e^{-sx}\big)\Big|_{s=t=0} \quad \text{for every } x, y \in \mathcal{E}^E_\Lambda. \tag{5}$$

Lemma 4.4.3. *Let $\Lambda \in \mathsf{Gr}$ and $x, y \in \mathcal{E}^E_\Lambda$.*

(i) *If $\lambda_I, \lambda_J \in \Lambda_{\overline{0}}$ then $[x \cdot \lambda_I, y \cdot \lambda_J]_\Lambda = [x,y]_\Lambda \cdot \lambda_I \lambda_J$.*

(ii) *If $\Lambda' \in \mathsf{Gr}$ and $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$ then*

$$[\mathcal{E}^E_\varrho(x), \mathcal{E}^E_\varrho(y)]_{\Lambda'} = \mathcal{E}^E_\varrho\big([x,y]_\Lambda\big).$$

Proof. Note that

$$[x,y]_\Lambda = \frac{\partial}{\partial s}\frac{\partial}{\partial t} h_\Lambda \circ g_\Lambda(s,t)\Big|_{s=t=0}$$

where $h : \mathcal{U} \times \mathcal{U} \to \mathcal{V}$ is the smooth morphism defined by

$$h_\Lambda(a,b) := a \bullet b \bullet a^{-1} \text{ for every } a, b, c \in \mathcal{U}_\Lambda$$

and

$$g_\Lambda : (-r, r) \times (-r, r) \to \mathcal{U}_\Lambda \times \mathcal{U}_\Lambda \ , \ g_\Lambda(s,t) := (e^{sx}, e^{ty})$$

for $r > 0$ sufficiently small. Lemma 2.1.3 implies that

$$[x,y]_\Lambda = \mathsf{d}^2 h_\Lambda\big((\mathbf{0}_\Lambda, \mathbf{0}_\Lambda)\big)\big((x, \mathbf{0}_\Lambda), (\mathbf{0}_\Lambda, y)\big).$$

Therefore (i) and (ii) follow from Lemma 3.6.5. □

4.5. A Lie superalgebra structure on E

Our next goal is to obtain a Lie superalgebra structure on $E = E_{\overline{0}} \oplus E_{\overline{1}}$ which is compatible with the Lie brackets $[\cdot,\cdot]_\Lambda$ on $\mathcal{E}^E_\Lambda$ for every $\Lambda \in \mathsf{Gr}$.

Lemma 4.5.1. *There exists a continuous bilinear map $\mathbf{c}_1 : E_{\overline{0}} \times E_{\overline{1}} \to E_{\overline{1}}$ such that*

$$[x, y \cdot \lambda_1]_{\Lambda_1} = \mathbf{c}_1(x,y) \cdot \lambda_1 \text{ for every } x \in E_{\overline{0}} \text{ and every } y \in E_{\overline{1}}.$$

There exists a continuous symmetric bilinear form $\mathbf{c}_2 : E_{\overline{1}} \times E_{\overline{1}} \to E_{\overline{0}}$ such that

$$[x \cdot \lambda_1, y \cdot \lambda_2]_{\Lambda_2} = \mathbf{c}_2(x,y) \cdot \lambda_1 \lambda_2 \text{ for every } x, y \in E_{\overline{1}}.$$

Proof. This is a consequence of Lemma 4.4.3(ii). For $\mathbf{c}_1$ we use $\varrho := \iota_{\Lambda_1} \circ \varepsilon_{\Lambda_1} \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_1, \Lambda_1)$. For $\mathbf{c}_2$, we use $\varrho := \varrho_i \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_2, \Lambda_2)$, where $i \in \{1,2\}$, defined by $\varrho_i(\lambda_j) := \delta_{i,j}\lambda_j$. To show that $\mathbf{c}_2$ is symmetric we use $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_2, \Lambda_2)$ given by $\varrho(\lambda_1) := \lambda_2$ and $\varrho(\lambda_2) := \lambda_1$. □

Consider the continuous skew-symmetric bilinear map

$$\mathbf{c}_0 : E_{\overline{0}} \times E_{\overline{0}} \to E_{\overline{0}} \ , \ \mathbf{c}_0(x,y) := [x,y]_{\Lambda_0}.$$

Using $\mathbf{c}_0, \mathbf{c}_1$, and $\mathbf{c}_2$ we define a Lie superalgebra structure on $E = E_{\overline{0}} \oplus E_{\overline{1}}$. For every $x_{\overline{0}}, y_{\overline{0}} \in E_{\overline{0}}$ and every $x_{\overline{1}}, y_{\overline{1}} \in E_{\overline{1}}$ we set

$$[x_{\overline{0}} + x_{\overline{1}}, y_{\overline{0}} + y_{\overline{1}}] := \mathbf{c}_0(x_{\overline{0}}, y_{\overline{0}}) + \mathbf{c}_1(x_{\overline{0}}, y_{\overline{1}}) - \mathbf{c}_1(y_{\overline{0}}, x_{\overline{1}}) - \mathbf{c}_2(x_{\overline{1}}, y_{\overline{1}}). \quad (6)$$

Bilinearity and continuity of $[\cdot,\cdot]$ are obvious. Example 4.5.2 below explains how to prove the super Jacobi identity for $[\cdot,\cdot]$ using the Jacobi identity of $[\cdot,\cdot]_\Lambda$.

Example 4.5.2. Assume $x, y \in E_{\overline{1}}$ and $z \in E_{\overline{0}}$. In this case we should prove that

$$[x,[y,z]] - [y,[z,x]] + [z,[x,y]] = 0. \tag{7}$$

Using (6) we can write (7) as

$$-\mathbf{c}_2(x,\mathbf{c}_1(z,y)) - \mathbf{c}_2(y,\mathbf{c}_1(z,x)) + \mathbf{c}_0(z,\mathbf{c}_2(x,y)) = 0. \tag{8}$$

From Lemma 4.5.1 and Lemma 4.4.3(ii) it follows that

$$[y\cdot\lambda_1, z]_{\Lambda_2} = -[z, y\cdot\lambda_1]_{\Lambda_2} = -\mathbf{c}_1(z,y)\cdot\lambda_1.$$

Again from Lemma 4.5.1 and Lemma 4.4.3(ii) it follows that

$$[x\cdot\lambda_2, [y\cdot\lambda_1, z]_{\Lambda_2}]_{\Lambda_2} = [x\cdot\lambda_2, -\mathbf{c}_1(z,y)\cdot\lambda_1]_{\Lambda_2} = \mathbf{c}_2(x,\mathbf{c}_1(z,y))\cdot\lambda_1\lambda_2.$$

In a similar way, it can be shown that $[y\cdot\lambda_1,[z,x\cdot\lambda_2]_{\Lambda_2}]_{\Lambda_2} = \mathbf{c}_2(y,\mathbf{c}_1(z,x))\cdot\lambda_1\lambda_2$ and $[z,[x\cdot\lambda_2,y\cdot\lambda_1]_{\Lambda_2}]_{\Lambda_2} = -\mathbf{c}_0(z,\mathbf{c}_2(x,y))\cdot\lambda_1\lambda_2$. Therefore (8) is a consequence of the Jacobi identity of $[\cdot,\cdot]_{\Lambda_2}$ for the elements $x\cdot\lambda_2, y\cdot\lambda_1$, and z.

For every $\Lambda \in \mathsf{Gr}$, the Lie superbracket on $E = E_{\overline{0}} \oplus E_{\overline{1}}$ defined in (6) induces a Lie superbracket

$$[\,\cdot,\cdot\,]'_\Lambda : (E\otimes\Lambda)\times(E\otimes\Lambda) \to E\otimes\Lambda$$

as follows. For homogeneous $v_1, v_2 \in E$ and $\lambda_{I_1}, \lambda_{I_2} \in \Lambda$ we set

$$[v_1\cdot\lambda_{I_1}, v_2\cdot\lambda_{I_2}]'_\Lambda := (-1)^{|\lambda_{I_1}|\cdot|v_2|}[v_1,v_2]\cdot\lambda_{I_1}\lambda_{I_2}.$$

Proposition 4.5.3. *Let $\Lambda \in \mathsf{Gr}$. Then $[x,y]_\Lambda = [x,y]'_\Lambda$ for every $x, y \in \mathcal{E}^E_\Lambda$.*

Proof. Follows from Lemma 4.4.3 and Lemma 4.5.1. □

4.6. The Harish–Chandra pair associated to $\mathcal{G}$

For every $\Lambda \in \mathsf{Gr}$, we define an action of $\mathcal{G}_{\Lambda_0}$ on $\mathcal{E}^E_\Lambda$ as follows:

$$\mathcal{G}_{\Lambda_0}\times\mathcal{E}^E_\Lambda \to \mathcal{E}^E_\Lambda\ ,\ (g,v)\mapsto g\star_\Lambda v := S_\Lambda^{-1}\big(\mathrm{Ad}_\Lambda(\mathcal{G}_{\iota_\Lambda}(g))(S_\Lambda(v))\big).$$

where S_Λ is the linear transformation defined in (4) and $\mathrm{Ad}_\Lambda(a)(x)$ denotes the adjoint action of $a \in \mathcal{G}_\Lambda$ on $x \in \mathrm{Lie}(\mathcal{G}_\Lambda)$. Note that by the Chain Rule,

$$g\star_\Lambda v = S_\Lambda^{-1}\big(\mathsf{d}(l_g\circ r_{g^{-1}})_\Lambda(\mathbf{1}_\Lambda)(S_\Lambda(v))\big) = \mathsf{d}(f^{-1}\circ l_g\circ r_{g^{-1}}\circ f)_\Lambda(\mathbf{0}_\Lambda)(v). \tag{9}$$

Lemma 4.6.1. *Let $g \in \mathcal{G}_{\Lambda_0}$, $\Lambda \in \mathsf{Gr}$, and $v \in \mathcal{E}^E_\Lambda$.*

(i) *If $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda,\Lambda')$ for some $\Lambda' \in \mathsf{Gr}$ then $g\star_{\Lambda'}\mathcal{E}^E_\varrho(v) = \mathcal{E}^E_\varrho(g\star_\Lambda v)$.*
(ii) *$g\star_\Lambda(v\cdot\lambda_I) = (g\star_\Lambda v)\cdot\lambda_I$ for every $\lambda_I \in \Lambda_{\overline{0}}$.*

Proof. (i) Differentiating $f_{\Lambda'} \circ \mathcal{U}_\varrho = \mathcal{G}_\varrho \circ f_\Lambda$ yields $S_{\Lambda'} \circ \mathcal{E}^E_\varrho(w) = \mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_{\Lambda'})(S_\Lambda(w))$. Similarly, differentiating $(l_g \circ r_{g^{-1}})_{\Lambda'} \circ \mathcal{G}_\varrho = \mathcal{G}_\varrho \circ (l_g \circ r_{g^{-1}})_\Lambda$ yields

$$\mathsf{d}(l_g \circ r_{g^{-1}})_{\Lambda'}(\mathbf{1}_{\Lambda'})\big(\mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)(w)\big) = \mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)\big(\mathsf{d}(l_g \circ r_{g^{-1}})_\Lambda(\mathbf{1}_\Lambda)(w)\big).$$

Therefore

$$\begin{aligned} g \star_{\Lambda'} \mathcal{E}^E_\varrho(v) &= S^{-1}_{\Lambda'}\Big(\mathsf{d}(l_g \circ r_{g^{-1}})_{\Lambda'}(\mathbf{1}_{\Lambda'})\big(S_{\Lambda'} \circ \mathcal{E}^E_\varrho(v)\big)\Big) \\ &= S^{-1}_{\Lambda'}\Big(\mathsf{d}(l_g \circ r_{g^{-1}})_{\Lambda'}(\mathbf{1}_{\Lambda'})\Big(\mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)(S_\Lambda(v))\Big)\Big) \\ &= S^{-1}_{\Lambda'}\Big(\mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)\Big(\mathsf{d}(l_g \circ r_{g^{-1}})_\Lambda(\mathbf{1}_\Lambda)\big(S_\Lambda(v)\big)\Big)\Big) \\ &= \mathcal{E}^E_\varrho \circ S^{-1}_\Lambda\Big(\mathsf{d}(l_g \circ r_{g^{-1}})_\Lambda(\mathbf{1}_\Lambda)\big(S_\Lambda(v)\big)\Big) = \mathcal{E}^E_\varrho(g \star_\Lambda v). \end{aligned}$$

(ii) Follows from (9) and $\Lambda_{\overline{0}}$-linearity of $v \mapsto \mathsf{d}_v(f^{-1} \circ l_g \circ r_{g^{-1}} \circ f)_\Lambda(\mathbf{0}_\Lambda)$. □

Theorem 4.6.2. *Let $G := \mathcal{G}_{\Lambda_0}$. Let $\mathfrak{g}_{\overline{0}} := \mathrm{Lie}(\mathcal{G}_{\Lambda_0})$ and*

$$A_{\overline{0}} : E_{\overline{0}} \to \mathfrak{g}_{\overline{0}} \ , \ A_{\overline{0}}(e_{\overline{0}}) := \mathsf{d}_{e_{\overline{0}}} f_{\Lambda_0}(\mathbf{0}_{\Lambda_0}).$$

Let $\mathfrak{g}_{\overline{1}} := \ker\big(\mathsf{d}\mathcal{G}_{\varepsilon_{\Lambda_1}}(\mathbf{1}_{\Lambda_1})\big) \subseteq \mathrm{Lie}(\mathcal{G}_{\Lambda_1})$ and

$$A_{\overline{1}} : E_{\overline{1}} \to \mathfrak{g}_{\overline{1}} \ , \ A(e_{\overline{1}}) := \mathsf{d}f_{\Lambda_1}(\mathbf{0}_{\Lambda_1})\,(e_{\overline{1}} \cdot \lambda_1)\,.$$

Set $\mathfrak{g} := \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ and

$$A : E \to \mathfrak{g} \ , \ e_{\overline{0}} \oplus e_{\overline{1}} \mapsto A_{\overline{0}}(e_{\overline{0}}) \oplus A_{\overline{1}}(e_{\overline{1}}).$$

Let $[\,\cdot,\cdot\,]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a Lie superbracket induced by the Lie superbracket $[\,\cdot,\cdot\,] : E \times E \to E$ given in (6) *as follows:*

$$[x, y]_{\mathfrak{g}} := A\big([A^{-1}(x), A^{-1}(y)]\big) \text{ for every } x, y \in \mathfrak{g}.$$

Define

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g} \ , \ \mathrm{Ad}(g)(x_{\overline{0}} \oplus x_{\overline{1}}) := \mathrm{Ad}_{\Lambda_0}(g)(x_{\overline{0}}) \oplus \mathrm{Ad}_{\Lambda_1}\big(\mathcal{G}_{\iota_{\Lambda_1}}(g)\big)\,(x_{\overline{1}}).$$

(i) *The superbracket $[\,\cdot,\cdot\,]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous and the map*

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$$

is smooth.

(ii) *The map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ for every $g \in G$.*

(iii) *Let $y \in \mathfrak{g}$ and $c_y : G \to \mathfrak{g}$ be the map defined by $c_y(g) := \mathrm{Ad}(g)(y)$. Then*

$$\mathsf{d}c_y(\mathbf{1}_{\Lambda_0})(x) = [x, y]_{\mathfrak{g}} \text{ for every } x \in \mathfrak{g}_{\overline{0}}\,.$$

Proof. (i) Continuity of $[\,\cdot,\cdot\,]_{\mathfrak{g}}$ follows from the fact that A is a bijective continuous linear transformation with a continuous inverse. Smoothness of Ad follows from smoothness of Ad_{Λ_0} and Ad_{Λ_1}.

(ii) The inclusion $\mathrm{Ad}(g)(\mathfrak{g}_{\overline{1}}) \subseteq \mathfrak{g}_{\overline{1}}$ follows from

$$g \star_{\Lambda_0} \mathcal{E}^E_{\varepsilon_{\Lambda_1}}(v) = \mathcal{E}^E_{\varepsilon_{\Lambda_1}}\big(g \star_{\Lambda_1} v\big)$$

which is a consequence of Lemma 4.6.1(i). Next we prove that

$$[\mathrm{Ad}(g)(x), \mathrm{Ad}(g)(y)]_{\mathfrak{g}} = \mathrm{Ad}(g)([x,y]_{\mathfrak{g}})$$

for every $x, y \in \mathfrak{g}$. It suffices to assume that x, y are homogenous. Depending on the parity of x and y there are four cases to consider, but the arguments are similar, and we will only give the argument when $x, y \in \mathfrak{g}_{\overline{1}}$. Set $\check{x} := A^{-1}(x)$ and $\check{y} := A^{-1}(y)$. From Lemma 4.6.1 it follows that

$$g \star_{\Lambda_2} (\check{x} \cdot \lambda_1) = A^{-1}\big(\mathrm{Ad}(g)(x)\big) \cdot \lambda_1 \;\text{ and }\; g \star_{\Lambda_2} (\check{y} \cdot \lambda_2) = A^{-1}\big(\mathrm{Ad}(g)(y)\big) \cdot \lambda_2.$$

Now on the one hand by Proposition 4.5.3,

$$\begin{aligned}[A^{-1}(\mathrm{Ad}(g)(x)) \cdot \lambda_1, A^{-1}(\mathrm{Ad}(g)(y)) \cdot \lambda_2]_{\Lambda_2} \\ = -[A^{-1}(\mathrm{Ad}(g)(x)), A^{-1}(\mathrm{Ad}(g)(y))] \cdot \lambda_1\lambda_2\end{aligned}$$

and on the other hand from Lemma 4.6.1 it follows that

$$\begin{aligned}[A^{-1}(\mathrm{Ad}(g)(x)) \cdot \lambda_1, A^{-1}(\mathrm{Ad}(g)(y)) \cdot \lambda_2]_{\Lambda_2} &= [g \star_{\Lambda_2} (\check{x} \cdot \lambda_1), g \star_{\Lambda_2} (\check{y} \cdot \lambda_2)]_{\Lambda_2} \\ &= g \star_{\Lambda_2} \big([\check{x} \cdot \lambda_1, \check{y} \cdot \lambda_2]_{\Lambda_2}\big) = g \star_{\Lambda_2} (-[\check{x}, \check{y}] \cdot \lambda_1\lambda_2) \\ &= -(g \star_{\Lambda_2} [\check{x}, \check{y}]) \cdot \lambda_1\lambda_2 = -A^{-1}\left(\mathrm{Ad}(g)([x,y])\right) \cdot \lambda_1\lambda_2.\end{aligned}$$

(iii) It suffices to prove the statement for $y \in \mathfrak{g}_{\overline{1}}$. In this case,

$$\mathsf{d}c_y(\mathbf{1}_{\Lambda_0})(x) = [\mathsf{d}\mathcal{G}_{\iota_{\Lambda_1}}(\mathbf{1}_{\Lambda_0})(x), y]_{\mathrm{Lie}(\mathcal{G}_{\Lambda_1})} = A\big([A^{-1}(x), A^{-1}(y)]\big) = [x,y]_{\mathfrak{g}}$$

which completes the proof. □

Theorem 4.6.2 links the abstract notion of a Lie supergroup to the more concrete notion of a Harish–Chandra pair.

In the next definition, $\mathrm{Aut}(\mathfrak{g})$ denotes the group of automorphisms of $\mathfrak{g}$ (not necessarily continuous) which preserve the $\mathbb{Z}_2$-grading.

Definition 4.6.3. A *Harish–Chandra pair* is a pair $(G, \mathfrak{g})$ satisfying the following properties.

(i) $\mathfrak{g} := \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a $\mathbb{Z}_2$-graded locally convex space endowed with a continuous Lie superbracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

(ii) G is a Lie group and $\mathfrak{g}_{\overline{0}} = \mathrm{Lie}(G)$.

(iii) There exists a group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ such that the map

$$G \times \mathfrak{g} \to \mathfrak{g}\ ,\ (g,x) \mapsto \mathrm{Ad}(g)(x)$$

is smooth.

(iv) If $c_y : G \to \mathfrak{g}$ is the map defined by $c_y(g) := \mathrm{Ad}(g)(y)$ then

$$\mathsf{d}_x c_y(\mathbf{1}_G) = [x,y] \text{ for every } x \in \mathfrak{g}_{\overline{0}} \text{ and every } y \in \mathfrak{g}$$

where $\mathbf{1}_G \in G$ denotes the identity element of G.

Corollary 4.6.4. *The pair $(G, \mathfrak{g})$ associated to $\mathcal{G}$ in Theorem 4.6.2 is a Harish–Chandra pair.*

Remark 4.6.5. A result analogous to Corollary 4.6.4 holds for analytic Lie supergroups, i.e., group objects in the category of analytic supermanifolds. The Harish–Chandra pair $(G, \mathfrak{g})$ associated to an analytic supermanifold satisfies two extra properties: G is an analytic Lie group and the adjoint action $G \times \mathfrak{g} \to \mathfrak{g}$ is analytic.

Definition 4.6.6. An *analytic Harish–Chandra pair* is a Harish–Chandra pair $(G, \mathfrak{g})$ where G is an analytic Lie group and the adjoint action $G \times \mathfrak{g} \to \mathfrak{g}$ is analytic.

Remark 4.6.7. From the results of this section it follows that $\mathcal{G}_\Lambda \simeq G \ltimes N_\Lambda$ for every $\Lambda \in \mathsf{Gr}$, where N_Λ is a nilpotent simply connected Lie group with Lie algebra $(\mathfrak{g} \otimes \Lambda^+)_{\overline{0}}$. If we identify N_Λ with its Lie algebra via the exponential map then the action of G on N_Λ is the canonical extension of the adjoint action of G to $(\mathfrak{g} \otimes \Lambda^+)_{\overline{0}}$.

5. Left invariant differential operators on Lie supergroups

To simplify our notation, in this section we assume that $\mathcal{G}$ is a Lie supergroup modeled on a locally convex space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ (that is, $\mathfrak{g}_{\overline{0}} = E_{\overline{0}}$ and $\mathfrak{g}_{\overline{1}} = E_{\overline{1}}$) and $[\,\cdot,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie superbracket defined by (6). Set $G := \mathcal{G}_{\Lambda_0}$. We will denote the identity element of G by $\mathbf{1}_G$.

Throughout this section $(\mathcal{U}, f)$ will denote the open chart of $\mathcal{G}$ obtained by Lemma 4.2.1 (note that $\mathcal{U} \sqsubseteq \mathcal{E}^{\mathfrak{g}}$).

Let $h_\Lambda : \mathcal{G}_\Lambda \to \mathcal{E}_\Lambda^{\mathbb{C}^{1|1}}$ be a smooth map. Recall that for every $v \in \mathrm{Lie}(\mathcal{G}_\Lambda)$ the left invariant differential operator L_v on $\mathcal{G}_\Lambda$ is defined by

$$L_v h_\Lambda(g) := \lim_{s \to 0} \frac{1}{s}\big(h_\Lambda(g e^{sv}) - h_\Lambda(g)\big).$$

The chain rule implies that $L_v h_\Lambda(g) = \mathsf{d}h_\Lambda(g)\big(\mathsf{d}_v(l_g)_\Lambda(\mathbf{1}_\Lambda)\big)$.

5.1. Some technical lemmas

Our next goal is to prove some basic properties of left invariant differential operators.

Lemma 5.1.1. *If $\Lambda \in \mathsf{Gr}$ then every $g \in \mathcal{G}_\Lambda$ can be written as $g = g_0 f_\Lambda(u_0)$ where $g_0 := \mathcal{G}_{\iota_\Lambda \circ \varepsilon_\Lambda}(g)$ and $u_0 \in \mathcal{U}_{\varepsilon_\Lambda}^{-1}(\mathbf{0}_{\Lambda_0}) \subseteq \mathcal{U}_\Lambda$.*

Proof. Follows immediately from Lemma 3.3.8. □

Lemma 5.1.2. *Let $\Lambda \in \mathsf{Gr}$ and $h_\Lambda : \mathcal{G}_\Lambda \to \mathcal{E}_\Lambda^{\mathbb{C}^{1|1}}$ be Λ-smooth.*

(i) *For every $v \in \mathrm{Lie}(\mathcal{G}_\Lambda)$ the map $L_v h_\Lambda : \mathcal{G}_\Lambda \to \mathcal{E}_\Lambda^{\mathbb{C}^{1|1}}$ is Λ-smooth.*

(ii) *Let $x_1, \ldots, x_k \in \mathcal{E}_\Lambda^{\mathfrak{g}}$ and $\lambda_{I_1}, \ldots, \lambda_{I_k} \in \Lambda_{\overline{0}}$. Set $\tilde{x}_i := \mathsf{d}_{x_i \cdot \lambda_{I_i}} f_\Lambda(\mathbf{0}_\Lambda)$ for $1 \le i \le k$. If $g \in \mathcal{G}_\Lambda$ then*

$$L_{\tilde{x}_1} \cdots L_{\tilde{x}_k} h_\Lambda(g) = \big(L_{\mathsf{d}_{x_1} f_\Lambda(\mathbf{0}_\Lambda)} \cdots L_{\mathsf{d}_{x_k} f_\Lambda(\mathbf{0}_\Lambda)} h_\Lambda(g)\big) \cdot \lambda_{I_1} \cdots \lambda_{I_k}.$$

Proof. (i) Fix $g \in \mathcal{G}_\Lambda$. By Lemma 5.1.1 we can write $g = g_0 f_\Lambda(u_0)$ where $g_0 := \mathcal{G}_{\iota_\Lambda \circ \varepsilon_\Lambda}(g)$ and $u_0 \in \mathcal{U}_\Lambda$. It suffices to prove that the map

$$\mathcal{U}_\Lambda \to \mathcal{E}_\Lambda^{\mathbb{C}^{1|1}} \ , \ x \mapsto (L_v h_\Lambda)(g_0 f_\Lambda(x))$$

is Λ-smooth at u_0. Fix $w \in \mathcal{E}_\Lambda^{\mathfrak{g}}$. Set $H_w(s,t) := h_\Lambda(g_0 f_\Lambda(u_0 + sw)e^{tv})$ for $s,t \in \mathbb{R}$ sufficiently close to 0. Observe that for every $s \in \mathbb{R}$,

$$\frac{\partial}{\partial t} H_w(s,t)\Big|_{t=0} = (L_v h_\Lambda)(g_0 f_\Lambda(u_0 + sw)).$$

Thus we need to prove that $\frac{\partial}{\partial s}\frac{\partial}{\partial t} H_w(s,t)\Big|_{s=t=0}$ is $\Lambda_{\overline{0}}$-linear in w. The map

$$\mathcal{U}_\Lambda \to \mathcal{E}_\Lambda^{\mathbb{C}^{1|1}} \ , \ x \mapsto \big(h_\Lambda \circ (l_{g_0})_\Lambda \circ (r_{e^{tv}})_\Lambda \circ f_\Lambda\big)(x)$$

is Λ-smooth and therefore $\frac{\partial}{\partial s} H_w(s,t)\Big|_{s=0} = \mathsf{d}_w\big(h_\Lambda \circ (l_{g_0})_\Lambda \circ (r_{e^{tv}})_\Lambda \circ f_\Lambda\big)(u_0)$ is $\Lambda_{\overline{0}}$-linear in w. Consequently, if $\lambda \in \Lambda_{\overline{0}}$ then

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t} H_{w\cdot\lambda}(s,t)\Big|_{s=t=0} &= \frac{\partial}{\partial t}\big(\frac{\partial}{\partial s} H_{w\cdot\lambda}(s,t)\Big|_{s=0}\big)\Big|_{t=0} \\ &= \frac{\partial}{\partial t}\big(\frac{\partial}{\partial s} H_w(s,t)\Big|_{s=0}\big)\Big|_{t=0}\cdot\lambda = \frac{\partial}{\partial s}\frac{\partial}{\partial t} H_w(0,0)\cdot\lambda.\end{aligned}$$

(ii) First assume $k = 1$. By the Chain Rule we have

$$\begin{aligned}L_{\mathsf{d}_{x_1\cdot\lambda} f_\Lambda(\mathbf{0}_\Lambda)} h_\Lambda(g) &= \mathsf{d}h_\Lambda(g)\Big(\mathsf{d}(l_g)_\Lambda(\mathbf{1}_\Lambda)\big(\mathsf{d}f_\Lambda(\mathbf{0}_\Lambda)(x_1\cdot\lambda)\big)\Big) \\ &= \mathsf{d}_{x_1\cdot\lambda}(h_\Lambda \circ (l_g)_\Lambda \circ f_\Lambda)(\mathbf{0}_\Lambda) = \big(\mathsf{d}_{x_1}(h_\Lambda \circ (l_g)_\Lambda \circ f_\Lambda)(\mathbf{0}_\Lambda)\big)\cdot\lambda\end{aligned}$$

and again by the Chain Rule the right hand side is equal to $L_{\mathsf{d}_{x_1} f_\Lambda(\mathbf{0}_\Lambda)} h_\Lambda(g)\cdot\lambda$. This completes the proof for the case $k = 1$. The case $k > 1$ follows from (i), (ii), and induction on k. □

Lemma 5.1.3. *Let $\Lambda, \Lambda' \in \mathsf{Gr}$, $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda')$, and $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$. Let $w_1, \ldots, w_k \in \mathcal{E}_\Lambda^{\mathfrak{g}}$ and for every $1 \le i \le k$ set $\tilde{w}_i := \mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})(\mathcal{E}_\varrho^{\mathfrak{g}}(w_i))$. If $g \in \mathcal{G}_\Lambda$ then*

$$L_{\tilde{w}_1}\cdots L_{\tilde{w}_k} h_{\Lambda'}\big(\mathcal{G}_\varrho(g)\big) = \mathcal{E}_\varrho^{\mathbb{C}^{1|1}}\big(L_{\mathsf{d}_{w_1} f_\Lambda(\mathbf{0}_\Lambda)}\cdots L_{\mathsf{d}_{w_k} f_\Lambda(\mathbf{0}_\Lambda)} h_\Lambda(g)\big).$$

Proof. We only give the argument for $k = 1$, as the case $k > 1$ follows by induction on k. Observe that $\mathcal{G}_\varrho \circ f_\Lambda = f_{\Lambda'} \circ \mathcal{U}_\varrho$ and $\mathsf{d}_w \mathcal{U}_\varrho(\mathbf{0}_\Lambda) = \mathcal{E}_\varrho^{\mathfrak{g}}(w)$. Thus

$$\mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})\big(\mathcal{E}_\varrho^{\mathfrak{g}}(w_1)\big) = \mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})\big(\mathsf{d}\mathcal{U}_\varrho(\mathbf{0}_\Lambda)(w_1)\big) = \mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)\big(\mathsf{d}f_\Lambda(\mathbf{0}_\Lambda)(w_1)\big)$$

and for every $s \in \mathbb{R}$ we have

$$e^{s\mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})(\mathcal{E}_\varrho^{\mathfrak{g}}(w_1))} = e^{s\mathsf{d}\mathcal{G}_\varrho(\mathbf{1}_\Lambda)(\mathsf{d}f_\Lambda(\mathbf{0}_\Lambda)(w_1))} = \mathcal{G}_\varrho\big(e^{s\mathsf{d}f_\Lambda(\mathbf{0}_\Lambda)(w_1)}\big).$$

It follows that

$$\begin{aligned}
L_{\tilde{w}_1} h_{\Lambda'}\big(\mathcal{G}_\varrho(g)\big) &= \lim_{s\to 0}\frac{1}{s}\Big(h_{\Lambda'}\big(\mathcal{G}_\varrho(g)e^{s\mathsf{d}f_{\Lambda'}(\mathbf{0}_{\Lambda'})(\mathcal{E}^{\mathfrak{g}}_\varrho(w_1))}\big) - h_{\Lambda'}\big(\mathcal{G}_\varrho(g)\big)\Big)\\
&= \lim_{s\to 0}\frac{1}{s}\Big(h_{\Lambda'}\big(\mathcal{G}_\varrho(ge^{s\mathsf{d}f_{\Lambda}(\mathbf{0}_{\Lambda})(w_1)})\big) - h_{\Lambda'}\big(\mathcal{G}_\varrho(g)\big)\Big)\\
&= \lim_{s\to 0}\frac{1}{s}\Big(\mathcal{E}^{\mathbb{C}^{1|1}}_\varrho\Big(h_{\Lambda}\big(ge^{s\mathsf{d}f_{\Lambda}(\mathbf{0}_{\Lambda})(w_1)}\big) - h_{\Lambda}(g)\Big)\Big)\\
&= \mathcal{E}^{\mathbb{C}^{1|1}}_\varrho\Big(\lim_{s\to 0}\frac{1}{s}\Big(h_{\Lambda}\big(ge^{s\mathsf{d}f_{\Lambda}(\mathbf{0}_{\Lambda})(w_1)}\big) - h_{\Lambda}(g)\Big)\Big)\\
&= \mathcal{E}^{\mathbb{C}^{1|1}}_\varrho\big(L_{\mathsf{d}_{w_1}f_\Lambda(\mathbf{0}_\Lambda)}h_\Lambda(g)\big)
\end{aligned}$$

which completes the proof. □

5.2. Left invariant differential operators on $\mathcal{G}$

Our next goal is to define left invariant differential operators

$$\mathrm{L}_x : C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}}) \to C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$$

for every $x \in \mathfrak{g}$ (see Remark 5.2.5). First we define L_x when x is homogeneous and then we extend it to all of $\mathfrak{g}$ by linearity.

Lemma 5.2.1. *Let $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$, $x \in \mathfrak{g}_{\overline{1}}$, and $n \geq 0$ be an integer. For every $m > n$ set $\tilde{x}_m := \mathsf{d}_{x\cdot\lambda_m} f_{\Lambda_m}(\mathbf{0}_{\Lambda_m})$. If $g \in \mathcal{G}_{\Lambda_n}$ then there exists a unique $w_g \in \mathcal{E}^{\mathbb{C}^{1|1}}_{\Lambda_n}$ such that*

$$\lambda_m \cdot \big(\mathcal{E}^{\mathbb{C}^{1|1}}_{\iota_{n,m}}(w_g)\big) = \lim_{t\to 0}\frac{1}{t}\big(h_{\Lambda_m}(ge^{t\tilde{x}_m}) - h_{\Lambda_m}(g)\big) \text{ for every } m > n.$$

Proof. For every $m > n$ set

$$w_{g,m} := \lim_{t\to 0}\frac{1}{t}\big(h_{\Lambda_m}(ge^{t\tilde{x}_m}) - h_{\Lambda_m}(g)\big).$$

For every $m' > m > n$ let $\varrho_{m,m'} \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda_m, \Lambda_{m'})$ by defined by

$$\varrho_{m,m'}(\lambda_i) := \begin{cases} \lambda_i & \text{if } 1 \leq i \leq m-1,\\ \lambda_{m'} & \text{if } i = m.\end{cases}$$

If $m' > m > n$ then $\mathsf{d}_{\tilde{x}_m}\mathcal{G}_{\varrho_{m,m'}}(\mathbf{1}_{\Lambda_m}) = \tilde{x}_{m'}$ and therefore

$$\mathcal{G}_{\varrho_{m,m'}}(e^{t\tilde{x}_m}) = e^{\mathsf{d}_{\tilde{x}_m}\mathcal{G}_{\varrho_{m,m'}}(\mathbf{1}_{\Lambda_m})} = e^{t\tilde{x}_{m'}} \text{ for every } t \in \mathbb{R}.$$

It follows that $h_{\Lambda_{m'}}(ge^{t\tilde{x}_{m'}}) = \mathcal{E}^{\mathbb{C}^{1|1}}_{\varrho_{m,m'}}\big(h_{\Lambda_m}(ge^{t\tilde{x}_m})\big)$ for every $g \in \mathcal{G}_\Lambda$ and thus $w_{g,m'} = \mathcal{E}^{\mathbb{C}^{1|1}}_{\varrho_{m,m'}}(w_{g,m})$. To complete the proof of the Lemma it is enough to show that $\mathcal{E}^{\mathbb{C}^{1|1}}_{\varepsilon_{n+1,n}}(w_{g,n+1}) = 0$. To prove the latter statement note that

$$\begin{aligned}
\mathcal{G}_{\varepsilon_{n+1,n}}(e^{t\tilde{x}_{n+1}}) &= e^{t\mathsf{d}\mathcal{G}_{\varepsilon_{n+1,n}}(\mathbf{1}_{\Lambda_{n+1}})(\tilde{x}_{n+1})} = e^{t\mathsf{d}\mathcal{G}_{\varepsilon_{n+1,n}}(\mathbf{1}_{\Lambda_{n+1}})\left(\mathsf{d}_{x\cdot\lambda_{n+1}}f_{\Lambda_{n+1}}(\mathbf{0}_{\Lambda_{n+1}})\right)}\\
&= e^{\mathsf{d}f_{\Lambda_n}(\mathbf{0}_{\Lambda_n})\left(\mathcal{E}^{\mathbb{C}^{1|1}}_{\varepsilon_{n+1,n}}(x\cdot\lambda_{n+1})\right)} = \mathbf{1}_{\Lambda_n}
\end{aligned}$$

and thus for every $t \in \mathbb{R}$ we have

$$\mathcal{E}^{\mathbb{C}^{1|1}}_{\varepsilon_{n+1,n}}\big(h_{\Lambda_{n+1}}(ge^{t\tilde{x}_{n+1}}) - h_{\Lambda_{n+1}}(g)\big) = h_{\Lambda_n}(g\mathcal{G}_{\varepsilon_{n+1,n}}(e^{t\tilde{x}_{n+1}})) - h_{\Lambda_n}(g) = 0.$$

This completes the proof. □

Definition 5.2.2. Let $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$ and $\Lambda \in \mathsf{Gr}$. If $x \in \mathfrak{g}_{\overline{0}}$ then set $\tilde{x} := \mathsf{d}_x f_\Lambda(\mathbf{0}_\Lambda)$ and define

$$(\mathrm{L}_x h)_\Lambda(g) := \lim_{t\to 0} \frac{1}{t}\left(h_\Lambda(ge^{t\tilde{x}}) - h_\Lambda(g)\right) \text{ for every } g \in \mathcal{G}_\Lambda.$$

For $x \in \mathfrak{g}_{\overline{1}}$ we define

$$(\mathrm{L}_x h)_{\Lambda_n}(g) := w_g$$

where $w_g \in \mathcal{E}^{\mathbb{C}^{1|1}}_{\Lambda_n}$ is given by Lemma 5.2.1.

Proposition 5.2.3. *Let $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$.*

(i) *If $x \in \mathfrak{g}$ then $\mathrm{L}_x h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$.*
(ii) *If $x \in \mathfrak{g}$ and $g \in \mathcal{G}_{\Lambda_0}$ then $\mathrm{L}_x(h \circ l_g) = (\mathrm{L}_x h) \circ l_g$ for every $g \in \mathcal{G}_{\Lambda_0}$.*

Proof. (i) We can assume x is homogeneous. If $x \in \mathfrak{g}_{\overline{0}}$ then the statement follows from Lemma 5.1.2(i) and Lemma 5.1.3. If $x \in \mathfrak{g}_{\overline{1}}$ then the statement follows from Lemma 5.1.2(i), Lemma 5.1.3, and the definition of L_x.

(ii) Straightforward from the definition. □

Lemma 5.2.4. *Let $x_1, \ldots, x_k \in \mathfrak{g}$ be homogeneous. Let $\Lambda \in \mathsf{Gr}$ and $\lambda_{I_1}, \ldots, \lambda_{I_k} \in \Lambda$ satisfy $|\lambda_{I_i}| = |x_i|$. Let $\tilde{x}_i := \mathsf{d}_{x_i \cdot \lambda_{I_i}} f_\Lambda(\mathbf{0}_\Lambda)$ for $1 \le i \le k$. Let $m \ge 0$ be an integer. Assume that*

$$I_i \cap \{r \in \mathbb{N} \ : \ r \le m\} = \varnothing \text{ for } 1 \le i \le k.$$

If $g \in \mathcal{G}_{\Lambda_m}$ then

$$L_{\tilde{x}_1} \cdots L_{\tilde{x}_k} h_\Lambda(g) = \lambda_{I_k} \cdots \lambda_{I_1} \cdot \big((\mathrm{L}_{x_1} \cdots \mathrm{L}_{x_k} h)_{\Lambda_m}(g)\big).$$

Proof. By Lemma 5.1.2(ii) the proof is reduced to the case where each I_i has at most one element.

Let $i_1 < \cdots < i_\ell$ be such that $I_i = \{m_i\}$ if and only if $i \in \{i_1, \ldots, i_\ell\}$. There are two cases to consider.

Case 1. The m_{i_j}'s are pairwise distinct numbers. Using Lemma 5.1.3 with a homomorphism $\varrho : \Lambda \to \Lambda$ which suitably permutes the generators of Λ, we can assume that $m_{i_1} < \cdots < m_{i_\ell}$. The case $k = 1$ follows from the definition of the differential operator L_x. Next we prove the case $k > 1$ by induction on k. Assume

that $I_1 \neq \varnothing$. (The argument for the case $I_1 = \varnothing$ is similar.) It follows that $I_i \cap \{r \in \mathbb{N} \,:\, r \leq m_{i_1}\} = \varnothing$ for $2 \leq i \leq k$. Set $\tilde{h} := \mathrm{L}_{x_2} \cdots \mathrm{L}_{x_k} h$. Then

$$\begin{aligned} L_{\tilde{x}_1} \cdots L_{\tilde{x}_k} h_\Lambda(g) &= \lim_{s \to 0} \frac{1}{s} \Big(L_{\tilde{x}_2} \cdots L_{\tilde{x}_k} h_\Lambda(g e^{s \mathsf{d}_{\tilde{x}_1} f_\Lambda(\mathbf{0}_\Lambda)}) - L_{\tilde{x}_2} \cdots L_{\tilde{x}_k} h_\Lambda(g) \Big) \\ &= \lim_{s \to 0} \frac{1}{s} \Big(\lambda_{m_{i_\ell}} \cdots \lambda_{m_{i_2}} \cdot \big(\tilde{h}_{\Lambda_{i_1}}(g e^{s \mathsf{d}_{\tilde{x}_1} f_\Lambda(\mathbf{0}_\Lambda)}) - \tilde{h}_{\Lambda_{i_1}}(g) \big) \Big) \\ &= \lambda_{m_{i_\ell}} \cdots \lambda_{m_{i_1}} \cdot (\mathrm{L}_{x_1} \tilde{h})_{\Lambda_{m_{i_1}}}(g) = \lambda_{m_{i_\ell}} \cdots \lambda_{m_{i_1}} \cdot (\mathrm{L}_{x_1} \tilde{h})_{\Lambda_m}(g) \\ &= \lambda_{m_{i_\ell}} \cdots \lambda_{m_{i_1}} \cdot (\mathrm{L}_{x_1} \cdots \mathrm{L}_{x_k} h)_{\Lambda_m}(g). \end{aligned}$$

Case 2. There exist $1 \leq a < b \leq \ell$ such that $m_{i_a} = m_{i_b}$. In this case we prove that $\big(L_{\tilde{x}_1} \cdots L_{\tilde{x}_k}\big) h_\Lambda(g) = 0$. This follows from Case 1 and taking $\varrho : \Lambda_{m+\ell} \to \Lambda$ defined by

$$\varrho(\lambda_j) = \begin{cases} \lambda_j & \text{if } 1 \leq j \leq m, \\ \lambda_{m_{i_j}} & \text{otherwise} \end{cases}$$

in Lemma 5.1.3. □

Remark 5.2.5. Since the definition of L_x is local, for every $\mathcal{Y} \sqsubseteq \mathcal{G}$ one can restrict L_x to a differential operator $\mathrm{L}_x : C^\infty(\mathcal{Y}, \mathcal{E}^{\mathbb{C}^{1|1}}) \to C^\infty(\mathcal{Y}, \mathcal{E}^{\mathbb{C}^{1|1}})$. Our statements regarding properties of L_x can be adapted suitably to hold for the restriction of L_x to $C^\infty(\mathcal{Y}, \mathcal{E}^{\mathbb{C}^{1|1}})$.

5.3. Extending L_x to the universal enveloping algebra

Let $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ and $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The next lemma shows that the definition of L_x can be extended to every $x \in U(\mathfrak{g}_\mathbb{C})$.

Lemma 5.3.1. *If $x, y \in \mathfrak{g}$ are homogeneous then*

$$\big(\mathrm{L}_x \mathrm{L}_y h - (-1)^{|x| \cdot |y|} \mathrm{L}_y \mathrm{L}_x h\big)_\Lambda(g) = (\mathrm{L}_{[x,y]} h)_\Lambda(g) \text{ for every } g \in \mathcal{G}_\Lambda.$$

Proof. We only give the argument for the case $x, y \in \mathfrak{g}_{\overline{1}}$. The remaining cases are similar. Let $\Lambda := \Lambda_n$ and set

$$\tilde{x} := \mathsf{d}_{x \cdot \lambda_{n+1}} f_{\Lambda_{n+2}}(\mathbf{0}_{\Lambda_{n+2}}) \text{ and } \tilde{y} := \mathsf{d}_{y \cdot \lambda_{n+2}} f_{\Lambda_{n+2}}(\mathbf{0}_{\Lambda_{n+2}}).$$

Thus $\tilde{x}, \tilde{y} \in \mathrm{Lie}(\mathcal{G}_{\Lambda_{n+2}})$ and $[\tilde{x}, \tilde{y}] = \mathsf{d} f_{\Lambda_{n+2}}(\mathbf{0}_{\Lambda_{n+2}})(-[x, y] \cdot \lambda_{n+1} \lambda_{n+2})$. From Lemma 5.2.4 it follows that

$$L_{\tilde{x}} L_{\tilde{y}} h_{\Lambda_{n+2}}(g) = \lambda_{n+2} \lambda_{n+1} \cdot (\mathrm{L}_x \mathrm{L}_y h)_\Lambda(g) = -\lambda_{n+1} \lambda_{n+2} \cdot (\mathrm{L}_x \mathrm{L}_y h)_\Lambda(g).$$

Similarly, $L_{\tilde{y}} L_{\tilde{x}} h_{\Lambda_{n+2}}(g) = \lambda_{n+1} \lambda_{n+2} \cdot (\mathrm{L}_y \mathrm{L}_x h)_\Lambda(g)$. Therefore

$$\begin{aligned} \lambda_{n+1} \lambda_{n+2} \cdot &\big((\mathrm{L}_x \mathrm{L}_y h)_\Lambda(g) + (\mathrm{L}_y \mathrm{L}_x h)_\Lambda(g) \big) \\ &= -\big(L_{\tilde{x}} L_{\tilde{y}} - L_{\tilde{y}} L_{\tilde{x}} \big) h_{\Lambda_{n+2}}(g) = -L_{[\tilde{x}, \tilde{y}]} h_{\Lambda_{n+2}}(g) \\ &= -\mathrm{L}_{-[x,y]} h_\Lambda(g) \cdot \lambda_{n+1} \lambda_{n+2} = \mathrm{L}_{[x,y]} h_\Lambda(g) \cdot \lambda_{n+1} \lambda_{n+2} \\ &= \lambda_{n+1} \lambda_{n+2} \cdot \mathrm{L}_{[x,y]} h_\Lambda(g) \end{aligned}$$

which implies that $(\mathrm{L}_x \mathrm{L}_y h)_\Lambda(g) + (\mathrm{L}_y \mathrm{L}_x h)_\Lambda(g) = \mathrm{L}_{[x,y]} h_\Lambda(g)$. □

5.4. Differentiating the exponential map

The next lemma will be used in the proof of Theorem 5.5.2.

Lemma 5.4.1. *Let $\Lambda \in \mathsf{Gr}$, $v_1, \ldots, v_n \in \mathfrak{g}$ be homogeneous, and $\lambda_{I_1}, \ldots, \lambda_{I_n} \in \Lambda$ satisfy the following properties.*

(i) $|\lambda_{I_i}| = |v_i|$ *for all* $1 \leq i \leq n$.
(ii) $I_i \cap I_j = \varnothing$ *for every* $1 \leq i \neq j \leq n$.

Then there exists a smooth map $\eta : \mathcal{U}_{\Lambda_0} \to \mathfrak{g}$ such that

$$\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} (u e^{t_1 v_1 \lambda_{I_1} + \cdots + t_n v_n \lambda_{I_n}}) \Big|_{t_1 = \cdots = t_n = 0} = \eta(u) \cdot \lambda_{I_1} \cdots \lambda_{I_n} \tag{10}$$

for every $u \in \mathcal{U}_{\Lambda_0}$. On the left hand side of (10) *we use the identification of $u \in \mathcal{U}_{\Lambda_0}$ with $\mathcal{U}_{\iota_\Lambda}(u) \in \mathcal{U}_\Lambda$ (see Remark 3.2.1).*

Proof. First observe that differentiation with respect to the t_i's commutes with $\mathcal{E}^{\mathfrak{g}}_\varrho$ for every $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda)$. Therefore by Lemma 4.4.2 it is enough to prove that, if $a \in I_i$ for some $1 \leq i \leq n$, and $\varrho \in \mathsf{Mor}_{\mathsf{Gr}}(\Lambda, \Lambda)$ is defined by

$$\varrho(\lambda_k) := \begin{cases} \lambda_k & \text{if } k \neq a, \\ 0 & \text{otherwise,} \end{cases}$$

then for all sufficiently small t_j, $j \neq i$, we have

$$\mathcal{E}^{\mathfrak{g}}_\varrho \left(\frac{\partial}{\partial t_i} (u e^{t_1 v_1 \lambda_{I_1} + \cdots + t_n v_n \lambda_{I_n}}) \Big|_{t_i = 0} \right) = 0. \tag{11}$$

Without loss of generality we can assume $i = 1$. By Lemma 4.4.2

$$\begin{aligned} \mathcal{U}_\varrho \big(u e^{t_1 v_1 \lambda_{I_1} + \cdots + t_n v_n \lambda_{I_n}} \big) &= u e^{\mathcal{E}^{\mathfrak{g}}_\varrho (t_1 v_1 \lambda_{I_1} + \cdots + t_n v_n \lambda_{I_n})} \\ &= u e^{\mathcal{E}^{\mathfrak{g}}_\varrho (t_2 v_2 \lambda_{I_2} + \cdots + t_n v_n \lambda_{I_n})} = \mathcal{U}_\varrho \big(u e^{t_2 v_2 \lambda_{I_2} + \cdots + t_n v_n \lambda_{I_n}} \big). \end{aligned}$$

Equality (11) now follows immediately. □

5.5. The algebra of smooth superfunctions revisited

Let $C^\infty(G, \mathbb{C})$ (resp., $C^\omega(G, \mathbb{C})$) be the space of smooth (resp., analytic) complex-valued functions on G. For every $x \in \mathfrak{g}_{\overline{0}}$ set $\tilde{x} := \mathsf{d}_x f_{\Lambda_0}(\mathbf{0}_{\Lambda_0})$. The space $C^\infty(G, \mathbb{C})$ is a $\mathfrak{g}_{\overline{0}}$-module via the action

$$x \cdot \psi(g) := L_{\tilde{x}} \psi(g) \tag{12}$$

for every $x \in \mathfrak{g}_{\overline{0}}$, every $\psi \in C^\infty(G, \mathbb{C})$, and every $g \in G$.

Let $\mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^\infty(G, \mathbb{C}))$ denote the complex vector space of $\mathbb{C}$-linear maps

$$\mathbf{h} : U(\mathfrak{g}_{\mathbb{C}}) \to C^\infty(G, \mathbb{C})$$

which satisfy

$$\big(\mathbf{h}(xy)\big)(g) = L_{\tilde{x}}(\mathbf{h}(y))(g) \tag{13}$$

for every $x \in \mathfrak{g}_{\overline{0}}$, every $y \in U(\mathfrak{g}_{\mathbb{C}})$, and every $g \in G$.

If $\mathbf{h} \in \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^\infty(G,\mathbb{C}))$, then for every $n \geq 1$ we set

$$\mathbf{h}^{[n]} : \mathfrak{g}^n \times G \to \mathbb{C}\ ,\ \mathbf{h}^{[n]}(x_1,\ldots,x_n,g) := \big(\mathbf{h}(x_1\cdots x_n)\big)(g). \tag{14}$$

Set

$$C^\infty(G,\mathfrak{g}) := \big\{\mathbf{h} \in \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^\infty(G,\mathbb{C}))\ :\ \mathbf{h}^{[n]} \text{ is smooth for every } n \geq 1\big\}.$$

Similarly, we define

$$C^\omega(G,\mathfrak{g}) := \big\{\mathbf{h} \in \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^\omega(G,\mathbb{C}))\ :\ \mathbf{h}^{[n]} \text{ is analytic for every } n \geq 1\big\}.$$

Remark 5.5.1. Let $Y \subseteq G$ be an open set. Then the space $C^\infty(Y,\mathbb{C})$ (resp., $C^\omega(Y,\mathbb{C})$) is also a $\mathfrak{g}_{\overline{0}}$-module via the action given in (12). In the definition of $C^\infty(G,\mathfrak{g})$ (resp., $C^\omega(G,\mathfrak{g})$), if we substitute $C^\infty(G,\mathbb{C})$ (resp., $C^\omega(G,\mathbb{C})$) by $C^\infty(Y,\mathbb{C})$ (resp., $C^\omega(Y,\mathbb{C})$), then we obtain the space $C^\infty(Y,\mathfrak{g})$ (resp., $C^\omega(Y,\mathfrak{g})$).

Theorem 5.5.2. *Let $\mathcal{Y} \sqsubseteq \mathcal{G}$ and $Y := \mathcal{Y}_{\Lambda_0}$. The map*

$$\Phi : C^\infty(\mathcal{Y}, \mathcal{E}^{\mathbb{C}^{1|1}}) \to C^\infty(Y,\mathfrak{g})\ ,\ \Phi(h)(x) := (\mathrm{L}_x h)_{\Lambda_0}$$

is a $\mathbb{C}$-linear isomorphism.

Proof. Throughout the proof we assume that $\mathcal{Y} = \mathcal{G}$. Slight changes render the proof applicable to arbitrary $\mathcal{Y} \sqsubseteq \mathcal{G}$.

The proof will be given in several steps.

Step 1. Let $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$. From the definition of L_x for $x \in \mathfrak{g}_{\overline{0}}$, Proposition 5.2.3, and Lemma 5.3.1 it follows that $\Phi(h) \in \mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_{\mathbb{C}}), C^\infty(G,\mathbb{C}))$. Lemma 5.2.4 and smoothness of the exponential map of $\mathcal{G}_\Lambda$ (see Proposition 4.3.2) imply that $\Phi(h)^{[n]}$ is smooth for every $n \geq 1$, i.e., $\Phi(h) \in C^\infty(G,\mathfrak{g})$. It remains to prove that Φ is a bijection.

Step 2. Fix $\Lambda \in \mathsf{Gr}$. By Lemma 5.1.1 every $g \in \mathcal{G}_\Lambda$ can be written as $g = g_0 f_\Lambda(u_0)$ where $g_0 = \mathcal{G}_{\iota_\Lambda \circ \varepsilon_\Lambda}(g)$. We can identify G with $\mathrm{im}(\mathcal{G}_{\iota_\Lambda})$ (see Remark 3.2.1) and consequently we will denote $\mathcal{G}_{\varepsilon_\Lambda}(g)$ by g_0 as well. For every $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$ we set $\check{h}^{\langle g\rangle} := h \circ l_{g_0} \circ f$. Observe that $\check{h}^{\langle g\rangle} \in C^\infty(\mathcal{U}, \mathcal{E}^{\mathbb{C}^{1|1}})$. Proving that Φ is an injection amounts to showing that if $\Phi(h) = 0$ then $\check{h}^{\langle g\rangle} = 0$ for every $g \in \mathcal{G}_\Lambda$.

Step 3. Let $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$ such that $\Phi(h) = 0$. Fix $g \in \mathcal{G}_\Lambda$. For every $v_1,\ldots,v_n \in \mathcal{E}^{\mathfrak{g}}_\Lambda$ and every $u \in \mathcal{U}_{\Lambda_0}$ set

$$\mathbf{A}(u; v_1,\ldots,v_n) := \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_n} h_\Lambda\big(g_0 f_\Lambda(u) e^{t_1 \mathbf{d}_{v_1} f_\Lambda(\mathbf{0}_\Lambda) + \cdots + t_n \mathbf{d}_{v_n} f_\Lambda(\mathbf{0}_\Lambda)}\big)\Big|_{t_1=\cdots=t_n=0}$$

where u is identified with $\mathcal{U}_{\iota_\Lambda}(u) \in \mathcal{U}_\Lambda$. From the definition of $\mathbf{A}(u; v_1,\ldots,v_n)$ it follows immediately that the map

$$\mathcal{U}_{\Lambda_0} \times \mathcal{E}^{\mathfrak{g}}_\Lambda \times \cdots \times \mathcal{E}^{\mathfrak{g}}_\Lambda \to \mathcal{E}^{\mathbb{C}^{1|1}}_\Lambda\ ,\ (u, v_1,\ldots,v_n) \mapsto \mathbf{A}(u; v_1,\ldots,v_n)$$

is smooth. Moreover, $\mathbf{A}(u; v_1, \dots, v_n)$ is linear in $v_1, \dots, v_n$. If we set $\tilde{v}_i := \mathsf{d}_{v_i} f_\Lambda(\mathbf{0}_\Lambda)$ for $1 \leq i \leq n$ then Lemma 2.2.1 implies that

$$\mathbf{A}(u; v_1, \dots, v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} L_{\tilde{v}_{\sigma(1)}} \cdots L_{\tilde{v}_{\sigma(n)}} h_\Lambda(g_0 f_\Lambda(u)). \tag{15}$$

Since $\Phi(h) = 0$, Lemma 5.2.4 implies that $\mathbf{A}(u; v_1, \dots, v_n) = 0$ for every $u \in \mathcal{U}_{\Lambda_0}$ and every $v_1, \dots, v_n \in \mathcal{E}^{\mathfrak{g}}_\Lambda$.

Step 4. Given a set $A = \{m_1, \dots, m_k\} \subseteq \mathbb{N}$, for every $u \in \mathcal{U}_{\Lambda_0}$ and every $v_{m_1}, \dots, v_{m_k} \in \mathcal{E}^{\mathfrak{g}}_\Lambda$ set

$$\mathbf{B}(u; v_A) := \frac{\partial}{\partial t_{m_1}} \cdots \frac{\partial}{\partial t_{m_k}} \big(u e^{t_{m_1} v_{m_1} + \cdots + t_{m_k} v_{m_k}}\big)\Big|_{t_{m_1} = \cdots = t_{m_k} = 0}.$$

Note that the smooth map

$$\mathcal{U}_{\Lambda_0} \times \mathcal{E}^{\mathfrak{g}}_\Lambda \times \cdots \times \mathcal{E}^{\mathfrak{g}}_\Lambda \to \mathcal{E}^{\mathfrak{g}}_\Lambda \ , \ (u, v_A) \mapsto \mathbf{B}(u; v_A)$$

is k-linear in $v_{m_1}, \dots, v_{m_k}$. Moreover, if $A = \{m_1\}$ then $\mathbf{B}(u; v_A) = \mathsf{d}_{v_{m_1}}(l_u)_\Lambda(\mathbf{0}_\Lambda)$. By Lemma 2.1.3 we can write

$$\begin{aligned} \mathbf{A}(u; v_1, \dots, v_n) &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} \check{h}^{\langle g \rangle}_\Lambda (u e^{t_1 v_1 + \cdots + t_n v_n})\Big|_{t_1 = \cdots = t_n = 0} \\ &= \sum_{\{A_1, \dots, A_k\} \in \mathscr{P}_n} \mathsf{d}^k \check{h}^{\langle g \rangle}_\Lambda (u)\big(\mathbf{B}(u; v_{A_1}), \dots, \mathbf{B}(u; v_{A_k})\big) \end{aligned}$$

and therefore

$$\begin{aligned} \mathbf{A}(u; v_1, \dots, v_n) &= \mathsf{d}^n \check{h}^{\langle g \rangle}_\Lambda (u)(w_1, \dots, w_n) \\ &\quad + \sum_{k<n} \sum_{\{A_1, \dots, A_k\} \in \mathscr{P}_n} \mathsf{d}^k \check{h}^{\langle g \rangle}_\Lambda (u)\big(\mathbf{B}(u; v_{A_1}), \dots, \mathbf{B}(u; v_{A_k})\big) \end{aligned} \tag{16}$$

where $w_i = \mathsf{d}_{v_i}(l_u)_\Lambda(\mathbf{0}_\Lambda)$ for every $1 \leq i \leq n$. The map

$$\mathcal{E}^{\mathfrak{g}}_\Lambda \to \mathcal{E}^{\mathfrak{g}}_\Lambda \ , \ v \mapsto \mathsf{d}_v(l_u)_\Lambda(\mathbf{0}_\Lambda)$$

is a bijective continuous linear map with a continuous inverse $v \mapsto \mathsf{d}_v(l_{u^{-1}})_\Lambda(u)$. Thus, $v_i = \mathsf{d}_{w_i}(l_{u^{-1}})_\Lambda(u)$. If we start from any $w_1, \dots, w_n \in \mathcal{E}^{\mathfrak{g}}_\Lambda$ and then recursively apply (16), we obtain a linear system with finitely many equations with an invertible triangular coefficient matrix. Since $\mathbf{A}(u, v_1, \dots, v_n) = 0$ for every n and every $v_1, \dots, v_n \in \mathcal{E}^{\mathfrak{g}}_\Lambda$, the linear system is homogeneous. Therefore the unique solution to the linear system is the trivial solution. It follows that

$$\mathsf{d}^n \check{h}^{\langle g \rangle}_\Lambda (u)(w_1, \dots, w_n) = 0 \text{ for every } u \in \mathcal{U}_{\Lambda_0} \text{ and every } w_1, \dots, w_n \in \mathcal{E}^{\mathfrak{g}}_\Lambda.$$

In particular, if $\Lambda = \Lambda_n$ and we set $w_i = x_i \lambda_i$ where $x_i \in \mathfrak{g}_{\overline{1}}$ for every $1 \leq i \leq n$, then Lemma 3.6.4 and Proposition 3.6.1 imply that $\check{h}^{\langle g \rangle} = 0$. This completes the proof of injectivity of Φ.

Step 5. We now proceed towards the proof of surjectivity of Φ. Lemma 5.4.1, the linear systems obtained by (16), and (15) lead to the following statement:

Statement A. Let $n \geq 1$ and $\mathscr{P}_n$ denote the collection of partitions of $\{1, \ldots, n\}$. Then, for every $S := \{A_1, \ldots, A_k\} \in \mathscr{P}_n$, there exists a family

$$\mathscr{C}_S := \{ \left(\mathbf{D}_{S,i}(u; x_{A_1}), \ldots, \mathbf{D}_{S,i}(u; x_{A_k})\right) \ : \ i = 1, \ldots, c(S) \}$$

of k-tuples of smooth maps

$$\mathcal{U}_{\Lambda_0} \times \mathfrak{g}_{\overline{1}}^{|A_j|} \to \mathfrak{g} \ , \ (u, x_{A_j}) \mapsto \mathbf{D}_{S,i}(u; x_{A_j})$$

where $|A_j|$ denotes the cardinality of A_j, such that the following statements hold:

(i) $\mathbf{D}_{S,i}(u; x_{A_j})$ is linear in $\{x_s \ : \ s \in A_j\}$.

(ii) If $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$, then for every $x_1, \ldots, x_n \in \mathfrak{g}_{\overline{1}}$ and every $u \in \mathcal{U}_{\Lambda_0}$ we have:

$$\begin{aligned}(-1)^{\frac{n(n-1)}{2}} \mathsf{d}^n \check{h}_\Lambda^{\langle g \rangle}(u)(x_1\lambda_1, \ldots, x_n\lambda_n) \qquad (17)\\ = \sum_{\substack{S \in \mathscr{P}_n \\ S = \{A_1, \ldots, A_k\}}} \sum_{i=1}^{c(S)} \left(\mathrm{L}_{\mathbf{D}_{S,i}(u; x_{A_1})} \cdots \mathrm{L}_{\mathbf{D}_{S,i}(u; x_{A_k})} h\right)_{\Lambda_0}(g_0 f_\Lambda(u)) \cdot \lambda_1 \cdots \lambda_n.\end{aligned}$$

(iii) Let $k = n$, i.e., $S = \{A_1, \ldots, A_n\}$ such that $A_j = \{j\}$ for every $1 \leq j \leq n$. Then $c(S) = 1$ and $\mathbf{D}_{S,1}(u; x_{\{j\}}) = \mathbf{f}(u; x_j)$ for every $1 \leq j \leq n$ where the smooth map

$$\mathcal{U}_{\Lambda_0} \times \mathfrak{g}_{\overline{1}} \to \mathfrak{g}_{\overline{1}} \ , \ (u, x) \mapsto \mathbf{f}(u; x)$$

is defined by the equality $\mathbf{f}(u; x) \cdot \lambda_1 = \mathsf{d}_{x \cdot \lambda_1}(l_{u^{-1}})_{\Lambda_1}(u)$. Moreover, for every $u \in \mathcal{U}_{\Lambda_0}$ the map

$$\mathfrak{g}_{\overline{1}} \to \mathfrak{g}_{\overline{1}} \ , \ x \mapsto \mathbf{f}(u; x)$$

is a bijective continuous linear transformation with a continuous inverse.

The proofs of (i)-(iii) are fairly straightforward. Part (i) follows from Lemma 5.4.1 and the fact that the $\mathbf{D}_{S,i}(u; x_A)$'s are obtained by superpositions of the $\mathbf{B}(u, x_B)$'s. Part (ii) follows from (16) and (15). Part (iii) follows from (16) and the fact that $v \mapsto \mathsf{d}_v(l_{u^{-1}})_{\Lambda_1}(u)$ is a bijective continuous linear transformation with a continuous inverse (see Step 4 above).

Step 6. For every $g_0 \in \mathcal{G}_{\Lambda_0}$ let $\mathcal{U}^{\langle g_0 \rangle} \sqsubseteq \mathcal{G}$ be defined by

$$\mathcal{U}_\Lambda^{\langle g_0 \rangle} := (l_{g_0} \circ f)_\Lambda(\mathcal{U}_\Lambda) \text{ for every } \Lambda \in \mathsf{Gr}.$$

Fix $\mathbf{h} \in C^\infty(G, \mathfrak{g})$. Our goal is to prove that $\mathbf{h} = \Phi(h)$ for some $h \in C^\infty(\mathcal{G}, \mathcal{E}^{\mathbb{C}^{1|1}})$. Lemma 5.2.4, equality (13), and a standard glueing argument show that it is enough to prove the following statement:

Statement B. For every $g_0 \in \mathcal{G}_{\Lambda_0}$ there exists a unique smooth morphism

$$h^{g_0} \in C^\infty(\mathcal{U}^{\langle g_0\rangle}, \mathcal{E}^{\mathbb{C}^{1|1}})$$

which satisfies $(\mathrm{L}_{x_1}\cdots\mathrm{L}_{x_n}h^{g_0})_{\Lambda_0}(g_1) = (\mathbf{h}(x_1\cdots x_n))(g_1)$ for every $n \geq 0$, every $x_1,\ldots,x_n \in \mathfrak{g}_{\overline{1}}$, and every $g_1 \in \mathcal{U}_{\Lambda_0}^{\langle g_0\rangle}$.

The proof of the uniqueness part of Statement B is similar to Steps 1-4 above. Next we give the proof of the existence part of Statement B.

Fix $g_0 \in \mathcal{G}_{\Lambda_0}$. For every $u \in \mathcal{U}_{\Lambda_0}$, every $n \geq 1$ and every $x_1,\ldots,x_n \in \mathfrak{g}_{\overline{1}}$ define $k_n^{\langle g_0\rangle}(u)(x_1,\ldots,x_n)$ as follows:

$$\begin{aligned} &k_n^{\langle g_0\rangle}(u)(x_1,\ldots,x_n) \qquad (18)\\ &:= \sum_{\substack{S\in\mathscr{P}_n\\ S=\{A_1,\ldots,A_k\}}} \sum_{i=1}^{c(S)} (-1)^{\frac{n(n-1)}{2}} \mathbf{h}\big(\mathbf{D}_{S,i}(u;x_{A_1})\cdots\mathbf{D}_{S,i}(u;x_{A_k})\big)(g_0 f_{\Lambda_0}(u)). \end{aligned}$$

Set $k_0^{\langle g_0\rangle}(u) := \mathbf{h}(1_{U(\mathfrak{g}_{\mathbb{C}})})(g_0 f_{\Lambda_0}(u))$ for every $u \in \mathcal{U}_{\Lambda_0}$, where $1_{U(\mathfrak{g}_{\mathbb{C}})}$ denotes the identity element of $U(\mathfrak{g}_{\mathbb{C}})$. By Proposition 3.6.1 the family $\{k_n^{\langle g_0\rangle} : n \geq 0\}$ corresponds to a smooth morphism $k^{\langle g_0\rangle} : \mathcal{U} \to \mathcal{E}^{\mathbb{C}^{1|1}}$. Let $h^{g_0} : \mathcal{U}^{\langle g_0\rangle} \to \mathcal{E}^{\mathbb{C}^{1|1}}$ be the unique smooth morphism which satisfies

$$k^{\langle g_0\rangle} = h^{g_0} \circ l_{g_0} \circ f.$$

Statement A and Lemma 3.6.4 imply that

$$\begin{aligned} (-1)^{\frac{n(n-1)}{2}} k_n^{\langle g_0\rangle}(x_1,\ldots,x_n) &= \big(\mathrm{L}_{\mathbf{f}(u;x_1)}\cdots\mathrm{L}_{\mathbf{f}(u;x_n)}h^{g_0}\big)_{\Lambda_0}(g_0 f_{\Lambda_0}(u)) \qquad (19)\\ &+ \sum_{\substack{S\in\mathscr{P}_n\\ S=\{A_1,\ldots,A_k\}\\ k<n}} \sum_{i=1}^{c(S)} \big(\mathrm{L}_{\mathbf{D}_{S,i}(u;x_{A_1})}\cdots\mathrm{L}_{\mathbf{D}_{S,i}(u;x_{A_k})}h^{g_0}\big)_{\Lambda_0}(g_0 f_{\Lambda_0}(u)) \end{aligned}$$

and (18) implies that

$$\begin{aligned} (-1)^{\frac{n(n-1)}{2}} k_n^{\langle g_0\rangle}(x_1,\ldots,x_n) &= \mathbf{h}\big(\mathbf{f}(u;x_1)\cdots\mathbf{f}(u;x_n)\big)(g_0 f_{\Lambda_0}(u)) \qquad (20)\\ &+ \sum_{\substack{S\in\mathscr{P}_n\\ S=\{A_1,\ldots,A_k\}\\ k<n}} \sum_{i=1}^{c(S)} \mathbf{h}\big(\mathbf{D}_{S,i}(u;x_{A_1})\cdots\mathbf{D}_{S,i}(u;x_{A_k})\big)(g_0 f_{\Lambda_0}(u)). \end{aligned}$$

Since the map $x \mapsto \mathbf{f}(u;x)$ is an invertible linear map (see Statement A), from (19), (20), and Lemma 3.6.4, and by induction on n we can prove that

$$(\mathrm{L}_{x_1}\cdots\mathrm{L}_{x_n}h^{g_0})_{\Lambda_0}(g_0 f_{\Lambda_0}(u)) = (\mathbf{h}(x_1\cdots x_n))(g_0 f_{\Lambda_0}(u))$$

for every $u \in \mathcal{U}_{\Lambda_0}$ and every $x_1,\ldots,x_n \in \mathfrak{g}_{\overline{1}}$. The proof of Statement B is now complete. □

Remark 5.5.3. Note that $C^\infty(Y,\mathfrak{g})$ is an associative $\mathbb{C}$-superalgebra with the multiplication

$$\big(\mathbf{h}\cdot\mathbf{h}'\big)(x)(g) := \big(\mathsf{m}\circ(\mathbf{h}\otimes\mathbf{h}')\circ\mathsf{c}(x)\big)(g)$$

where $\mathsf{m}: C^\infty(Y,\mathbb{C})\otimes C^\infty(Y,\mathbb{C})\to C^\infty(Y,\mathbb{C})$ denotes the standard pointwise multiplication and $\mathsf{c}: U(\mathfrak{g}_\mathbb{C})\to U(\mathfrak{g}_\mathbb{C})\otimes U(\mathfrak{g}_\mathbb{C})$ denotes the standard co-multiplication. It can be shown that the map Φ of Theorem 5.5.2 is an isomorphism of $\mathbb{C}$-superalgebras. We will not need this fact and therefore we omit its proof.

6. The GNS construction

Theorem 5.5.2 allows us to substitute a Lie supergroup $\mathcal{G}$ by its associated Harish–Chandra pair. Therefore in the rest of this article we will not need the functorial formalism of the previous sections and we can concentrate on Harish–Chandra pairs. Throughout this section $(G,\mathfrak{g})$ will denote a Harish–Chandra pair.

6.1. Smooth and analytic unitary representations

We recall the definition of smooth and analytic unitary representations of a Harish–Chandra pair (see [CCTV06], [MeNeSa12], and [NeSa13]).

Definition 6.1.1. Let $(G,\mathfrak{g})$ be a Harish–Chandra pair (resp., an analytic Harish–Chandra pair). A *smooth unitary representation* (resp., an *analytic unitary representation*) of $(G,\mathfrak{g})$ is a triple $(\pi,\rho^\pi,\mathscr{H})$ satisfying the following properties.

(i) $(\pi,\mathscr{H})$ is a smooth (resp., analytic) unitary representation of G on the $\mathbb{Z}_2$-graded Hilbert space $\mathscr{H}=\mathscr{H}_{\overline{0}}\oplus\mathscr{H}_{\overline{1}}$ such that for every $g\in G$, the operator $\pi(g)$ preserves the $\mathbb{Z}_2$-grading.
(ii) $\rho^\pi:\mathfrak{g}\to\mathrm{End}_\mathbb{C}(\mathscr{B})$ is a representation of the Lie superalgebra $\mathfrak{g}$, where $\mathscr{B}=\mathscr{H}^\infty$ (resp., $\mathscr{B}=\mathscr{H}^\omega$).
(iii) For every $x\in\mathfrak{g}_{\overline{0}}$, if $\mathsf{d}\pi(x)$ denotes the infinitesimal generator of the one-parameter group $t\mapsto\pi(e^{tx})$, then $\rho^\pi(x)=\mathsf{d}\pi(x)\big|_{\mathscr{B}}$.
(iv) $e^{-\frac{\pi i}{4}}\rho^\pi(x)$ is a symmetric operator for every $x\in\mathfrak{g}_{\overline{1}}$.
(v) Every element of the component group G/G° has a coset representative $g\in G$ such that $\pi(g)\rho^\pi(x)\pi(g)^{-1}=\rho^\pi(\mathrm{Ad}(g)x)$ for every $x\in\mathfrak{g}_{\overline{1}}$.

Lemma 6.1.2. *Let $(G,\mathfrak{g})$ be a Harish–Chandra pair, $(\pi,\rho^\pi,\mathscr{H})$ be a smooth unitary representation of $(G,\mathfrak{g})$, and $v\in\mathscr{H}^\infty$. Assume that $\mathfrak{g}$ is a Fréchet–Lie superalgebra. Then for every $n\geq 1$ the map*

$$\mathfrak{g}^n\to\mathscr{H}\ ,\ (x_1,\ldots,x_n)\mapsto\rho^\pi(x_1)\cdots\rho^\pi(x_n)v$$

is continuous.

Proof. The proof is by induction on n. For $n=0$ there is nothing to prove. Let $n\geq 1$. First we prove that the map

$$\mathfrak{g}_{\overline{0}}\times\mathfrak{g}^{n-1}\to\mathscr{H}\ ,\ (x,x_1\ldots,x_{n-1})\mapsto\rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v \tag{21}$$

is continuous. For every nonzero $t\in\mathbb{R}$ define a map $f_t:\mathfrak{g}_{\overline{0}}\times\mathfrak{g}^{n-1}\to\mathscr{H}$ by

$$f_t(x,x_1,\ldots,x_{n-1}) := \frac{1}{t}\left(\pi(e^{tx})\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})-\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})\right)v.$$

The maps f_t are continuous from a Baire space into a metric space. Moreover,

$$\lim_{t\to 0} f_t(x, x_1, \ldots, x_{n-1}) = \rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v.$$

It follows from [Bo74, Ch. IX, §5, Ex. 22(a)] that the set of discontinuity points of the map (21) is of first category, and therefore its set of continuity points is nonempty. Since the map (21) is n-linear, [NeSa13, Lemma 4.8] implies that it is continuous.

To complete the proof it is enough to show that the map

$$\mathfrak{g}_{\overline{1}} \times \mathfrak{g}^{n-1} \to \mathscr{H}\ ,\ (x, x_1, \ldots, x_{n_1}) \mapsto \rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v \tag{22}$$

is continuous at $(0, \ldots, 0) \in \mathfrak{g}_{\overline{1}} \times \mathfrak{g}^{n-1}$. If $(x, x_1, \ldots, x_{n-1}) \in \mathfrak{g}_{\overline{1}} \times \mathfrak{g}^{n-1}$ then

$$\begin{aligned}
&\|\rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v\|^2 \\
&\qquad = \langle \rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v, \rho^\pi(x)\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v\rangle \\
&\qquad = |\langle \rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v, \rho^\pi(x)^2\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v\rangle| \\
&\qquad \leq \frac{1}{2}\|\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v\| \cdot \|\rho^\pi([x,x])\rho^\pi(x_1)\cdots\rho^\pi(x_{n-1})v\|.
\end{aligned}$$

Therefore continuity of (22) follows from the the induction hypothesis, continuity of (21), and continuity of the superbracket of $\mathfrak{g}$. □

Lemma 6.1.3. *Let $(G, \mathfrak{g})$ be an analytic Harish–Chandra pair. Assume that $\mathfrak{g}$ is a Fréchet–Lie superalgebra. Let $(\pi, \mathscr{H}, \rho^\pi)$ be a smooth unitary representation of $(G, \mathfrak{g})$. If $v \in \mathscr{H}^\omega$ then $\rho^\pi(x)v \in \mathscr{H}^\omega$ for every $x \in \mathfrak{g}$.*

Proof. The argument appears in the proof of [NeSa13, Thm 6.13], but for the reader's convenience we explain the details. Since $\mathscr{H}^\omega$ is $\mathfrak{g}_{\overline{0}}$-invariant, it suffices to prove that $\rho^\pi(x)v \in \mathscr{H}^\omega$ for every $x \in \mathfrak{g}_{\overline{1}}$. Set $w := \rho^\pi(x)v$. By [Ne11, Thm 5.2] it suffices to prove that the map

$$G \to \mathbb{C}\ ,\ g \mapsto \langle \pi(g)w, w\rangle \tag{23}$$

is analytic. Note that $\langle \pi(g)w, w\rangle = \sqrt{-1}\langle \pi(g)v, \rho^\pi(g\cdot x)w\rangle$. Since $w \in \mathscr{H}^\infty$, Lemma 6.1.2 implies that the linear map $z \mapsto \rho^\pi(z)w$ is continuous, and therefore the map $g \mapsto \rho^\pi(g\cdot x)w$ is analytic. Since the map $g \mapsto \pi(g)v$ is analytic, the map (23) is also analytic. □

6.2. The involutive monoid associated to $(G, \mathfrak{g})$

The anti-linear map

$$\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}\ ,\ x \mapsto x^*$$

defined by

$$x^* := \begin{cases} -x & \text{if } x \in \mathfrak{g}_{\overline{0}}, \\ -\sqrt{-1}\, x & \text{if } x \in \mathfrak{g}_{\overline{1}}. \end{cases}$$

is an anti-automorphism. It extends to an anti-linear anti-automorphism

$$U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}}) \ , \ D \mapsto D^* \tag{24}$$

in a canonical way. Consider the monoid S with underlying set $G \times U(\mathfrak{g}_{\mathbb{C}})$ and multiplication

$$(g_1, D_1)(g_2, D_2) = (g_1 g_2, (g_2^{-1} \cdot D_1) D_2)$$

where $g \cdot D$ denotes the adjoint action of $g \in G$ on $D \in U(\mathfrak{g}_{\mathbb{C}})$. The neutral element of S is $1_{\mathsf{S}} := (\mathbf{1}_G, 1_{U(\mathfrak{g}_{\mathbb{C}})})$. The map

$$\mathsf{S} \to \mathsf{S} \, , \, s \mapsto s^*$$

defined by

$$(g, D)^* := (g^{-1}, g \cdot (D^*))$$

is an involution of S.

Recall that $U(\mathfrak{g}_{\mathbb{C}})$ is an associative superalgebra. An element $(g, D) \in \mathsf{S}$ is called *odd* (resp. *even*) if D is an odd (resp. even) element of $U(\mathfrak{g}_{\mathbb{C}})$.

6.3. Smooth and analytic superfunctions on Harish–Chandra pairs

Similar to Section 5.5, let $C^\infty(G, \mathfrak{g})$ (resp., $C^\omega(G, \mathfrak{g})$) denote the set of $\mathbb{C}$-linear maps

$$f : U(\mathfrak{g}_{\mathbb{C}}) \to C^\infty(G, \mathbb{C})$$

which satisfy the following two properties.

(i) $f(xD)(g) = L_x(f(D))(g)$ for every $x \in \mathfrak{g}_{\overline{0}}$, every $D \in U(\mathfrak{g}_{\mathbb{C}})$, and every $g \in G$.

(ii) For every $n \geq 0$ the map

$$f^{[n]} : \mathfrak{g}^n \times G \to \mathbb{C} \ , \ f^{[n]}(x_1, \dots, x_n, g) := \big(f(x_1 \cdots x_n)\big)(g) \tag{25}$$

is smooth (resp., analytic).

Remark 6.3.1. Because of Theorem 5.5.2, the spaces $C^\infty(G, \mathfrak{g})$ and $C^\omega(G, \mathfrak{g})$ deserve to be called the spaces of smooth and analytic superfunctions on the Harish–Chandra pair $(G, \mathfrak{g})$.

For every $f \in C^\infty(G, \mathfrak{g})$ we set

$$\breve{f} : \mathsf{S} \to \mathbb{C} \ , \ \breve{f}(g, D) := f(D)(g)$$

for every $g \in G$ and every $D \in U(\mathfrak{g}_{\mathbb{C}})$.

Lemma 6.3.2. *Let $f \in C^\infty(G, \mathfrak{g})$, $(g, D) \in \mathsf{S}$, and $x \in \mathfrak{g}_{\overline{0}}$. Then*

$$\lim_{t \to 0} \frac{1}{t} \left(\breve{f}(g e^{tx}, e^{-tx} \cdot D) - \breve{f}(g, D) \right) = \breve{f}(g, Dx).$$

Proof. By linearity of the map $D \mapsto f(D)$ we can assume that D is a monomial of degree n. By the Chain Rule we have

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\big(\breve{f}(ge^{tx}, e^{-tx}\cdot D) - \breve{f}(g,D)\big)&\\ = \frac{d}{dt}\Big|_{t=0} f(e^{-tx}\cdot D)(ge^{tx}) &= f(-xD+Dx)(g) + L_x\big(f(D)\big)(g)\\ = f(-xD+Dx)(g) + f(xD)(g) &= f(Dx)(g) = \breve{f}(g,Dx)\end{aligned}$$

which is the desired equality. □

Observe that S acts on $\mathbb{C}^{\mathsf{S}}$ (the space of complex-valued functions on S) by right translation, that is, $(s\cdot\psi)(t) := \psi(ts)$ for every $s,t\in\mathsf{S}$ and every $\psi\in\mathbb{C}^{\mathsf{S}}$. The next lemma shows that $C^\infty(G,\mathfrak{g})$ is an invariant subspace of $\mathbb{C}^{\mathsf{S}}$ under this action.

Lemma 6.3.3. *Let $f\in C^\infty(G,\mathfrak{g})$ and $(g_\circ, D_\circ)\in\mathsf{S}$. Then the map*

$$h : U(\mathfrak{g}_{\mathbb{C}}) \to C^\infty(G,\mathbb{C})\ ,\ h(D)(g) := f\big((g_\circ^{-1}\cdot D)D_\circ\big)(gg_\circ)$$

belongs to $C^\infty(G,\mathfrak{g})$.

Proof. The only nontrivial statement is that $L_x(h(D))(g) = h(xD)(g)$ for $x\in\mathfrak{g}_{\overline{0}}$ and $D\in U(\mathfrak{g}_{\mathbb{C}})$. Observe that

$$\begin{aligned}L_x(h(D))(g) &= \lim_{t\to 0}\frac{1}{t}\big(h(D)(ge^{tx}) - h(D)\big)\\ &= \lim_{t\to 0}\frac{1}{t}\Big(f\big((g_\circ^{-1}\cdot D)D_\circ\big)(ge^{tx}g_\circ) - f\big((g_\circ^{-1}\cdot D)D_\circ\big)(gg_\circ)\Big)\\ &= \lim_{t\to 0}\frac{1}{t}\Big(f\big((g_\circ^{-1}\cdot D)D_\circ\big)(gg_\circ e^{t(g_\circ^{-1}\cdot x)}) - f\big((g_\circ^{-1}\cdot D)D_\circ\big)(gg_\circ)\Big)\\ &= L_{g_\circ^{-1}\cdot x}\big(f((g_\circ^{-1}\cdot D)D_\circ)\big)(gg_\circ) = f\big((g_\circ^{-1}\cdot(xD))D_\circ\big)(gg_\circ)\\ &= h(xD)(g)\end{aligned}$$

which completes the proof. □

6.4. Positive definite smooth superfunctions

We can now define positive definite smooth superfunctions on a Harish–Chandra pair using the involutive monoid S introduced in Section 6.2.

Definition 6.4.1. An $f\in C^\infty(G,\mathfrak{g})$ is called *even* if $\breve{f}(g,D)=0$ for every odd element $(g,D)\in\mathsf{S}$. An $f\in C^\infty(G,\mathfrak{g})$ is called *positive definite* if f is even and $\breve{f}$ is a positive definite function on S, i.e.,

$$\sum_{1\le i,j\le n}\overline{c_i}c_j\breve{f}(s_i^*s_j)\ge 0 \ \text{ for } n\ge 1,\ c_1,\dots,c_n\in\mathbb{C},\ \text{ and } s_1,\dots,s_n\in\mathsf{S}.$$

6.5. Matrix coefficients of unitary representations

For smooth and analytic unitary representations of a Harish–Chandra pair $(G,\mathfrak{g})$, the matrix coefficients are defined as follows.

Definition 6.5.1. Let $(\pi, \mathscr{H}, \rho^\pi)$ be a (smooth or analytic) unitary representation of $(G, \mathfrak{g})$. For every $v, w \in \mathscr{H}^\infty$ the function

$$\varphi_{v,w} : \mathsf{S} \to \mathbb{C}\,,\ \varphi_{v,w}(g, D) := \langle \pi(g)\rho^\pi(D)v, w\rangle$$

is called the *matrix coefficient* of the vectors v, w.

Proposition 6.5.2. *Let $(G, \mathfrak{g})$ be a Harish–Chandra pair such that $\mathfrak{g}$ is a Fréchet–Lie superalgebra.*

(i) *Let $(\pi, \rho^\pi, \mathscr{H})$ be a smooth unitary representation of $(G, \mathfrak{g})$ and $v, w \in \mathscr{H}^\infty$ be homogeneous vectors such that $|v| = |w|$. Then there exists an even $f \in C^\infty(G, \mathfrak{g})$ such that $\breve{f} = \varphi_{v,w}$.*

(ii) *Assume that $(G, \mathfrak{g})$ is an analytic Harish–Chandra pair. Let $(\pi, \rho^\pi, \mathscr{H})$ be an analytic unitary representation of $(G, \mathfrak{g})$ and $v, w \in \mathscr{H}^\omega$ be homogeneous vectors such that $|v| = |w|$. Then there exists an even $f \in C^\omega(G, \mathfrak{g})$ such that $\breve{f} = \varphi_{v,w}$.*

Proof. (i) It is fairly straightforward to check that the map

$$f : U(\mathfrak{g}_\mathbb{C}) \to C^\infty(G, \mathbb{C})\ ,\ f(D)(g) := \varphi_{v,w}(g, D)$$

is in $\mathrm{Hom}_{\mathfrak{g}_{\overline{0}}}(U(\mathfrak{g}_\mathbb{C}), C^\infty(G, \mathbb{C}))$. To complete the proof, we need to show that for every $n \geq 0$ the map

$$\varphi_{n,v,w} : \mathfrak{g}^n \times G \to \mathbb{C}\ ,\ \varphi_{n,v,w}(x_1, \ldots, x_n, g) := \langle \pi(g)\rho^\pi(x_1 \cdots x_n)v, w\rangle \qquad (26)$$

is smooth. Note that

$$\varphi_{n,v,w}(x_1, \ldots, x_n, g) = \langle \rho^\pi(x_1 \cdots x_n)v, \pi(g^{-1})w\rangle.$$

By Lemma 6.1.2 the n-linear map $(x_1, \ldots, x_n) \mapsto \rho^\pi(x_1 \cdots x_n)v$ is continuous, hence smooth. Since $w \in \mathscr{H}^\infty$, the map $g \mapsto \pi(g^{-1})w$ is smooth. Therefore the map (26) is also smooth.

(ii) The proof is similar to (i), as a continuous multilinear map is analytic. □

In Theorem 6.7.5 we prove a converse for Proposition 6.5.2.

Remark 6.5.3. Assume $\mathfrak{g}_{\overline{1}} = \{0\}$. Then the condition of Definition 6.4.1 a priori seems to be stronger than the condition which is classically used to define positive definite functions on a Lie group: classically, a map $f : G \to \mathbb{C}$ is called positive definite if

$$\sum_{1 \leq i,j \leq n} c_i \overline{c}_j f(g_i g_j^{-1}) \geq 0 \ \text{ for } n \geq 1,\ g_1, \ldots, g_n \in G\,,\ \text{and } c_1, \ldots, c_n \in \mathbb{C}.$$

However, for smooth maps $f : G \to \mathbb{C}$, the classical definition and Definition 6.4.1 are equivalent. In fact, if $f \in C^\infty(G, \mathbb{C})$ is positive definite in the classical sense, then by the GNS construction [Ne00, III.1.22] we have $f(g) = \langle \pi(g)v, v\rangle$ for some unitary representation $(\pi, \mathscr{H})$ of G. Since f is smooth, from [Ne10, Thm 7.2] it follows that $v \in \mathscr{H}^\infty$, and therefore the map

$$\mathsf{S} \to \mathbb{C}\ ,\ (g, D) \mapsto \langle \pi(g)\rho^\pi(D)v, v\rangle$$

is well-defined. It is easy to check that the latter map is positive definite in the sense of Definition 6.4.1.

6.6. The reproducing kernel Hilbert space

To every smooth unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $(G, \mathfrak{g})$ we can associate a representation $\widetilde{\rho^\pi}$ of the monoid S as follows:

$$\widetilde{\rho^\pi} : \mathsf{S} \to \mathrm{End}(\mathscr{H}^\infty)\ ,\ \widetilde{\rho^\pi}(g, D) := \pi(g)\rho^\pi(D) \text{ for every } (g, D) \in \mathsf{S}. \tag{27}$$

Observe that $(\widetilde{\rho^\pi}, \mathscr{H}^\infty)$ is a $*$-*representation*, i.e., $\langle \widetilde{\rho^\pi}(s)v, w\rangle = \langle v, \widetilde{\rho^\pi}(s^*)w\rangle$ for every $s \in \mathsf{S}$ and every $v, w \in \mathscr{H}^\infty$. It is easy to check that for every $v \in \mathscr{H}^\infty$ the matrix coefficient $\varphi_{v,v}$ is positive definite.

Conversely, one can associate a $*$-representation of S to a positive definite function $\varphi : \mathsf{S} \to \mathbb{C}$ as follows. Set

$$\mathscr{D}_\varphi := \mathrm{Span}_{\mathbb{C}}\{\varphi_s \ :\ s \in \mathsf{S}\} \subseteq \mathbb{C}^{\mathsf{S}} \tag{28}$$

where $\varphi_s : \mathsf{S} \to \mathbb{C}$ is defined by $\varphi_s(t) := \varphi(ts)$. Observe that $\mathscr{D}_\varphi$ has a pre-Hilbert space structure given by

$$\langle \varphi_s, \varphi_t\rangle := \varphi(t^*s).$$

Set $K(s,t) := \varphi(st^*)$ and define $K_s : \mathsf{S} \to \mathbb{C}$ by $K_s(t) := K(t,s) = \varphi_{s^*}(t)$. The completion $\mathscr{H}_\varphi$ of $\mathscr{D}_\varphi$ is a reproducing kernel Hilbert space with kernel $K(s,t)$. In other words, one can identify $\mathscr{H}_\varphi$ with a space of complex valued functions on S such that

$$h(s) = \langle h, K_s\rangle \text{ for every } h \in \mathscr{H}_\varphi \text{ and every } s \in \mathsf{S}. \tag{29}$$

The monoid S acts on $\mathscr{D}_\varphi$ by right translation, yielding a $*$-representation $(\widetilde{\rho_\varphi}, \mathscr{D}_\varphi)$ of S. More precisely,

$$\big(\widetilde{\rho_\varphi}(s)\psi\big)(t) = \psi(ts) \text{ for every } s, t \in \mathsf{S} \text{ and every } \psi \in \mathscr{D}_\varphi.$$

Remark 6.6.1. If an element $s \in \mathsf{S}$ satisfies $ss^* = s^*s = 1_{\mathsf{S}}$, then $\widetilde{\rho_\varphi}(s) : \mathscr{D}_\varphi \to \mathscr{D}_\varphi$ is an isometry and therefore extends to an isometry $\widetilde{\rho_\varphi}(s) : \mathscr{H}_\varphi \to \mathscr{H}_\varphi$, yielding a unitary representation of the abstract group $\{s \in \mathsf{S} \ :\ ss^* = s^*s = 1_{\mathsf{S}}\}$.

6.7. Cyclic representations and the GNS construction

Our next goal is to prove Theorem 6.7.5 below, which is the analogue of the GNS construction for Lie supergroups.

Definition 6.7.1. A smooth (resp. analytic) unitary representation $(\pi, \rho^\pi, \mathscr{H})$ is called *cyclic* if there exists a vector $v_\circ \in \mathscr{H}_{\overline{0}}^\infty$ (resp. $v_\circ \in \mathscr{H}_{\overline{0}}^\omega$) such that the set

$$\mathrm{Span}_{\mathbb{C}}\{\pi(g)\rho^\pi(D)v_\circ \ :\ g \in G \text{ and } D \in U(\mathfrak{g}_{\mathbb{C}})\}$$

is dense in $\mathscr{H}$. The vector $v_\circ$ is called a *cyclic vector* of $(\pi, \rho^\pi, \mathscr{H})$.

Remark 6.7.2. To indicate that a unitary representation $(\pi, \rho^\pi, \mathscr{H})$ is cyclic with a cyclic vector $v_\circ$, we write $(\pi, \rho^\pi, \mathscr{H}, v_\circ)$.

Definition 6.7.3. We say a Lie group G has the *Trotter property* if for every $x, y \in \mathrm{Lie}(G)$ the equality

$$e^{t(x+y)} = \lim_{n\to\infty} \big(e^{\frac{t}{n}x} e^{\frac{t}{n}y}\big)^n$$

holds in the sense of uniform convergence on compact subsets of $\mathbb{R}$.

Example 6.7.4. We now mention some examples of Lie supergroups $\mathcal{G}$ for which $\mathcal{G}_{\Lambda_0}$ has the Trotter property. Proofs are given in [NeSa13].

(i) Every locally exponential Lie group (and in particular every Banach–Lie group) has the Trotter property. Therefore if $\mathcal{G}$ is a Lie supergroup modeled on a $\mathbb{Z}_2$-graded Banach space, then $\mathcal{G}_{\Lambda_0}$ has the Trotter property.

(ii) From (i) it follows that if M is a compact smooth manifold and K is a finite-dimensional Lie group, then the mapping group $C^\infty(M,K)$ and its central extensions have the Trotter property. These Lie groups appear as $\mathcal{G}_{\Lambda_0}$ of mapping Lie supergroups $\mathcal{G} := C^\infty(M,\mathcal{K})$, where M is a compact manifold and $\mathcal{K}$ is a finite-dimensional Lie supergroup, or mapping Lie supergroups $\mathcal{G} := C^\infty(\mathcal{M},K)$, where $\mathcal{M}$ is a compact supermanifold and K is a finite-dimensional Lie group.

(iii) If M is a compact smooth manifold then the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M and its central extensions have the Trotter property. This implies that if $\mathcal{G}$ is the Lie supergroup of smooth diffeomorphisms of certain compact supermanifolds, such as the supercircle $S^{1|1}$, then $\mathcal{G}_{\Lambda_0}$ has the Trotter property. We expect the latter statement to hold for other classes of compact supermanifolds.

Theorem 6.7.5. *Let $(G,\mathfrak{g})$ be a Harish–Chandra pair such that $\mathfrak{g}$ is a Fréchet–Lie superalgebra and G has the Trotter property. Let $f \in C^\infty(G,\mathfrak{g})$ be positive definite.*

(i) *There exists a cyclic smooth unitary representation $(\pi,\rho^\pi,\mathscr{H},v_\circ)$ of $(G,\mathfrak{g})$ such that $\breve{f} = \varphi_{v_\circ,v_\circ}$.*

(ii) *Let $(\sigma,\rho^\sigma,\mathscr{K},w_\circ)$ be another cyclic smooth unitary representation of $(G,\mathfrak{g})$ such that $\breve{f} = \varphi_{w_\circ,w_\circ}$. Then $(\pi,\rho^\pi,\mathscr{H})$ and $(\sigma,\rho^\sigma,\mathscr{K})$ are unitarily equivalent via an intertwining operator*

$$T : (\sigma,\rho^\sigma,\mathscr{K}) \to (\pi,\rho^\pi,\mathscr{H})$$

which maps $w_\circ$ to $v_\circ$.

(iii) *Assume that $(G,\mathfrak{g})$ is an analytic Harish–Chandra pair and the map*

$$G \to \mathbb{C}\ ,\ g \mapsto f(1_{U(\mathfrak{g}_\mathbb{C})})(g)$$

is analytic. Then the representation $(\pi,\mathscr{H},\rho^\pi,v_\circ)$ obtained in (i) *is an analytic representation of $(G,\mathfrak{g})$.*

Proof. (i) Set $\varphi := \breve{f}$. Let $\mathscr{H} := \mathscr{H}_\varphi$ be the Hilbert completion of $\mathscr{D}_\varphi$ defined in (28), and $v_\circ := \varphi \in \mathscr{H}$. For every $s := (g,D) \in \mathsf{S} = G \times U(\mathfrak{g})$ we have

$$\varphi_{v_\circ,v_\circ}(s) = \langle \varphi_s, \varphi\rangle = \varphi(s) = \breve{f}(s).$$

By Remark 6.6.1, if $g \in G$ then $\widetilde{\rho_\varphi}\big(g, 1_{U(\mathfrak{g}_\mathbb{C})}\big)$ extends to an isometry of $\mathscr{H}_\varphi$. Setting $\pi(g) := \widetilde{\rho_\varphi}\big(g, 1_{U(\mathfrak{g}_\mathbb{C})}\big)$ we obtain a representation π of G on $\mathscr{H}$ by unitary operators.

Our next goal is to prove that $(\pi, \mathscr{H})$ is a smooth unitary representation of G and $\mathscr{D}_\varphi \subseteq \mathscr{H}^\infty$, where $\mathscr{H}^\infty$ is the subspace of smooth vectors of $(\pi, \mathscr{H})$. By [Ne10, Theorem 7.2] it suffices to prove that for every $s \in \mathsf{S}$ the map

$$G \to \mathbb{C}\ ,\ g \mapsto \langle \pi(g)\varphi_s, \varphi_s\rangle \tag{30}$$

is smooth. Fix $s = (g_\circ, D_\circ) \in \mathsf{S}$ and let $\tilde{g} := (g, 1_{U(\mathfrak{g}_\mathbb{C})}) \in \mathsf{S}$ for every $g \in G$. Note that

$$\langle \pi(g)\varphi_s, \varphi_s\rangle = \langle \varphi_{\tilde{g}s}, \varphi_s\rangle = \varphi(s^*\tilde{g}s) = f\big((g_\circ^{-1}g^{-1}g_\circ \cdot D_\circ^*)D_\circ\big)(g_\circ^{-1}gg_\circ)$$

and therefore smoothness of (30) follows from smoothness of the map (14).

Next we prove that if $x \in \mathfrak{g}_{\overline{0}}$ and $v \in \mathscr{D}_\varphi$ then $\mathsf{d}\pi(x)v = \widetilde{\rho_\varphi}\big(\mathbf{1}_G, x\big)v$. It suffices to take $v = \varphi_s$ for some $s := (g_\circ, D_\circ) \in \mathsf{S}$. Let $s' := (g, D) \in \mathsf{S}$. Then $\varphi_s(s') = \varphi(s's) = f\big((g_\circ^{-1}\cdot D)D_\circ\big)(gg_\circ)$. Lemma 6.3.3 implies that $\varphi_s = \breve{h}$ for some $h \in C^\infty(G, \mathfrak{g})$. From (29) and Lemma 6.3.2 it follows that

$$\begin{aligned}\big(\mathsf{d}\pi(x)\varphi_s\big)(s') &= \langle \mathsf{d}\pi(x)\breve{h}, K_{s'}\rangle = \langle \lim_{t\to 0}\frac{1}{t}(\pi(e^{tx})\breve{h} - \breve{h}), K_{s'}\rangle \\ &= \lim_{t\to 0}\frac{1}{t}\langle \pi(e^{tx})\breve{h} - \breve{h}, K_{s'}\rangle = \lim_{t\to 0}\frac{1}{t}(\breve{h}(ge^{tx}, e^{-tx}\cdot D) - \breve{h}(g, D)) \\ &= \breve{h}(g, Dx) = \varphi_s(g, Dx) = \varphi_s\big((g, D)(\mathbf{1}_G, x)\big) = \big(\widetilde{\rho_\varphi}\big(\mathbf{1}_G, x\big)\varphi_s\big)(s').\end{aligned}$$

Finally, to complete the proof of existence of $(\pi, \rho^\pi, \mathscr{H}, v_\circ)$ set $\rho^{\mathscr{D}_\varphi}(x) := \widetilde{\rho_\varphi}(\mathbf{1}_G, x)$ for every $x \in \mathfrak{g}$. From linearity of $\varphi_s(g, D)$ in D it follows directly that the $\mathbb{Z}_2$-grading of $U(\mathfrak{g}_\mathbb{C})$ induces a $\mathbb{Z}_2$-grading on $\mathscr{D}_\varphi$ (and hence on $\mathscr{H}_\varphi$) and the actions of G and $\mathfrak{g}$ are compatible with the $\mathbb{Z}_2$-grading. From [MeNeSa12, Lem. 2.2(a)] it follows that for every $x \in \mathfrak{g}_{\overline{0}}$, the operator $\rho^{\mathscr{D}_\varphi}(x)$ is essentially skew-adjoint. Consequently, the 4-tuple $(\pi, \mathscr{H}, \mathscr{D}_\varphi, \rho^{\mathscr{D}_\varphi})$ is a pre-representation of $(\mathfrak{g}, G)$ in the sense of [NeSa13, Def. 6.4]. Therefore the existence of $(\pi, \rho^\pi, \mathscr{H})$ follows from [NeSa13, Thm 6.13].

(ii) In principle, the proof is similar to the standard uniqueness proofs of the GNS construction (e.g., see [Ne00, Thm III.1.22]). Nevertheless, [Ne00, Thm III.1.22] is not directly applicable because for example the representation of the semigroup S is not bounded. The new technical issues that arise in the super context will be addressed below.

Let $\mathscr{H}^\infty$ (resp., $\mathscr{K}^\infty$) denote the set of smooth vectors of $(\pi, \mathscr{H})$ (resp., $(\sigma, \mathscr{K})$). As in (27) we obtain $*$-representations $(\widetilde{\rho^\pi}, \mathscr{H}^\infty)$ and $(\widetilde{\rho^\sigma}, \mathscr{K}^\infty)$ of S. Set

$$\mathscr{D}_{v_\circ} := \mathrm{Span}_\mathbb{C}\{\widetilde{\rho^\pi}(s)v_\circ\ :\ s \in \mathsf{S}\} \text{ and } \mathscr{D}_{w_\circ} := \mathrm{Span}_\mathbb{C}\{\widetilde{\rho^\sigma}(s)w_\circ\ :\ s \in \mathsf{S}\}.$$

Define a $\mathbb{C}$-linear map $T : \mathscr{D}_{w_\circ} \to \mathscr{D}_{v_\circ}$ by

$$T\widetilde{\rho^\sigma}(s)w_\circ := \widetilde{\rho^\pi}(s)v_\circ \text{ for all } s \in \mathsf{S}.$$

It is straightforward to check that T is well-defined and extends to an isometry $T : \mathscr{K} \to \mathscr{H}$. From the definition of T it follows that

$$T\widetilde{\rho^\sigma}(s)u = \widetilde{\rho^\pi}(s)Tu \text{ for every } u \in \mathscr{D}_{w_\circ} \text{ and every } s \in \mathsf{S}. \tag{31}$$

Since $\mathscr{D}_{w_\circ}$ is dense in $\mathscr{K}$, from (31) it follows that

$$T\sigma(g)u = \pi(g)Tu \text{ for every } u \in \mathscr{K} \text{ and every } g \in G.$$

Consequently, $T\mathscr{K}^\infty = \mathscr{H}^\infty$. Next we prove that

$$T\rho^\sigma(x)u = \rho^\pi(x)Tu \text{ for every } u \in \mathscr{K}^\infty \text{ and every } x \in \mathfrak{g}.$$

It suffices to prove the latter statement for $x \in \mathfrak{g}_{\overline{1}}$. Set

$$P_1 := e^{-\frac{\pi i}{4}}\rho^\pi(x),\ P_2 := e^{-\frac{\pi i}{4}}T\rho^\sigma(x)T^{-1}\Big|_{\mathscr{H}^\infty} \text{ and } \mathscr{L} := T\mathscr{D}_{w_\circ} = \mathscr{D}_{v_\circ}.$$

The linear operators P_1 and P_2 are symmetric with common domain $\mathscr{H}^\infty$ such that

$$P_1\Big|_{\mathscr{L}} = P_2\Big|_{\mathscr{L}}.$$

Since $\mathscr{L}$ is G-invariant, from [MeNeSa12, Lem. 2.2(a)] it follows that $(P_1\Big|_{\mathscr{L}})^2$ is essentially self-adjoint, and therefore by [MeNeSa12, Lem. 2.4] the operator $P_1\Big|_{\mathscr{L}}$ is essentially self-adjoint. Consequently, by [MeNeSa12, Lem. 2.5] we have $P_1 = P_2$.

(iii) Let $(\pi, \rho^\pi, \mathscr{H})$ be the smooth unitary representation obtained in (i). By [NeSa13, Thm 6.13(b)] it is enough to show that $\mathscr{D}_\varphi \subseteq \mathscr{H}^\omega$. Since $\langle \pi(g)\varphi, \varphi\rangle = f(1_{U(\mathfrak{g}_\mathbb{C})})(g)$, from [Ne11, Thm 5.2] it follows that $\varphi \in \mathscr{H}^\omega$. From Lemma 6.1.3 it follows that $\rho^\pi(D_\circ)\varphi \in \mathscr{H}^\omega$ for every $D_\circ \in U(\mathfrak{g}_\mathbb{C})$. Finally, for every $s := (g_\circ, D_\circ) \in \mathsf{S}$ we have

$$\varphi_s = \widetilde{\rho_\varphi}(s)\varphi = \widetilde{\rho_\varphi}(g_\circ, 1_{U(\mathfrak{g}_\mathbb{C})})\widetilde{\rho_\varphi}(\mathbf{1}_G, D_\circ)\varphi = \pi(g_\circ)\rho^\pi(D_\circ)\varphi \in \mathscr{H}^\omega$$

because $\mathscr{H}^\omega$ is G-invariant. □

The next corollary, which is of independent interest, is in a sense an automatic analyticity criterion for smooth superfunctions in odd directions.

Corollary 6.7.6. *Let $\mathcal{G}$ be an analytic Lie supergroup modeled on a $\mathbb{Z}_2$-graded Fréchet space and $(G, \mathfrak{g})$ be the Harish–Chandra pair associated to $\mathcal{G}$. Assume that G has the Trotter property. Let $f \in C^\infty(G, \mathfrak{g})$ be positive definite. If $f(1_{U(\mathfrak{g}_\mathbb{C})}) \in C^\omega(G, \mathbb{C})$ then $f \in C^\omega(G, \mathfrak{g})$.*

Proof. From Theorem 6.7.5(iii) it follows that $\breve{f} = \varphi_{v,v}$, where v is an analytic vector. Thus, by Proposition 6.5.2, we have $f \in C^\omega(G, \mathfrak{g})$. □

7. A characterisation of integrable linear functionals

Let $(G, \mathfrak{g})$ be an analytic Harish–Chandra pair such that G is 1-connected (that is, connected and simply connected). In this section we give a characterisation of $\mathbb{C}$–linear functionals $\lambda : U(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ which are *integrable* in the sense that there exists an analytic unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $(G, \mathfrak{g})$ such that $\lambda(D) = \langle \rho^\pi(D)v, v\rangle$ for some $v \in \mathscr{H}^\omega$. For Lie groups, this question is addressed in detail in [Ne11, Sec. 6].

7.1. Weak and strong analyticity of linear functionals

We begin by defining the notion of analyticity of a linear functional on $U(\mathfrak{g}_{\mathbb{C}})$.

Definition 7.1.1. Let $\lambda : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ be a $\mathbb{C}$-linear map. We say λ is *even* if $\lambda(D) = 0$ for every $D \in U(\mathfrak{g}_{\mathbb{C}})_{\overline{1}}$. We say λ is *positive* if λ is even and $\lambda(D^*D) \geq 0$ for every $D \in U(\mathfrak{g}_{\mathbb{C}})$, where $D \mapsto D^*$ is the map given in (24). We say λ is *continuous* if for every $n \geq 0$ the map

$$\mathfrak{g}^n \to \mathbb{C}\ ,\ (x_1, \ldots, x_n) \mapsto \lambda(x_1 \cdots x_n)$$

is continuous. A continuous $\mathbb{C}$-linear map $\lambda : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ is called *weakly analytic* if for every $D_1, D_2 \in U(\mathfrak{g}_{\mathbb{C}})$ there exists a 0-neighborhood $U_{D_1,D_2} \subseteq \mathfrak{g}_{\mathbb{C},\overline{0}} := \mathfrak{g}_{\overline{0}} \otimes_{\mathbb{R}} \mathbb{C}$ such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} |\lambda(D_1 x^n D_2)|$$

converges for every $x \in U_{D_1,D_2}$. A continuous $\mathbb{C}$-linear map $\lambda : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ is called *strongly analytic* if there exists a 0-neighborhood $U \subseteq \mathfrak{g}_{\mathbb{C},\overline{0}}$ such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} |\lambda(D_1 x^n D_2)|$$

converges for every $D_1, D_2 \in U(\mathfrak{g}_{\mathbb{C}})$ and every $x \in U$.

7.2. Characterisations of integrability

We recall the definition of a BCH–Lie group.

Definition 7.2.1. An analytic Lie group G is called a *BCH–Lie group* if the exponential map is an analytic diffeomorphism in an open 0-neighborhood.

Every Banach–Lie group is a BCH Lie group. For a detailed study and several interesting examples of BCH–Lie groups see [Gl02] and [Ne06, Sec. IV].

Theorem 7.2.2. *Let $(G, \mathfrak{g})$ be an analytic Harish–Chandra pair such that $\mathfrak{g}$ is a Fréchet–Lie superalgebra and G is a 1-connected BCH–Lie group. Let*

$$\lambda : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$

be a $\mathbb{C}$-linear map. The following statements are equivalent.

(i) *λ is positive and strongly analytic.*

(ii) *There exists an analytic unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $(G, \mathfrak{g})$ and a homogeneous vector $v \in \mathscr{H}^\omega$ such that $\lambda(D) = \langle \rho^\pi(D)v, v\rangle$ for every $D \in U(\mathfrak{g}_{\mathbb{C}})$.*

Proof. (i)⇒(ii): Let $U(\mathfrak{g}_{\mathbb{C}})^*$ denote the algebraic dual of $U(\mathfrak{g}_{\mathbb{C}})$ and

$$\rho : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(U(\mathfrak{g}_{\mathbb{C}})^*)\ ,\ \big(\rho(x)\mu\big)(D) := \mu(Dx) \text{ for every } D \in U(\mathfrak{g}_{\mathbb{C}})$$

be the right regular representation of $\mathfrak{g}_{\mathbb{C}}$ on $U(\mathfrak{g}_{\mathbb{C}})^*$. Set $\mathscr{D}_\lambda := \rho(U(\mathfrak{g}_{\mathbb{C}}))\lambda$. We denote the restriction of ρ to $\mathscr{D}_\lambda$ by $(\rho_\lambda, \mathscr{D}_\lambda)$. We endow $\mathscr{D}_\lambda$ with a pre-Hilbert structure as follows:

$$\langle \rho_\lambda(D_1)\lambda, \rho_\lambda(D_2)\lambda\rangle := \lambda(D_2^* D_1) \text{ for every } D_1, D_2 \in U(\mathfrak{g}_{\mathbb{C}}).$$

It is easily checked that $\langle \rho_\lambda(D)\mu_1, \mu_2\rangle = \langle \mu_1, \rho_\lambda(D^*)\mu_2\rangle$ for every $\mu_1, \mu_2 \in \mathscr{D}_\lambda$, i.e., $(\rho_\lambda, \mathscr{D}_\lambda)$ is a $*$-representation. Since λ is even, the $\mathbb{Z}_2$-grading of $U(\mathfrak{g}_\mathbb{C})$ induces a $\mathbb{Z}_2$-grading on $\mathscr{D}_\lambda$ with perpendicular homogeneous components. The rest of the proof is given in the following four steps:

Step 1. Since λ is continuous, $(\rho_\lambda, \mathscr{D}_\lambda)$ is a strongly continuous representation of $\mathfrak{g}_{\overline{0}}$, i.e., for every $\mu \in \mathscr{D}_\lambda$ the map

$$\mathfrak{g}_{\overline{0}} \to \mathscr{D}_\lambda \ , \ x \mapsto \rho_\lambda(x)\mu$$

is continuous.

Step 2. We claim that $\mathscr{D}_\lambda$ is equianalytic in the sense of [Ne11, Def. 6.4], that is, there exists a 0-neighborhood $V \subseteq \mathfrak{g}_{\overline{0}}$ such that for every $\mu \in \mathscr{D}_\lambda$ the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|\rho_\lambda(x)^n \mu\|$$

converges for all $x \in V$. Let $U \subseteq \mathfrak{g}_{\overline{0}}$ be a 0-neighborhood such that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} |\lambda(D_1 x^n D_2)|$$

converges for every $D_1, D_2 \in U(\mathfrak{g}_\mathbb{C})$ and every $x \in U$. Assume that $\mu = \rho_\lambda(D_\circ)\lambda$ for some $D_\circ \in U(\mathfrak{g}_\mathbb{C})$. Then

$$||\rho_\lambda(x)^n \mu||^2 = \langle \rho_\lambda(x^n D_\circ)\lambda, \rho_\lambda(x^n D_\circ)\lambda\rangle = |\lambda(D_\circ^* x^{2n} D_\circ)|.$$

Set $V = \frac{1}{r}U := \{x \in \mathfrak{g}_{\overline{0}} \ : \ rx \in U\}$ where $r > 2$. If $x \in V$ then by the Cauchy–Schwarz inequality

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \|\rho_\lambda(x)^n \mu\| &= \sum_{n=0}^{\infty} \frac{1}{n!} |\lambda(D_\circ^* x^{2n} D_\circ)|^{\frac{1}{2}} \\ &\leq \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!} |\lambda(D_\circ^* (rx)^{2n} D_\circ)|\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(2n)!}{n!n!r^{2n}}\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

Step 3. By [Ne11, Thm 6.8] there exists a unitary representation $(\pi_\lambda, \mathscr{H}_\lambda)$ of G, where $\mathscr{H}_\lambda$ is the completion of $\mathscr{D}_\lambda$, such that $\mathsf{d}\pi_\lambda(x)\Big|_{\mathscr{D}_\lambda} = \rho_\lambda(x)$ for every $x \in \mathfrak{g}_{\overline{0}}$.

Step 4. From the previous steps and [MeNeSa12, Lem. 2.2(b)] it follows that $(\pi_\lambda, \mathscr{H}_\lambda, \mathscr{D}_\lambda, \rho_\lambda)$ is a pre-representation of $(G, \mathfrak{g})$. By [NeSa13, Thm 6.13(b)], the latter pre-representation corresponds to an analytic unitary representation of $(G, \mathfrak{g})$.

(ii)⇒(i): To check that λ is positive is routine. Next we prove that λ is strongly analytic. Let P denote the set of seminorms that define the topology of $\mathfrak{g}_{\mathbb{C},\overline{0}}$. For every $p \in \mathsf{P}$ and every $r > 0$ we set

$$U_{p,r} := \{x \in \mathfrak{g}_{\mathbb{C},\overline{0}} \ : \ p(x) < r\}$$

and

$$\mathscr{H}^{\omega,p,r} := \left\{ w \in \mathscr{H}^\infty \ : \ \sum_{n=0}^\infty \frac{1}{n!} \mathsf{d}\pi(x)^n w \text{ converges for every } x \in U_{p,r} \right\}.$$

Choose $p_\circ \in \mathsf{P}$ and $r_\circ > 0$ such that the restriction of the exponential map of G to $U_{p_\circ,r_\circ} \cap \mathfrak{g}_{\overline{0}}$ is an analytic diffeomorphism, the Baker–Campbell–Hausdorff product defines an analytic function $U_{p_\circ,r_\circ} \times U_{p_\circ,r_\circ} \to \mathfrak{g}_{\mathbb{C},\overline{0}}$, and the map

$$\mathfrak{g}_{\overline{0}} \times \mathfrak{g} \to \mathfrak{g} \ , \ (x,y) \mapsto \mathrm{Ad}(e^x)(y)$$

extends to a complex analytic map

$$U_{p_\circ,r_\circ} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}. \tag{32}$$

If $p \in \mathsf{P}$ then we write $p \geq p_\circ$ if $U_{p,r} \subseteq U_{p_\circ,r}$ for some (equivalently, every) $r > 0$. The rest of the proof is given in the following four steps:

Step 5. Let $p \geq p_\circ$, $0 < r < r_\circ$, and $v \in \mathscr{H}^\infty$. Then $v \in \mathscr{H}^{\omega,p,r}$ if and only if the orbit map

$$\mathfrak{g}_{\overline{0}} \mapsto \mathscr{H} \ , \ x \mapsto \pi(e^x)v$$

extends to an analytic function on $U_{p,r}$. The proof of the latter statement is similar to the proof of [MeNeSa12, Lem. 3.3]. (Here the main point is that $U_{p,r}$ is a balanced 0-neighborhood.)

Step 6. Let $p \geq p_\circ$, $0 < r < r_\circ$, $v \in \mathscr{H}^{\omega,p,r}$, and $a \in \mathfrak{g}_{\overline{0}}$. Then

$$\sum_{n=0}^\infty \frac{1}{n!} \mathsf{d}\pi(x)^n v \in \mathscr{H}^\omega \text{ for every } x \in U_{p,r}$$

and the map

$$u_a : U_{p,r} \to \mathscr{H} \ , \ u_a(x) := \mathsf{d}\pi(a) \left(\sum_{n=0}^\infty \frac{1}{n!} \mathsf{d}\pi(x)^n v \right)$$

is an analytic function. The latter statement is an extension of [MeNeSa12, Lem. 3.4] to Fréchet–Lie groups. The proof of [MeNeSa12, Lem. 3.4] is still valid because we are assuming that the BCH product formula locally defines an analytic function.

Step 7. If $0 < r < r_\circ$ and $p \geq p_\circ$, then $\mathscr{H}^{\omega,p,r}$ is $\mathfrak{g}_{\mathbb{C}}$-invariant. The latter statement is an extension of [MeNeSa12, Prop. 4.9] and its proof is an adaptation of the proof of [MeNeSa12, Prop. 4.9]. For the reader's convenience we briefly explain the necessary modifications. Instead of [MeNeSa12, Lem. 3.3] and [MeNeSa12, Lem 3.4] one uses Steps 5-6 above. In order to prove that the map given in [MeNeSa12, Eq. (30)] is analytic, one can substitute the norm estimates given in [MeNeSa12] by the analyticity of the map (32).

Step 8. Choose $0 < r < r_\circ$ and $p \geq p_\circ$ such that $v \in \mathscr{H}^{\omega,p,r}$. By $\mathfrak{g}_{\mathbb{C}}$-invariance of $\mathscr{H}^{\omega,p,r}$ the series

$$\sum_{n=0}^\infty \frac{1}{n!} \rho^\pi(x)^n \rho^\pi(D_2) v$$

converges for every $x \in U_{p,r}$ and every $D_2 \in U(\mathfrak{g}_{\mathbb{C}})$. Therefore the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} |\lambda(D_1 x^n D_2)| = \sum_{n=0}^{\infty} \frac{1}{n!} |\langle \rho^\pi(x)^n \rho^\pi(D_2) v, \rho^\pi(D_1^*) v \rangle|$$

converges for every $D_1, D_2 \in U(\mathfrak{g}_{\mathbb{C}})$ and every $x \in U_{p,\frac{r}{2}}$. □

Corollary 7.2.3. *Let $(G, \mathfrak{g})$ be an analytic Harish–Chandra pair such that $\mathfrak{g}$ is a Banach–Lie superalgebra and G is 1-connected. Let $\lambda : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ be a $\mathbb{C}$-linear map. The following statements are equivalent.*

(i) *λ is positive and weakly analytic.*
(ii) *λ is positive and strongly analytic.*
(iii) *There exists an analytic unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $(G, \mathfrak{g})$ and a homogeneous vector $v \in \mathscr{H}^\omega$ such that $\lambda(D) = \langle \rho^\pi(D) v, v \rangle$ for every $D \in U(\mathfrak{g}_{\mathbb{C}})$.*

Proof. (iii)⇒(ii) follows from Theorem 7.2.2 and (ii)⇒(i) is trivial. For (i)⇒(iii) the proof of Theorem 7.2.2 still remains valid, because for Banach–Lie groups the conclusion of [Ne11, Thm 6.8] remains true without assuming equianalyticity. □

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