Lie Supergroups, Unitary Representations, and Invariant Cones

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1 Introduction

The goal of this article is twofold. First, it presents an application of the theory of invariant convex cones of Lie algebras to the study of unitary representations of Lie supergroups. Second, it provides an exposition of recent results of the second author on the classification of irreducible unitary representations of nilpotent Lie supergroups using the method of orbits.

In relation to the first goal, it is shown that there is a close connection between unitary representations of Lie supergroups and dissipative unitary representations of Lie groups (in the sense of [20]). It will be shown that for a large class of Lie supergroups the only irreducible unitary representations are highest weight modules in a suitable sense. This circle of ideas leads to explicit necessary conditions for determining when a Lie supergroup has faithful unitary representations. These necessary conditions are then used to analyze the situation for simple and semisimple Lie supergroups.

Pertaining to the second goal, the main results in [27] are explained in a more reader friendly style. Complete proofs of the results are given in [27], and will not be repeated. However, wherever appropriate, ideas of the proofs are sketched.

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2 Algebraic Background

We start by introducing the notation and stating several facts which are used in this article. The reader is assumed to be familiar with basics of the theory of superalgebras, and therefore this section is rather terse. For more detailed accounts of the subject the reader is referred to [1, 13, 28].

Let $\mathbb{S}$ be an arbitrary associative unital ring. A possibly nonassociative $\mathbb{S}$-algebra $\mathfrak{s}$ is called a *superalgebra* if it is $\mathbb{Z}_2$-graded, i.e., $\mathfrak{s} = \mathfrak{s}_{\overline{0}} \oplus \mathfrak{s}_{\overline{1}}$ where $\mathfrak{s}_i \mathfrak{s}_j \subseteq \mathfrak{s}_{i+j}$. The degree of a homogeneous element $a \in \mathfrak{s}$ is denoted by $|a|$.

A superalgebra $\mathfrak{s}$ is called *supercommutative* if

$$ab = (-1)^{|a|\cdot|b|} ba$$

for every two homogeneous elements $a, b \in \mathfrak{s}$.

A *Lie superalgebra* is a superalgebra whose multiplication, usually called its *superbracket*, satisfies graded analogues of antisymmetry and the Jacobi identity. This means that if A, B, C are homogeneous elements of a Lie superalgebra, then

$$[A, B] = -(-1)^{|A|\cdot|B|}[B, A]$$

and

$$(-1)^{|A|\cdot|C|}[A, [B, C]] + (-1)^{|B|\cdot|A|}[B, [C, A]] + (-1)^{|C|\cdot|B|}[C, [A, B]] = 0.$$

Let $\mathbb{K}$ be a field and $\mathfrak{g}$ be a Lie superalgebra over $\mathbb{K}$. If $\mathfrak{h}$ is a Lie subsuperalgebra of $\mathfrak{g}$ then $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{h})$ denotes the *supercommutant* of $\mathfrak{h}$ in $\mathfrak{g}$, i.e.,

$$\mathscr{Z}_{\mathfrak{g}}(\mathfrak{h}) = \{\, X \in \mathfrak{g} \mid [\mathfrak{h}, X] = \{0\} \,\}.$$

The *center* of $\mathfrak{g}$ is the supercommutant of $\mathfrak{g}$ in $\mathfrak{g}$ and is denoted by $\mathscr{Z}(\mathfrak{g})$. The *universal enveloping algebra* of $\mathfrak{g}$, which is defined in [13, Sect. 1.1.3], is denoted by $\mathscr{U}(\mathfrak{g})$. The group of $\mathbb{K}$-linear (even) automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g})$. Finally, recall that the definitions of nilpotent and solvable Lie superalgebras are the same as the ones for Lie algebras (see [13, Sect. 1]).

2.1 Centroid, Derivations, and Differential Constants

Let $\mathbb{K}$ be an arbitrary field and $\mathfrak{s}$ be a finite dimensional superalgebra over $\mathbb{K}$. The *multiplication algebra* of $\mathfrak{s}$, denoted by $\mathscr{M}(\mathfrak{s})$, is the associative unital superalgebra over $\mathbb{K}$ which is generated by the elements R_x and L_x of $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, for all homogeneous $x \in \mathfrak{s}$, where

$$\mathrm{L}_x(y) = xy \quad \text{and} \quad \mathrm{R}_x(y) = (-1)^{|x|\cdot|y|}yx \quad \text{for every homogeneous } y \in \mathfrak{s}.$$

As usual, the superbracket on $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ is defined by

$$[A, B] = AB - (-1)^{|A|\cdot|B|}BA$$

for homogeneous elements $A, B \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, and is then extended to $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ by linearity. The *centroid* of $\mathfrak{s}$, denoted by $\mathscr{C}(\mathfrak{s})$, is the supercommutant of $\mathscr{M}(\mathfrak{s})$ in $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$, i.e,

$$\mathscr{C}(\mathfrak{s}) = \left\{ A \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s}) \;\middle|\; [A, B] = 0 \text{ for every } B \in \mathscr{M}(\mathfrak{s}) \right\}.$$

Obviously $\mathscr{C}(\mathfrak{s})$ is a unital associative superalgebra over $\mathbb{K}$. If $\mathfrak{s}^2 = \mathfrak{s}$ then $\mathscr{C}(\mathfrak{s})$ is supercommutative (see [6, Proposition 2.1] for a proof).

If $s \in \{\overline{0}, \overline{1}\}$, a *homogeneous derivation of degree s* of $\mathfrak{s}$ is an element $D \in \mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ such that for every two homogeneous elements $a, b \in \mathfrak{s}$,

$$D(ab) = D(a)b + (-1)^{|a|\cdot s}aD(b).$$

The subspace of $\mathrm{End}_{\mathbb{K}}(\mathfrak{s})$ which is spanned by homogeneous derivations of $\mathfrak{s}$ is a Lie superalgebra over $\mathbb{K}$ and is denoted by $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s})$. The *ring of differential constants*, denoted by $\mathscr{R}(\mathfrak{s})$, is the supercommutant of $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s})$ in $\mathscr{C}(\mathfrak{s})$.

Suppose that $\mathfrak{s}$ is *simple*, i.e., $\mathfrak{s}^2 \neq \{0\}$ and $\mathfrak{s}$ does not have proper two-sided ideals. By Schur's Lemma every nonzero homogeneous element of $\mathscr{C}(\mathfrak{s})$ is invertible. Since $\mathfrak{s}^2$ is always a two-sided ideal, $\mathfrak{s}^2 = \mathfrak{s}$ and therefore $\mathscr{C}(\mathfrak{s})$ is supercommutative. It follows that $\mathscr{C}(\mathfrak{s})_{\overline{1}} = \{0\}$, $\mathscr{C}(\mathfrak{s})_{\overline{0}}$ is a field, and $\mathscr{R}(\mathfrak{s})$ is a subfield of $\mathscr{C}(\mathfrak{s})_{\overline{0}}$ containing $\mathbb{K}$.

2.2 Derivations of Base Extensions

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbf{\Lambda}(n, \mathbb{K})$ be the *Graßmann superalgebra over* $\mathbb{K}$ *in n indeterminates*, i.e., the associative unital superalgebra over $\mathbb{K}$ generated by odd elements $\xi_1, \ldots, \xi_n$ modulo the relations

$$\xi_i\xi_j + \xi_j\xi_i = 0 \quad \text{for every } 1 \leq i, j \leq n.$$

Let $\mathfrak{s}$ be a superalgebra over $\mathbb{K}$. The tensor product $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})$ is a superalgebra over $\mathbb{K}$. Note that since $\mathbf{\Lambda}(n, \mathbb{K})$ is supercommutative, if $\mathfrak{s}$ is a Lie superalgebra then so is $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})$.

It is proved in [6, Proposition 7.1] that

$$\mathrm{Der}_{\mathbb{K}}\big(\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K})\big) = \mathrm{Der}_{\mathbb{K}}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{\Lambda}(n, \mathbb{K}) + \mathscr{C}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{W}(n, \mathbb{K}), \qquad (1)$$

where

$$\mathbf{W}(n,\mathbb{K}) = \mathrm{Der}_{\mathbb{K}}\big(\mathbf{\Lambda}(n,\mathbb{K})\big).$$

The right hand side of (1) acts on $\mathfrak{s} \otimes_{\mathbb{K}} \mathbf{\Lambda}(n,\mathbb{K})$ via

$$(D_{\mathfrak{s}} \otimes_{\mathbb{K}} a)(X \otimes_{\mathbb{K}} b) = (-1)^{|a|\cdot|X|} D_{\mathfrak{s}}(X) \otimes_{\mathbb{K}} ab$$

and

$$(T \otimes_{\mathbb{K}} D_{\mathbf{\Lambda}})(X \otimes_{\mathbb{K}} a) = (-1)^{|D_{\mathbf{\Lambda}}|\cdot|X|} T(X) \otimes_{\mathbb{K}} D_{\mathbf{\Lambda}}(a).$$

Note that the right hand side of (1) is indeed a direct sum of the two summands. This follows from the fact that every element of $\mathscr{C}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{W}(n,\mathbb{K})$ vanishes on $\mathfrak{s} \otimes_{\mathbb{K}} 1_{\mathbf{\Lambda}(n,\mathbb{K})}$, but an element of $\mathrm{Der}_{\mathbb{K}}(\mathfrak{s}) \otimes_{\mathbb{K}} \mathbf{\Lambda}(n,\mathbb{K})$ which vanishes on $\mathfrak{s} \otimes_{\mathbb{K}} 1_{\mathbf{\Lambda}(n,\mathbb{K})}$ must be zero.

2.3 Cartan Subsuperalgebras

Let $\mathbb{K}$ be a field of characteristic zero and $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\mathbb{K}$. A Lie subsuperalgebra of $\mathfrak{g}$ which is nilpotent and self normalizing is called a *Cartan subsuperalgebra*.

An important property of Cartan subsuperalgebras of $\mathfrak{g}$ is that they are uniquely determined by their intersections with $\mathfrak{g}_{\overline{0}}$. Our next goal is to state this fact more formally.

For every subset $\mathcal{W}_{\overline{0}}$ of $\mathfrak{g}_{\overline{0}}$, let

$$\mathscr{N}_{\mathfrak{g}}(\mathcal{W}_{\overline{0}}) = \big\{ X \in \mathfrak{g} \,\big|\, \text{for every } W \in \mathcal{W}_{\overline{0}}, \text{ if } k \gg 0 \text{ then } \mathrm{ad}(W)^k(X) = 0 \big\}.$$

One can easily prove that $\mathscr{N}_{\mathfrak{g}}(\mathcal{W}_{\overline{0}})$ is indeed a subsuperalgebra of $\mathfrak{g}$. The next proposition is stated in [29, Proposition 1] (see also [24, Proposition 1]).

Proposition 2.3.1. *If $\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}}$ is a Cartan subsuperalgebra of $\mathfrak{g}$ then $\mathfrak{h}_{\overline{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$. Conversely, if $\mathfrak{h}_{\overline{0}}$ is a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$ then $\mathscr{N}_{\mathfrak{g}}(\mathfrak{h}_{\overline{0}})$ is a Cartan subsuperalgebra of $\mathfrak{g}$. The correspondence*

$$\mathfrak{h}_{\overline{0}} \longleftrightarrow \mathscr{N}_{\mathfrak{g}}(\mathfrak{h}_{\overline{0}})$$

is a bijection between Cartan subalgebras of $\mathfrak{g}_{\overline{0}}$ and Cartan subsuperalgebras of $\mathfrak{g}$.

2.4 Compactly Embedded Subalgebras

Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over $\mathbb{R}$. The group $\mathrm{Aut}(\mathfrak{g})$ is a (possibly disconnected) Lie subgroup of $\mathrm{GL}(\mathfrak{g})$, the group of invertible elements of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$. The subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by $e^{\mathrm{ad}(\mathfrak{g}_{\overline{0}})}$ is denoted by $\mathrm{Inn}(\mathfrak{g})$.

If $\mathfrak{h}_{\overline{0}}$ is a Lie subalgebra of $\mathfrak{g}_{\overline{0}}$ then $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{h}_{\overline{0}})$ denotes the closure in $\mathrm{Aut}(\mathfrak{g})$ of the subgroup generated by $e^{\mathrm{ad}(\mathfrak{h}_{\overline{0}})}$. When $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{h}_{\overline{0}})$ is compact $\mathfrak{h}_{\overline{0}}$ is said to be *compactly embedded in* $\mathfrak{g}$.

Cartan subalgebras of $\mathfrak{g}_{\overline{0}}$ which are compactly embedded in $\mathfrak{g}$ are especially interesting because they yield root decompositions of the complexification of $\mathfrak{g}$. The next proposition states this fact formally. In the next proposition, let τ denote the usual complex conjugation of elements of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $\tau(X + iY) = X - iY$ for every $X, Y \in \mathfrak{g}$.

Proposition 2.4.1. *Let $\mathfrak{t}_{\overline{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. Then the following statements hold.*

(i) *$\mathfrak{t}_{\overline{0}}$ is abelian.*
(ii) *One can decompose $\mathfrak{g}^{\mathbb{C}}$ as*

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C},\alpha}, \tag{2}$$

where

$$\Delta = \left\{ \alpha \in \mathfrak{t}_{\overline{0}}^{*} \;\middle|\; \mathfrak{g}^{\mathbb{C},\alpha} \neq \{0\} \right\}$$

and

$$\mathfrak{g}^{\mathbb{C},\alpha} = \left\{ X \in \mathfrak{g}^{\mathbb{C}} \;\middle|\; [H, X] = i\alpha(H)X \text{ for every } H \in \mathfrak{t}_{\overline{0}} \right\}.$$

(iii) *If $\alpha \in \Delta$ then $-\alpha \in \Delta$ as well, and if $X \in \mathfrak{g}^{\mathbb{C},\alpha}$ then $\tau(X) \in \mathfrak{g}^{\mathbb{C},-\alpha}$.*
(iv) $\mathfrak{g}_{\overline{0}} = \mathfrak{t}_{\overline{0}} \oplus [\mathfrak{t}_{\overline{0}}, \mathfrak{g}_{\overline{0}}]$.

Proof. The proof of [20, Theorem VII.2.2] can be adapted to prove (i), (ii), and (iii). Part (iv) can be proved using the fact that $\mathfrak{t}_{\overline{0}} = \mathscr{Z}_{\mathfrak{g}_{\overline{0}}}(\mathfrak{t}_{\overline{0}})$ (see [3, Chap. VII]).

More generally, if $\mathfrak{g}_{\overline{0}}$ has a Cartan subalgebra which is compactly embedded in $\mathfrak{g}$, then any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ yields a root decomposition. This is the content of the following proposition.

Proposition 2.4.2. *Assume that $\mathfrak{g}_{\overline{0}}$ has a Cartan subalgebra which is compactly embedded in $\mathfrak{g}$. If $\mathfrak{h}^{\mathbb{C}}$ is an arbitrary Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ then $\mathfrak{h}_{\overline{0}}^{\mathbb{C}}$ is abelian and there exists a root decomposition of $\mathfrak{g}^{\mathbb{C}}$ associated to $\mathfrak{h}^{\mathbb{C}}$, i.e.,*

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta(\mathfrak{h}^{\mathbb{C}})} \mathfrak{g}^{\mathbb{C},\alpha},$$

where

$$\Delta(\mathfrak{h}^{\mathbb{C}}) = \left\{ \alpha \in (\mathfrak{h}_{\overline{0}}^{\mathbb{C}})^{*} \mid \mathfrak{g}^{\mathbb{C},\alpha} \neq \{0\} \right\}$$

and

$$\mathfrak{g}^{\mathbb{C},\alpha} = \{\, X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = i\alpha(H)X \;\; \text{for every } H \in \mathfrak{h}_{\overline{0}}^{\mathbb{C}} \,\}.$$

Moreover, $\Delta(\mathfrak{h}^{\mathbb{C}}) = -\Delta(\mathfrak{h}^{\mathbb{C}})$.

Proof. Let $\mathfrak{t}_{\overline{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. Then $\mathfrak{t}_{\overline{0}}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\overline{0}}^{\mathbb{C}}$, and by Proposition 2.3.1 it corresponds to a Cartan subsuperalgebra $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$. Proposition 2.4.1 implies that there is a root decomposition of $\mathfrak{g}^{\mathbb{C}}$ associated to $\mathfrak{t}^{\mathbb{C}}$, and if $\Delta(\mathfrak{t}^{\mathbb{C}})$ denotes the corresponding set of roots then $\Delta(\mathfrak{t}^{\mathbb{C}}) = -\Delta(\mathfrak{t}^{\mathbb{C}})$. It is known that any two Cartan subalgebras of $\mathfrak{g}_{\overline{0}}^{\mathbb{C}}$ are conjugate under inner automorphisms of $\mathfrak{g}_{\overline{0}}^{\mathbb{C}}$. Using Proposition 2.3.1 one can show that any two Cartan subsuperalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate under the group of $\mathbb{C}$-linear automorphisms of $\mathfrak{g}^{\mathbb{C}}$ generated by $e^{\mathrm{ad}(\mathfrak{g}_{\overline{0}}^{\mathbb{C}})}$. By conjugacy, the root decomposition associated to $\mathfrak{t}^{\mathbb{C}}$ turns into one associated to $\mathfrak{h}^{\mathbb{C}}$.

2.5 Simple and Semisimple Lie Superalgebras

The classification of finite dimensional complex simple Lie superalgebras and their real forms is known from [13] and [31]. Every complex simple Lie superalgebra is isomorphic to one of the following types.

(i) A Lie superalgebra of *classical* type, i.e., $\mathbf{A}(m|n)$ where $m, n > 0$, $\mathbf{B}(m|n)$ where $m \geq 0$ and $n > 0$, $\mathbf{C}(n)$ where $n > 1$, $\mathbf{D}(m|n)$ where $m > 1$ and $n > 0$, $\mathbf{G}(3)$, $\mathbf{F}(4)$, $\mathbf{D}(2|1, \alpha)$ where $\alpha \in \mathbb{C} \setminus \{0, -1\}$, $\mathbf{P}(n)$ where $n > 1$, or $\mathbf{Q}(n)$ where $n > 1$.
(ii) A Lie superalgebra of *Cartan type,* i.e., $\mathbf{W}(n)$ where $n \geq 3$, $\mathbf{S}(n)$ where $n \geq 4$, $\widetilde{\mathbf{S}}(n)$ where n is even and $n \geq 4$, or $\mathbf{H}(n)$ where $n \geq 5$.
(iii) A complex simple Lie algebra.

Let $\mathfrak{s}$ be a finite dimensional real simple Lie superalgebra with nontrivial odd part, i.e., $\mathfrak{s}_{\overline{1}} \neq \{0\}$. Since $\mathscr{C}(\mathfrak{s})$ is a finite dimensional field extension of $\mathbb{R}$, we have $\mathscr{C}(\mathfrak{s}) = \mathbb{R}$ or $\mathscr{C}(\mathfrak{s}) = \mathbb{C}$. If $\mathscr{C}(\mathfrak{s}) = \mathbb{C}$, then $\mathfrak{s}$ is a complex simple Lie superalgebra which is considered as a real Lie superalgebra. If $\mathscr{C}(\mathfrak{s}) = \mathbb{R}$, then $\mathfrak{s}$ is a real form of the complex simple Lie superalgebra $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$. The classification of these real forms is summarized in Table 1.

A Lie superalgebra is called *semisimple* if it has no nontrivial solvable ideals. Semisimple Lie superalgebras are not necessarily direct sums of simple Lie superalgebras. In fact the structure theory of semisimple Lie superalgebras is rather complicated. The following statement can be obtained by a slight modification of the arguments in [6].

Theorem 2.5.1. *If a real Lie superalgebra $\mathfrak{g}$ is semisimple then there exist real simple Lie superalgebras $\mathfrak{s}_1, \ldots, \mathfrak{s}_k$ and nonnegative integers $n_1, \ldots, n_k$ such that*

$$\bigoplus_{i=1}^{k} \left(\mathfrak{s}_i \otimes_{\mathbb{K}_i} \mathbf{\Lambda}(n_i, \mathbb{K}_i)\right) \subseteq \mathfrak{g} \subseteq \bigoplus_{i=1}^{k} \left(\mathrm{Der}_{\mathbb{K}_i}(\mathfrak{s}_i) \otimes_{\mathbb{K}_i} \mathbf{\Lambda}(n_i, \mathbb{K}_i) + \mathbb{L}_i \otimes_{\mathbb{K}_i} \mathbf{W}(n_i, \mathbb{K}_i)\right)$$

where $\mathbb{K}_i = \mathscr{R}(\mathfrak{s}_i)$ and $\mathbb{L}_i = \mathscr{C}(\mathfrak{s}_i)$ for every $1 \leq i \leq k$.

Table 1 Simple real Lie superalgebras with nontrivial odd part

$\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathfrak{s}$		$\mathfrak{s}_{\overline{0}}/\mathrm{rad}(\mathfrak{s}_{\overline{0}})$
	$\mathfrak{su}(p,q\|r,s)$	$(p+q=m, r+s=n)$	$\mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s)$
$\mathbf{A}(m-1\|n-1)$	$\mathfrak{su}^*(2p\|2q)$	$(m=2p, n=2q$ even$)$	$\mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2q)$
$m>n>1$	$\mathfrak{sl}(m\|n,\mathbb{R})$		$\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$
	$\mathfrak{psu}(p,q\|r,s)$	$(p+q=r+s=m)$	$\mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s)$
	$\mathfrak{psu}^*(2p\|2p)$	$(m=2p$ even$)$	$\mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2p)$
	$\mathfrak{psl}(m\|m,\mathbb{R})$		$\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(m,\mathbb{R})$
$\mathbf{A}(m-1\|m-1)$	$\mathfrak{p}\overline{\mathfrak{q}}(m)$		$\mathfrak{sl}(m,\mathbb{C})$
$n>1$	$\mathfrak{usp}(m)$		$\mathfrak{sl}(m,\mathbb{C})$
	$\mathfrak{osp}(p,q\|2n)$	$(p+q=2m+1)$	$\mathfrak{so}(p,q) \oplus \mathfrak{sp}(2n,\mathbb{R})$
$\mathfrak{osp}(m\|2n,\mathbb{C})$	$\mathfrak{osp}^*(m\|p,q)$	$(p+q=n)$	$\mathfrak{so}^*(m) \oplus \mathfrak{sp}(p,q)$
	$D(2\|1,\alpha,2)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$
	$D(2\|1,\alpha,0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathbf{D}(2\,\|\,1,\alpha)$	$D(2\|1,\frac{1}{\alpha},0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\alpha=\overline{\alpha}$ or	$D(2\|1,-\frac{\alpha}{1+\alpha},0)$	$\alpha \in \mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\alpha=-1-\overline{\alpha}$	$D(2\|1,\alpha,1)$	$\alpha=-1-\overline{\alpha}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$
	$\mathbf{F}(4,0)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(7)$
	$\mathbf{F}(4,1)$		$\mathfrak{su}(2) \oplus \mathfrak{so}(1,6)$
	$\mathbf{F}(4,2)$		$\mathfrak{su}(2) \oplus \mathfrak{so}(2,5)$
$\mathbf{F}(4)$	$\mathbf{F}(4,3)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3,4)$
	$\mathbf{G}(3,1)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathrm{Der}_{\mathbb{R}}(\mathbb{O})$
$\mathbf{G}(3)$	$\mathbf{G}(3,2)$		$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathrm{Der}_{\mathbb{R}}(\mathbb{O}_{\mathsf{split}})$
	$\mathfrak{sp}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\mathbf{P}(n-1)$	$\mathfrak{sp}^*(n)$	$(n$ even$)$	$\mathfrak{su}^*(n)$
	$\mathfrak{psq}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
	$\mathfrak{psq}(p,q)$	$(p+q=n)$	$\mathfrak{su}(p,q)$
$\mathbf{Q}(n-1)$	$\mathfrak{psq}^*(n)$	$(n$ even$)$	$\mathfrak{su}^*(n)$
$\mathbf{W}(n)$	$\mathbf{W}(n,\mathbb{R})$		$\mathfrak{gl}(n,\mathbb{R})$
$\mathbf{S}(n)$	$\mathbf{S}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\widetilde{\mathbf{S}}(n)$ n even	$\widetilde{\mathbf{S}}(n,\mathbb{R})$		$\mathfrak{sl}(n,\mathbb{R})$
$\mathbf{H}(n)$	$\mathbf{H}(p,q)$	$(p+q=n)$	$\mathfrak{so}(p,q)$

3 Geometric Background

Since we are interested in studying unitary representations from an analytic viewpoint, we need to realize them as representations of Lie supergroups on $\mathbb{Z}_2$-graded Hilbert spaces. To this end, we first need to make precise what we mean by Lie supergroups.

One can define Lie supergroups abstractly as *group objects* in the category of supermanifolds. To give sense to this definition, one needs to define the category of supermanifolds. It will be seen below that this can be done by means of sheaves and ringed spaces.

Nevertheless, the above abstract definition of Lie supergroups is not well-suited for the study of unitary representations, and a more explicit description of Lie supergroups is necessary. The aim of this section is to explain the latter description,

which is based on the notion of Harish–Chandra pairs, and to clarify the relation between Harish–Chandra pairs and the categorical definition of Lie supergroups.

This section starts with a quick review of the theory of supermanifolds. The reader who is not familiar with the basics of this subject and is interested in further detail is referred to [9, 16–18, 33].

We remind the reader that in the study of unitary representations only the simple point of view of Harish–Chandra pairs will be used. Therefore the reader may also skip the review of supergeometry and continue reading from Sect. 3.4, where Harish–Chandra pairs are introduced.

3.1 Supermanifolds

Let p and q be nonnegative integers, and let $\mathscr{O}_{\mathbb{R}^p}$ denote the sheaf of smooth real valued functions on $\mathbb{R}^p$. The *smooth* $(p|q)$*-dimensional superspace* $\mathbb{R}^{p|q}$ is the ringed space $(\mathbb{R}^p, \mathscr{O}_{\mathbb{R}^{p|q}})$ where $\mathscr{O}_{\mathbb{R}^{p|q}}$ is the sheaf of smooth *superfunctions* in q odd coordinates. The latter statement simply means that for every open $U \subseteq \mathbb{R}^p$ one has

$$\mathscr{O}_{\mathbb{R}^{p|q}}(U) = \mathscr{O}_{\mathbb{R}^p}(U) \otimes_{\mathbb{R}} \mathbf{\Lambda}(q, \mathbb{R})$$

and the restriction maps of $\mathscr{O}_{\mathbb{R}^{p|q}}$ are obtained by base extensions of the restriction maps of $\mathscr{O}_{\mathbb{R}^p}$.

The ringed space $(\mathbb{R}^p, \mathscr{O}_{\mathbb{R}^{p|q}})$ is an object of the category $\mathsf{Top}_{\mathsf{s-alg}}$ of topological spaces which are endowed with sheaves of associative unital superalgebras over $\mathbb{R}$. If $\mathcal{X} = (\mathcal{X}_\circ, \mathscr{O}_\mathcal{X})$ and $\mathcal{Y} = (\mathcal{Y}_\circ, \mathscr{O}_\mathcal{Y})$ are objects in $\mathsf{Top}_{\mathsf{s-alg}}$ then a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ is a pair $(\varphi_\circ, \varphi^\#)$ such that $\varphi_\circ : \mathcal{X}_\circ \to \mathcal{Y}_\circ$ is a continuous map and

$$\varphi^\# : \mathscr{O}_\mathcal{Y} \to (\varphi_\circ)_* \mathscr{O}_\mathcal{X}$$

is a morphism of sheaves of associative unital superalgebras over $\mathbb{R}$, where $(\varphi_\circ)_* \mathscr{O}_\mathcal{X}$ is the direct image[1] of $\mathscr{O}_\mathcal{X}$.

An object of $\mathsf{Top}_{\mathsf{s-alg}}$ is called a *supermanifold* of dimension $(p|q)$ if it is locally isomorphic to $\mathbb{R}^{p|q}$. supermanifolds constitute objects of a full subcategory of $\mathsf{Top}_{\mathsf{s-alg}}$.

3.2 Some Basic Constructions for Supermanifolds

If $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_\mathcal{M})$ is a supermanifold of dimension $(p|q)$ then the nilpotent sections of $\mathscr{O}_\mathcal{M}$ generate a sheaf of ideals $\mathscr{I}_\mathcal{M}$. Indeed the underlying space $\mathcal{M}_\circ$

[1] In [9] the authors define morphisms based on pullback. Since pullback and direct image are adjoint functors, the definition of [9] is equivalent to the definition given in this article, which is also used in [17].

is an ordinary smooth manifold whose sheaf of smooth functions is $\mathscr{O}_{\mathcal{M}}/\mathscr{I}_{\mathcal{M}}$. One can also show that if $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}_\circ, \mathscr{O}_{\mathcal{N}})$ are two supermanifolds and $\varphi : \mathcal{M} \to \mathcal{N}$ is a morphism then the map $\varphi_\circ : \mathcal{M}_\circ \to \mathcal{N}_\circ$ is smooth (see [17, Sect. 2.1.5]).

Locally, $\mathscr{O}_{\mathcal{M}}/\mathscr{I}_{\mathcal{M}}$ is isomorphic to $\mathscr{O}_{\mathbb{R}^p}$. Therefore, if $U \subseteq \mathcal{M}_\circ$ is an open set, then for every section $f \in \mathscr{O}_{\mathcal{M}}(U)$ and every point $m \in U$ the value $f(m)$ is well defined. In this fashion, from any section f one obtains a smooth map

$$\widetilde{f} : U \to \mathbb{R}.$$

Nevertheless, because of the existence of nilpotent sections, f is not uniquely determined by $\widetilde{f}$.

Supermanifolds resemble ordinary manifolds in many ways. For example, one can prove the existence of finite direct products in the category of supermanifolds. Moreover, for a supermanifold $\mathcal{M}$ of dimension $(p|q)$ the sheaf $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$ of $\mathbb{R}$-linear derivations of the structural sheaf $\mathscr{O}_{\mathcal{M}}$ is a locally free sheaf of $\mathscr{O}_{\mathcal{M}}$-modules of rank $(p|q)$. Sections of the latter sheaf are called *vector fields* of $\mathcal{M}$. The space of vector fields is closed under the superbracket induced from $\mathrm{End}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$.

If $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ is a supermanifold and $m \in \mathcal{M}_\circ$, then there exists an obvious morphism

$$\delta_m : \mathbb{R}^{0|0} \to \mathcal{M},$$

where $(\delta_m)_\circ : \mathbb{R}^0 \to \mathcal{M}_\circ$ maps the unique point of $\mathbb{R}^0$ to m, and for every open set $U \subseteq \mathcal{M}_\circ$ if $f \in \mathscr{O}_{\mathcal{M}}(U)$ then

$$(\delta_m)^\#(f) = \begin{cases} \widetilde{f}(m) & \text{if } m \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\mathbb{R}^{0|0}$ is a terminal object in the category of supermanifolds. Indeed for every supermanifold $\mathcal{M} = (\mathcal{M}_\circ, \mathscr{O}_{\mathcal{M}})$ there exists a morphism

$$\kappa_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{0|0}$$

such that $(\kappa_{\mathcal{M}})_\circ : \mathcal{M}_\circ \to \mathbb{R}^0$ maps every point of $\mathcal{M}_\circ$ to the unique point of $\mathbb{R}^0$ and for every $t \in \mathscr{O}_{\mathbb{R}^{0|0}}(\mathbb{R}^0) \simeq \mathbb{R}$ one has $(\kappa_{\mathcal{M}})^\#(t) = t \cdot 1_{\mathcal{M}}$.

3.3 Lie Supergroups and Their Lie Superalgebras

Recall that by a *Lie supergroup* we mean a group object in the category of supermanifolds. In other words, a supermanifold $\mathcal{G} = (\mathcal{G}_\circ, \mathscr{O}_{\mathcal{G}})$ is a Lie supergroup if there exist morphisms

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}\,, \quad \varepsilon : \mathbb{R}^{0|0} \to \mathcal{G}\,, \quad \text{and } \iota : \mathcal{G} \to \mathcal{G},$$

which satisfy the standard relations that describe associativity, existence of an identity element, and inversion. It follows that $\mathcal{G}_\circ$ is a Lie group whose multiplication is given by $\mu_\circ : \mathcal{G}_\circ \times \mathcal{G}_\circ \to \mathcal{G}_\circ$.

To a Lie supergroup $\mathcal{G}$ one can associate a Lie superalgebra $\mathrm{Lie}(\mathcal{G})$ which is the subspace of $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{G}})$ consisting of left invariant vector fields of $\mathcal{G}$. The only subtle point in the definition of $\mathrm{Lie}(\mathcal{G})$ is the definition of left invariant vector fields. Left invariant vector fields can be defined in several ways. For example, in [9] the authors use the functor of points. We would like to mention a different method which is also described in [4]. For every $g \in \mathcal{G}_\circ$, one can define left translation morphisms $\lambda_g : \mathcal{G} \to \mathcal{G}$ by

$$\lambda_g = \mu \circ ((\delta_g \circ \kappa_{\mathcal{G}}) \times \mathrm{id}_{\mathcal{G}}),$$

where $\mathrm{id}_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ is the identity morphism. Similarly, one can define right translation morphisms

$$\rho_g = \mu \circ (\mathrm{id}_{\mathcal{G}} \times (\delta_g \circ \kappa_{\mathcal{G}})).$$

A vector field D is called *left invariant* if it commutes with left translation, i.e.,

$$(\lambda_g)^{\#} \circ D = D \circ (\lambda_g)^{\#}.$$

It is easily checked that $\mathrm{Lie}(\mathcal{G})$, the space of left invariant vector fields of $\mathcal{G}$, is closed under the superbracket of $\mathrm{Der}_{\mathbb{R}}(\mathscr{O}_{\mathcal{M}})$. Moreover, there is an action of $\mathcal{G}_\circ$ on $\mathrm{Lie}(\mathcal{G})$ given by

$$D \mapsto (\rho_g)^{\#} \circ D \circ (\rho_{\iota_\circ(g)})^{\#}. \tag{3}$$

Because of Part (ii) of Proposition 3.3.1 below it is natural to denote this action by $\mathrm{Ad}(g)$.

Proposition 3.3.1. *For a Lie supergroup $\mathcal{G} = (\mathcal{G}_\circ, \mathscr{O}_{\mathcal{G}})$ the following statements hold.*

(i) $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G})_{\bar{0}} \oplus \mathrm{Lie}(\mathcal{G})_{\bar{1}}$ *is a Lie superalgebra over* $\mathbb{R}$.

(ii) *The action of $\mathcal{G}_\circ$ on* $\mathrm{Lie}(\mathcal{G})$ *given by* (3) *yields a smooth homomorphism of Lie groups*

$$\mathrm{Ad} : \mathcal{G}_\circ \to \mathrm{GL}(\mathrm{Lie}(\mathcal{G}))$$

such that $\mathrm{Ad}(\mathcal{G}_\circ) \subseteq \mathrm{End}_{\mathbb{R}}(\mathrm{Lie}(\mathcal{G}))_{\bar{0}}$.

(iii) $\mathrm{Lie}(\mathcal{G})_{\bar{0}}$ *is the Lie algebra of $\mathcal{G}_\circ$ and if* $\mathsf{d}(\mathrm{Ad})$ *denotes the differential of the above map* Ad*, then*

$$\mathsf{d}(\mathrm{Ad})(X)(Y) = \mathrm{ad}(X)(Y)$$

for every $X \in \mathrm{Lie}(\mathcal{G})_{\bar{0}}$ and every $Y \in \mathrm{Lie}(\mathcal{G})$, where

$$\mathrm{ad}(X)(Y) = [X, Y].$$

3.4 Harish–Chandra Pairs

Proposition 3.3.1 states that to a Lie supergroup $\mathcal{G}$ one can associate an ordered pair $(\mathcal{G}_\circ, \mathrm{Lie}(\mathcal{G}))$, where $\mathcal{G}_\circ$ is a real Lie group and $\mathrm{Lie}(\mathcal{G})$ is a Lie superalgebra over $\mathbb{R}$, which satisfy certain properties. Such an ordered pair is a *Harish–Chandra pair*.

Definition 3.4.1. A Harish–Chandra pair is a pair $(G, \mathfrak{g})$ consisting of a Lie group G and a Lie superalgebra $\mathfrak{g}$ which satisfy the following properties.

(i) $\mathfrak{g}_{\bar{0}}$ is the Lie algebra of G.
(ii) G acts on $\mathfrak{g}$ smoothly by $\mathbb{R}$-linear automorphisms.
(iii) The differential of the action of G on $\mathfrak{g}$ is equal to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}$.

One can prove that

$$\mathcal{G} \mapsto (\mathcal{G}_\circ, \mathrm{Lie}(\mathcal{G}))$$

is an equivalence of categories from the category of Lie supergroups to the category of Harish–Chandra pairs. Under this equivalence of categories, a morphism $\psi : \mathcal{G} \to \mathcal{H}$ in the category of Lie supergroups corresponds to a pair $(\psi_\circ, \psi_{\mathrm{Lie}})$ where $\psi_\circ : \mathcal{G}_\circ \to \mathcal{H}_\circ$ is a homomorphism of Lie groups,

$$\psi_{\mathrm{Lie}} : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{H})$$

is a homomorphism of Lie superalgebras, and

$$\mathsf{d}\psi_\circ = \psi_{\mathrm{Lie}}\big|_{\mathrm{Lie}(\mathcal{G})_{\bar{0}}}.$$

Remark 3.4.2. Using Harish–Chandra pairs one can study Lie supergroups and their representations without any reference to the structural sheaves. In the rest of this article, Lie supergroups will always be realized as Harish–Chandra pairs.

4 Unitary Representations

According to [9, Sect. 4.4] one can define a finite dimensional super Hilbert space as a finite dimensional complex $\mathbb{Z}_2$-graded vector space which is endowed with an even super Hermitian form. Nevertheless, since the even super Hermitian form is generally indefinite, in the infinite dimensional case one should address the issues of topological completeness and separability. For the purpose of studying unitary representations it would be slightly more convenient to take an equivalent approach which is more straightforward, but less canonical.

4.1 Super Hilbert Spaces

A *super Hilbert space* is a $\mathbb{Z}_2$-graded complex Hilbert space $\mathscr{H} = \mathscr{H}_{\bar{0}} \oplus \mathscr{H}_{\bar{1}}$ such that $\mathscr{H}_{\bar{0}}$ and $\mathscr{H}_{\bar{1}}$ are mutually orthogonal closed subspaces. If $\langle \cdot, \cdot \rangle$ denotes the inner

product of $\mathscr{H}$, then for every two homogeneous elements $v, w \in \mathscr{H}$ the even super Hermitian form (v, w) of $\mathscr{H}$ is defined by

$$(v,w) = \begin{cases} 0 & \text{if } v \text{ and } w \text{ have opposite parity,} \\ \langle v, w\rangle & \text{if } v \text{ and } w \text{ are even,} \\ i\langle v, w\rangle & \text{if } v \text{ and } w \text{ are odd.} \end{cases}$$

One can check that $(\cdot,\cdot)$ satisfies the properties stated in [9, Sect. 4.4]. In this article the latter sesquilinear form will not be used.

4.2 The Definition of a Unitary Representation

In order to obtain an analytic theory of unitary representations of Lie supergroups one should deal with the same sort of analytic difficulties that exist in the case of Lie groups. One of the main difficulties is that in general one cannot define the differential of an infinite dimensional representation of a Lie group on the entire representation space. However, one can always define the differential on certain invariant dense subspaces, such as the space of *smooth vectors*.

In the rest of this article, the reader is assumed to be familiar with classical results in the theory of unitary representations of Lie groups. For a detailed and readable treatment of this subject see [32].

If $\mathscr{H}$ is a (possibly $\mathbb{Z}_2$-graded) complex Hilbert space, the group of unitary operators of $\mathscr{H}$ is denoted by $\mathbf{U}(\mathscr{H})$. As usual, if $\pi : G \to \mathbf{U}(\mathscr{H})$ is a unitary representation of a Lie group G, then the space of smooth vectors (respectively, analytic vectors) of $(\pi, \mathscr{H})$ is denoted by $\mathscr{H}^\infty$ (respectively, $\mathscr{H}^\omega$).

Definition 4.2.1. Let $(G, \mathfrak{g})$ be a Lie supergroup. A unitary representation of $(G, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathscr{H})$ satisfying the following properties.

(i) $\mathscr{H} = \mathscr{H}_{\bar{0}} \oplus \mathscr{H}_{\bar{1}}$ is a super Hilbert space.
(ii) $(\pi, \mathscr{H})$ is a unitary representation of G and $\pi(g) \in \mathrm{End}_{\mathbb{C}}(\mathscr{H})_{\bar{0}}$ for every $g \in G$.
(iii) $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$ is an $\mathbb{R}$-linear $\mathbb{Z}_2$-graded map, where $\mathscr{H}^\infty$ denotes the space of smooth vectors of $(\pi, \mathscr{H})$. Moreover, for every $X, Y \in \mathfrak{g}_{\bar{1}}$,

$$\rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X) = -i\rho^\pi([X,Y]).$$

(iv) $\rho^\pi(X) = \mathsf{d}\pi(X)\big|_{\mathscr{H}^\infty}$ for every $X \in \mathfrak{g}_{\bar{0}}$.
(v) For every $X \in \mathfrak{g}_{\bar{1}}$ the operator $\rho^\pi(X)$ is symmetric, i.e., if $v, w \in \mathscr{H}^\infty$ then

$$\langle \rho^\pi(X)v, w\rangle = \langle v, \rho^\pi(X)w\rangle.$$

(vi) For every $g \in G$ and every $X \in \mathfrak{g}$,

$$\rho^\pi\big(\mathrm{Ad}(g)(X)\big) = \pi(g)\rho^\pi(X)\pi(g)^{-1}.$$

Remark 4.2.2. It is easy to see that by letting an element $X_{\bar{0}} + X_{\bar{1}} \in \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ act on $\mathscr{H}^\infty$ as

$$\rho^\pi(X_{\bar{0}}) + e^{\frac{\pi}{4}i}\rho^\pi(X_{\bar{1}})$$

one obtains from ρ^π a homomorphism of Lie superalgebras from $\mathfrak{g}$ into $\mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$.

Remark 4.2.3. Subrepresentations, irreducibility, and unitary equivalence of unitary representations of Lie supergroups are defined similar to unitary representations of Lie groups (see [5]). Note that in the definition of unitary equivalence, intertwining operators are assumed to preserve the grading. This means that in general a unitary representation is not necessarily unitarily equivalent to the one obtained by parity change.

Lemma 4.2.4. For each $X \in \mathfrak{g}$, the operator $\rho^\pi(X)\colon \mathscr{H}^\infty \to \mathscr{H}^\infty$ is continuous with respect to the Fréchet topology on $\mathscr{H}^\infty$. Moreover, the bilinear map

$$\mathfrak{g} \times \mathscr{H}^\infty \to \mathscr{H}^\infty, \quad (X, v) \mapsto \rho^\pi(X)v \tag{4}$$

is continuous.

Proof. Since $\mathfrak{g}$ is finite dimensional, it suffices to show that each operator $\rho^\pi(X)$ is continuous. For $X \in \mathfrak{g}_{\bar{0}}$, this follows from the definition of the Fréchet topology on $\mathscr{H}^\infty$.

For $X \in \mathfrak{g}_{\bar{1}}$, the operator $\rho^\pi(X)$ on $\mathscr{H}^\infty$ is symmetric (see Definition 4.2.1(v)), hence the graph of $\rho^\pi(X)$ is closed. Now the Closed Graph Theorem for Fréchet spaces (see [26, Theorem 2.15]) implies its continuity. □

From now on we assume that $\mathscr{H}$ is separable. Although this assumption is not needed in Definition 4.2.1, it helps in avoiding technical conditions in various constructions, e.g., when induced representations are defined. Note that if $(\pi, \rho^\pi, \mathscr{H})$ is irreducible then $\mathscr{H}$ is separable.

In Definition 4.2.1 the fact that $\mathscr{H}^\infty$ is chosen as the space of the representation of $\mathfrak{g}$ is not a limitation. In fact it is shown in [5, Proposition 2] that in some sense any reasonable choice of the space of the representation of $\mathfrak{g}$, i.e., one which is dense in $\mathscr{H}$ and satisfies natural invariance properties under the actions on G and $\mathfrak{g}$, would yield a definition equivalent to the one given above. This fact also plays a role in showing that restriction and induction functors are well defined. Another useful fact, which follows from [5, Proposition 3], is that the space $\mathscr{H}^\omega$ of analytic vectors of $(\pi, \mathscr{H})$ is invariant under $\rho^\pi(\mathfrak{g})$.

4.3 Restriction and Induction

Suppose that $\mathcal{G} = (G, \mathfrak{g})$ is a Lie supergroup, and $\mathcal{H} = (H, \mathfrak{h})$ is a Lie subsupergroup of $\mathcal{G}$. Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. A priori it is not clear how to restrict $(\pi, \rho^\pi, \mathscr{H})$ to $\mathcal{H}$. The difficulty is that in general the space of smooth

vectors of the restriction of $(\pi, \mathscr{H})$ to H will be larger than $\mathscr{H}^\infty$. To circumvent this issue one can use [5, Proposition 2] to show that the action of $\mathcal{H}$ on $\mathscr{H}^\infty$ determines a unique unitary representation of $\mathcal{H}$ on $\mathscr{H}$. This representation is called the restriction of $(\pi, \rho^\pi, \mathscr{H})$ to $\mathcal{H}$, and is denoted by

$$\mathrm{Res}^{\mathcal{G}}_{\mathcal{H}}(\pi, \rho^\pi, \mathscr{H}).$$

Inducing from $\mathcal{H}$ to $\mathcal{G}$ is more delicate. Let $(\sigma, \rho^\sigma, \mathscr{K})$ be a unitary representation of $\mathcal{H}$. The first step towards defining a representation $(\pi, \rho^\pi, \mathscr{H})$ of $\mathcal{G}$ that is induced from $(\sigma, \rho^\sigma, \mathscr{K})$ is to identify the super Hilbert space $\mathscr{H}$. By analogy with the case of Lie groups one expects the super Hilbert space $\mathscr{H}$ to be a space of $\mathscr{K}$-valued functions on $\mathcal{G}$ which satisfy an equivariance property with respect to the left regular action of $\mathcal{H}$. One can then describe the action of $\mathcal{G}$ by formal relations, hoping that a unitary representation, as defined in Definition 4.2.1, is obtained. This formal approach leads to technical complications and it is not clear how to get around some of them. Nevertheless, at least in the special case that the homogeneous super space $\mathcal{H}\backslash\mathcal{G}$ is purely even, i.e., when $\dim \mathfrak{g}_{\overline{1}} = \dim \mathfrak{h}_{\overline{1}}$, it is shown in [5, Sect. 3] that the induced representation can be defined rigorously. In this article, only the special case when both G and H are unimodular groups is used, and in this case the induced representation is defined as follows. Since the homogeneous space $\mathcal{H}\backslash\mathcal{G}$ is purely even, there is a natural isomorphism $\mathcal{H}\backslash\mathcal{G} \simeq H\backslash G$. Choose an invariant measure μ on $H\backslash G$, and let $\mathscr{H}$ be the space of measurable functions $f : G \to \mathscr{K}$ which satisfy the following properties.

(i) $f(hg) = \sigma(h) f(g)$ for every $g \in G$ and every $h \in H$.
(ii) $\int_{H\backslash G} ||f||^2 d\mu < \infty$

The action of G on $\mathscr{H}$ is the right regular representation, i.e.,

$$\big(\pi(g) f\big)(g_1) = f(g_1 g) \quad \text{for every} \quad g, g_1 \in G,$$

and one can easily check that it is unitary with respect to the standard inner product of $\mathscr{H}$. The most natural way to define the action of an element $X \in \mathfrak{g}_{\overline{1}}$ on an element $f \in \mathscr{H}^\infty$ is via the formula

$$\big(\rho^\pi(X) f\big)(g) = \rho^\sigma\big(\mathrm{Ad}(g)(X)\big)\big(f(g)\big). \tag{5}$$

It is known that every $f \in \mathscr{H}^\infty$ is a smooth function from G to $\mathscr{K}$ and $f(g) \in \mathscr{K}^\infty$ for every $g \in G$ [25, Theorem 5.1]. Consequently, the right hand side of (5) is well defined. However, a priori it is not obvious why for an element $X \in \mathfrak{g}_{\overline{1}}$ the right hand side of (5) belongs to $\mathscr{H}^\infty$. One can prove the weaker statement that $\rho^\pi(X) f \in \mathscr{H}$ using a trick which is based on the ideas used in [5]. Since this trick sheds some light on the situation, it may be worthwhile to mention it. One can prove that the operator $\rho^\pi(X)$ is essentially self-adjoint. Let $\overline{\rho^\pi(X)}$ denote the closure of $\rho^\pi(X)$. The operator $I + \overline{\rho^\pi(X)}^2$ has a bounded inverse whose domain is all of $\mathscr{H}$

(this follows for instance from [7, Chap. X, Proposition 4.2]). For every $f \in \mathscr{H}^\infty$,

$$\rho^\pi(X)f = \overline{\rho^\pi(X)}f = \overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}(I + \overline{\rho^\pi(X)}^2)f.$$

Using the spectral theory of self-adjoint operators one can show that the operator $\overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}$ is bounded. Moreover,

$$(I + \overline{\rho^\pi(X)}^2)f = \big(I - \frac{i}{2}\mathrm{d}\pi([X,X])\big)f \in \mathscr{H}^\infty.$$

Finally, boundedness of $\overline{\rho^\pi(X)}(I + \overline{\rho^\pi(X)}^2)^{-1}$ implies that $\rho^\pi(X)f \in \mathscr{H}$.

To prove that indeed $\rho^\pi(X)f \in \mathscr{H}^\infty$ requires more effort. This is proved in [5, Sect. 3] in an indirect way. The idea of the proof is to find a dense subspace $\mathscr{B} \subseteq \mathscr{H}^\infty$ such that $\rho^\pi(\mathfrak{g})\mathscr{B} \subseteq \mathscr{B}$. As shown in [5, Sect. 3], one can take $\mathscr{B}$ to be the subspace of $\mathscr{H}^\infty$ consisting of functions from G to $\mathscr{K}$ with compact support modulo H. That $(\pi, \rho^\pi, \mathscr{H})$ is well defined then follows from [5, Proposition 2].

The representation $(\pi, \rho^\pi, \mathscr{H})$ induced from $(\sigma, \rho^\sigma, \mathscr{K})$ is denoted by

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}).$$

It can be shown [27, Proposition 3.2.1] that induction may be done in stages, i.e., if $\mathcal{H}$ is a Lie subsupergroup of $\mathcal{G}$, $\mathcal{K}$ is a Lie subsupergroup of $\mathcal{H}$, and $(\sigma, \rho^\sigma, \mathscr{K})$ is a unitary representation of $\mathcal{K}$, then

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}\mathrm{Ind}_{\mathcal{K}}^{\mathcal{H}}(\sigma, \rho^\sigma, \mathscr{K}) \simeq \mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}).$$

5 Invariant Cones in Lie Algebras

The goal of this section is to take a brief look at convex cones in finite dimensional real Lie algebras which are invariant under the adjoint action. A natural reduction to the case where the cone is pointed and generating leads to an interesting class of Lie algebras with a particular structure that will be discussed below.

A closed convex cone C in a finite dimensional vector space V is said to be *pointed* if $C \cap -C = \{0\}$, i.e., if C contains no affine lines. It is said to be *generating* if $C - C = V$ or equivalently if $Int(C)$ is nonempty, where $Int(C)$ denotes the set of interior points of C. If C is a cone in a finite dimensional vector space V then $C^\star$ denotes the cone in V^* consisting of all $\lambda \in V^*$ such that $\lambda(x) \geq 0$ for every $x \in C$.

5.1 Pointed Generating Invariant Cones

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. A cone $C \subseteq \mathfrak{g}$ is called *invariant* if it is closed, convex, and invariant under $\mathrm{Inn}(\mathfrak{g})$.

Suppose that C is an invariant cone in $\mathfrak{g}$ and set $\mathrm{H}(C) = C \cap -C$ and $\mathfrak{g}(C) = C - C$. The subspaces $\mathrm{H}(C)$ and $\mathfrak{g}(C)$ are ideals of $\mathfrak{g}$ and $C/\mathrm{H}(C)$ is a pointed generating invariant cone in the quotient Lie algebra $\mathfrak{g}(C)/\mathrm{H}(C)$. The main concern of the theory of invariant cones is to understand the situation when C is pointed and generating.

The existence of pointed generating invariant cones in a Lie algebra has the following simple but useful consequence.

Lemma 5.1.1. Let C be a pointed generating invariant cone in $\mathfrak{g}$. If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$ then $\mathfrak{a} \subseteq \mathscr{Z}(\mathfrak{g})$.

Proof. If $X \in \mathit{Int}(C)$, then $C \supseteq e^{\mathrm{ad}(\mathfrak{a})}X = X + [\mathfrak{a}, X]$. Since C contains no affine lines, $[\mathfrak{a}, X] = \{0\}$. Since $X \in \mathit{Int}(C)$ is arbitrary, $\mathfrak{a} \subseteq \mathscr{Z}(\mathfrak{g})$. □

To study invariant cones further, we need the following lemma.

Lemma 5.1.2. Let V be a finite dimensional vector space, $S \subseteq V$ be a convex subset, and $K \subseteq \mathrm{GL}(V)$ be a subgroup which leaves S invariant. Suppose that the closure of K in $\mathrm{GL}(V)$ is compact. If S is open or closed, then it contains K-fixed points.

Proof. Let $\overline{K}$ be the closure of K, and $\mu_{\overline{K}}$ be a normalized Haar measure on $\overline{K}$. For every $v \in S$, the point $v_\circ = \int_{\overline{K}} (k \cdot v)\, d\mu_{\overline{K}}(k)$ is K-fixed, and it is easily verified that $v_\circ \in S$. □

The preceding lemma has the following interesting consequence for invariant cones.

Lemma 5.1.3. Let $C \subseteq \mathfrak{g}$ be a pointed generating invariant cone. Then a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) \cap \mathit{Int}(C) \neq \emptyset$.

Proof. If $\mathfrak{k} \subseteq \mathfrak{g}$ is compactly embedded in $\mathfrak{g}$ then Lemma 5.1.2 implies that $\mathit{Int}(C)$ contains fixed points for $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$, i.e.,

$$\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) \cap \mathit{Int}(C) \neq \emptyset. \tag{6}$$

Conversely, if $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) \cap \mathit{Int}(C) \neq \emptyset$ then set $K = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$ and observe that K is a subgroup of $\mathrm{Inn}(\mathfrak{g})$ with a fixed point $X_0 \in \mathit{Int}(C)$. The set $C \cap (X_0 - C)$ is a compact K-invariant subset of $\mathfrak{g}$ with interior points. This implies that K is bounded in $\mathrm{GL}(\mathfrak{g})$ and therefore it has compact closure in $\mathrm{Aut}(\mathfrak{g})$. □

5.2 Compactly Embedded Cartan Subalgebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. Our next goal is to show that the existence of a pointed generating invariant cone in $\mathfrak{g}$ implies that $\mathfrak{g}$ has compactly embedded Cartan subalgebras. The next lemma shows how such a Cartan subalgebra can be obtained explicitly.

Lemma 5.2.1. Let $C \subseteq \mathfrak{g}$ be a pointed generating invariant cone. Suppose that $Y \in Int(C)$ is a regular element of $\mathfrak{g}$, i.e., the subspace

$$\mathscr{N}_{\mathfrak{g}}(Y) = \bigcup_n \ker(\mathrm{ad}(Y)^n)$$

has minimal dimension. If $\mathfrak{t} = \ker(\mathrm{ad}(Y))$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ which is compactly embedded in $\mathfrak{g}$.

Proof. For any such Y, the subspace $\mathfrak{t} = \mathscr{N}_{\mathfrak{g}}(Y)$ is a Cartan subalgebra of $\mathfrak{g}$ (see [3, Chap. VII]). Since $Y \in \mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}_{\mathfrak{g}}(\mathbb{R}Y))$, Lemma 5.1.3 implies that $\mathscr{Z}_{\mathfrak{g}}(\mathbb{R}Y)$ is compactly embedded in $\mathfrak{g}$. It follows immediately that $\mathbb{R}Y$ is compactly embedded in $\mathfrak{g}$. Therefore the endomorphism $\mathrm{ad}(Y) : \mathfrak{g} \to \mathfrak{g}$ is semisimple and

$$\mathfrak{t} = \ker(\mathrm{ad}(Y)) = \mathscr{Z}_{\mathfrak{g}}(Y)$$

from which it follows that $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$. □

Remark 5.2.2. It is known that the set of regular elements of $\mathfrak{g}$ is dense (see [3, Chap. VII]). Since $Int(C) \neq \emptyset$, the intersection of $Int(C)$ with the set of regular elements of $\mathfrak{g}$ is nonempty.

5.3 Characterization of Lie Algebras with Invariant Cones

The material in this section is meant to shed light on the connection between invariant cones and Hermitian Lie algebras. The reader is assumed to be familiar with the classification of real semisimple Lie algebras.

The study of invariant cones in finite-dimensional Lie algebras was initiated by Kostant, Segal and Vinberg [30,34]. A structure theory of invariant cones in general finite dimensional Lie algebras was developed by Hilgert and Hofmann in [11]. The characterization of those finite dimensional Lie algebras containing pointed generating invariant cones was obtained in [19] in terms of certain symplectic modules called of convex type, whose classification can be found in [22]. A self-contained exposition of this theory is available in [20], where the Lie algebras $\mathfrak{g}$ for which there exist pointed generating invariant cones in $\mathfrak{g} \oplus \mathbb{R}$ are called *admissible*.

Example 5.3.1. (*cf.* [34]) Suppose that $\mathfrak{g}$ is a real simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\mathfrak{p}$ is a simple nontrivial $\mathfrak{k}$-module, $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) = \mathscr{Z}(\mathfrak{k})$. If C is a pointed generating invariant cone in $\mathfrak{g}$, then from Lemma 5.1.3 it follows that

$$Int(C) \cap \mathscr{Z}(\mathfrak{k}) \neq \emptyset.$$

In particular $\mathscr{Z}(\mathfrak{k}) \neq \{0\}$, i.e., $\mathfrak{g}$ is *Hermitian*. Conversely, assume that $\mathfrak{g}$ is Hermitian and $0 \neq Z \in \mathscr{Z}(\mathfrak{k})$. If $(\cdot,\cdot)$ denotes the Killing form of $\mathfrak{g}$, then from the Cartan decomposition $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{k})e^{\mathrm{ad}(\mathfrak{p})}$ it follows that

$$(\mathrm{Inn}(\mathfrak{g})Z, Z) = (e^{\mathrm{ad}(\mathfrak{p})}Z, Z).$$

If $P \in \mathfrak{p}$ then $(e^{\mathrm{ad}(P)}Z, Z) < 0$ because

$$(e^{\mathrm{ad}(P)}Z, Z) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}(P)^{2n}(Z), Z)}{(2n)!}$$

and the linear transformations $\mathrm{ad}(P)^{2n} : \mathfrak{k} \to \mathfrak{k}$ are positive definite with respect to $(\cdot, \cdot)$. It follows that $\mathrm{Inn}(\mathfrak{g})Z$ lies in a proper invariant cone $C \subseteq \mathfrak{g}$. Since $\mathfrak{g}$ is simple, C is pointed and generating.

A slight refinement of the above arguments shows that a reductive Lie algebra $\mathfrak{g}$ is admissible if and only if $\mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}(\mathfrak{k})) = \mathfrak{k}$ holds for a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Lie algebras satisfying this property are called *quasihermitian*. This is equivalent to all simple ideals of $\mathfrak{g}$ being either compact or Hermitian. A reductive admissible Lie algebra contains pointed generating invariant cones if and only if it is not compact semisimple. This clarifies the structure of reductive Lie algebras with invariant cones.

Below we shall need the following lemma.

Lemma 5.3.2. Let $\mathfrak{g}$ be a quasihermitian Lie algebra, $\mathfrak{k} \subseteq \mathfrak{g}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$, and $p_{\mathfrak{z}} : \mathfrak{g} \to \mathscr{Z}(\mathfrak{k})$ be the fixed point projection for the compact group $e^{\mathrm{ad}\,\mathfrak{k}}$. Then every closed invariant convex subset $C \subseteq \mathfrak{g}$ satisfies $p_{\mathfrak{z}}(C) = C \cap \mathscr{Z}(\mathfrak{k})$.

Proof. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a $\mathfrak{k}$-invariant complement and recall that $\mathfrak{g}$ is said to be quasihermitian if $\mathfrak{k} = \mathscr{Z}_{\mathfrak{g}}(\mathscr{Z}(\mathfrak{k}))$. This condition implies in particular that $\mathfrak{p}$ contains no non-zero trivial $\mathfrak{k}$-submodule, so that $\mathscr{Z}_{\mathfrak{g}}(\mathfrak{k}) = \mathscr{Z}(\mathfrak{k})$. The assertion now follows from the proof of Lemma 5.1.2. □

In the case of an arbitrary Lie algebra $\mathfrak{g}$ having a pointed generating invariant cone, one can use Lemma 5.1.1 to show that the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ is two-step nilpotent, i.e., a generalized Heisenberg algebra. Moreover, $\mathfrak{n}$ clearly contains $\mathscr{Z}(\mathfrak{g})$, which is contained in any compactly embedded Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let $\mathfrak{a} \subseteq \mathfrak{t}$ be a complement to $\mathscr{Z}(\mathfrak{g})$ and $\mathfrak{s}$ be a $\mathfrak{t}$-invariant Levi complement to $\mathfrak{n}$ in $\mathfrak{g}$ (which always exists), and set $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{s}$. Then $\mathfrak{l}$ is reductive, $\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{n}$, and $\mathfrak{l}$ is an admissible reductive Lie algebra (see [20, Proposition VII.1.9]). At this point the structure of $\mathfrak{n}$ and $\mathfrak{l}$ is quite clear. However, to derive a classification of Lie algebras with invariant cones from this semidirect decomposition, one has to analyze the possibilities for the $\mathfrak{l}$-module structure on $\mathfrak{n}$ in some detail. This is done in [19] and [22].

6 Unitary Representations and Invariant Cones

A Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called $\star$-*reduced* if for every nonzero $X \in \mathfrak{g}$ there exists a unitary representation $(\pi, \rho^{\pi}, \mathcal{H})$ of $\mathcal{G}$ such that $\rho^{\pi}(X) \neq 0$. Note that when $\mathfrak{g}$ is simple, $\mathcal{G}$ is $\star$-reduced if and only if it has a nontrivial unitary representation. In

this section we study properties of $\star$-reduced Lie supergroups via methods based on the theory of invariant cones. We obtain necessary conditions for a Lie supergroup $\mathcal{G}$ to be $\star$-reduced. It turns out that these necessary conditions are strong enough for the classification of $\star$-reduced simple Lie supergroups.

Let $\mathcal{G} = (G, \mathfrak{g})$ be an arbitrary Lie supergroup, and let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. Fix an element $X \in \mathfrak{g}_{\overline{1}}$. From

$$\rho^\pi([X, X]) = i[\rho^\pi(X), \rho^\pi(X)] = 2i\rho^\pi(X)^2$$

and the fact that the operator $\rho^\pi(X)$ is symmetric it follows that

$$\langle i\rho^\pi([X, X])v, v\rangle \leq 0 \text{ for every } v \in \mathscr{H}^\infty.$$

Let $Cone(\mathcal{G})$ denote the invariant cone in $\mathfrak{g}_{\overline{0}}$ which is generated by elements of the form $[X, X]$ where $X \in \mathfrak{g}_{\overline{1}}$. Linearity of ρ^π implies that

$$\langle i\rho^\pi(Y)v, v\rangle \leq 0 \text{ for every } v \in \mathscr{H}^\infty \text{ and every } Y \in Cone(\mathcal{G}). \tag{7}$$

This means that π is $Cone(\mathcal{G})$-dissipative in the sense of [20].

6.1 Properties of $\star$-Reduced Lie Supergroups

Unlike Lie groups, which are known to have faithful unitary representations, certain Lie supergroups do not have such representations. The next proposition, which is given in [27, Lemma 4.1.1], shows how this can happen. The proof of this proposition is based on the fact that for every $X \in \mathfrak{g}_1$, the spectrum of $-i\rho^\pi([X, X])$ is nonnegative, so that a sum of such operators vanishes if and only if all summands vanish.

Proposition 6.1.1. *Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G} = (G, \mathfrak{g})$. Suppose that elements $X_1, \ldots, X_m \in \mathfrak{g}_{\overline{1}}$ satisfy*

$$[X_1, X_1] + \cdots + [X_m, X_m] = 0.$$

Then $\rho^\pi(X_1) = \cdots = \rho^\pi(X_m) = 0$.

The next proposition provides necessary conditions for a Lie supergroup to be $\star$-reduced.

Proposition 6.1.2. *If $\mathcal{G} = (G, \mathfrak{g})$ is $\star$-reduced, then the following statements hold.*

(i) *$Cone(\mathcal{G})$ is pointed.*
(ii) *For every $\lambda \in Int(Cone(\mathcal{G})^\star)$, the symmetric bilinear form*

$$\Omega_\lambda : \mathfrak{g}_{\overline{0}} \times \mathfrak{g}_{\overline{0}} \to \mathbb{R} \quad \textit{defined by} \quad \Omega_\lambda(X,Y) = \lambda([X,Y])$$

is positive definite .

(iii) *Let $\mathfrak{k}_{\overline{0}}$ be a Lie subalgebra of $\mathfrak{g}_{\overline{0}}$. If $\mathfrak{k}_{\overline{0}}$ is compactly embedded in $\mathfrak{g}_{\overline{0}}$, then $\mathfrak{k}_{\overline{0}}$ is compactly embedded in $\mathfrak{g}$ as well.*

(iv) *If $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ then $\mathfrak{g}_{\overline{0}}$ has a Cartan subalgebra which is compactly embedded in $\mathfrak{g}$.*

(v) *Assume that there exists a Cartan subalgebra $\mathfrak{h}_{\overline{0}}$ of $\mathfrak{g}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. Let $\mathbf{p} : \mathfrak{g}_{\overline{0}} \to \mathfrak{h}_{\overline{0}}$ be the projection map corresponding to the decomposition*

$$\mathfrak{g}_{\overline{0}} = \mathfrak{h}_{\overline{0}} \oplus [\mathfrak{h}_{\overline{0}}, \mathfrak{g}_{\overline{0}}]$$

(see Proposition 2.4.1) and $\mathbf{p}^* : \mathfrak{h}_{\overline{0}}^* \to \mathfrak{g}_{\overline{0}}^*$ be the corresponding dual map. Then

$$\textit{Int}(\textit{Cone}(\mathcal{G})^\star) \cap \mathbf{p}^*(\mathfrak{h}_{\overline{0}}^*) \neq \emptyset.$$

Proof. (i) Suppose, on the contrary, that $Y, -Y \in \textit{Cone}(\mathcal{G})$ for some nonzero Y. Let $(\pi, \rho^\pi, \mathscr{H})$ be a unitary representation of $\mathcal{G}$. For every $v \in \mathscr{H}^\infty$,

$$0 \leq \langle i\rho^\pi(Y)v, v\rangle \leq 0,$$

which implies that $\langle i\rho^\pi(Y)v, v\rangle = 0$. Therefore for every $v, w \in \mathscr{H}^\infty$ and every $z \in \mathbb{C}$,

$$\begin{aligned} 0 &= \langle \big(i\rho^\pi(Y)\big)(v+zw), v+zw\rangle \\ &= \langle i\rho^\pi(Y)v, v\rangle + \overline{z}\langle i\rho^\pi(Y)v, w\rangle + z\langle i\rho^\pi(Y)w, v\rangle + |z|^2\langle i\rho^\pi(Y)w, w\rangle \\ &= \overline{z}\langle i\rho^\pi(Y)v, w\rangle + z\langle i\rho^\pi(Y)w, v\rangle \end{aligned}$$

and since z is arbitrary, $\langle i\rho^\pi(Y)v, w\rangle = 0$ for every $v, w \in \mathscr{H}^\infty$. This means that $\rho^\pi(Y) = 0$, hence $Y = 0$ because $\mathcal{G}$ is $\star$-reduced.

(ii) That Ω_λ is positive semidefinite is immediate from the definition of $\textit{Cone}(\mathcal{G})^\star$. If $X \in \mathfrak{g}_1$ satisfies $\Omega_\lambda(X,X) = 0$ then from $\lambda \in \textit{Int}(\textit{Cone}(\mathcal{G})^\star)$ it follows that $[X,X] = 0$. Since $\mathcal{G}$ is $\star$-reduced, Proposition 6.1.1 implies that $X = 0$.

(iii) Part (i) implies that $\textit{Cone}(\mathcal{G})$ is pointed, and therefore $\textit{Int}(\textit{Cone}(\mathcal{G})^\star)$ is nonempty [20, Proposition V.1.5]. The action of the compact group $\mathrm{INN}_{\mathfrak{g}_{\overline{0}}}(\mathfrak{k}_{\overline{0}})$ on $\textit{Cone}(\mathcal{G})^\star$ leaves $\textit{Int}(\textit{Cone}(\mathcal{G})^\star)$ invariant. By Lemma 5.1.2, this action has a fixed point $\lambda \in \textit{Int}(\textit{Cone}(\mathcal{G})^\star)$. Therefore the symmetric bilinear form Ω_λ of Part (ii) is positive definite and invariant with respect to $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{k}_{\overline{0}})$. From the inclusion $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g}_{\overline{0}}) \times \mathrm{GL}(\mathfrak{g}_{\overline{1}})$ it follows that $\mathrm{INN}_{\mathfrak{g}}(\mathfrak{k}_{\overline{0}})$ is compact.

(iv) By Part (iii) it is enough to prove the existence of a Cartan subalgebra which is compactly embedded in $\mathfrak{g}_{\overline{0}}$. Part (i) implies that $\textit{Cone}(\mathcal{G})$ is pointed. The equality $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ means that $\textit{Cone}(\mathcal{G})$ is generating. Therefore Lemma 5.2.1 completes the proof.

(v) Part (i) implies that $Int(Cone(\mathcal{G})^\star) \neq \emptyset$. Since $\mathrm{INN}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{h}_{\bar{0}})$ is compact and leaves $Int(Cone(\mathcal{G})^\star)$ invariant, Lemma 5.1.2 implies that there exists a $\mu \in Int(Cone(\mathcal{G})^\star)$ which is fixed by $\mathrm{INN}_{\mathfrak{g}_{\bar{0}}}(\mathfrak{h}_{\bar{0}})$, i.e., contained in $\mathbf{p}^*(\mathfrak{h}_{\bar{0}}^*)$. □

Proposition 6.1.3. *Suppose that $\mathcal{G} = (G, \mathfrak{g})$ is a $\star$-reduced Lie supergroup. Let*

(i) *$\mathfrak{h}_{\bar{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ which is compactly embedded in $\mathfrak{g}$,*
(ii) *Δ be the root system associated to $\mathfrak{h}_{\bar{0}}$ (see Proposition 2.4.1),*
(iii) *$\mu \in Int(Cone(\mathcal{G})^\star) \cap \mathbf{p}^*(\mathfrak{h}_{\bar{0}}^*)$, where $\mathbf{p}^*$ is the map defined in the statement of Proposition 6.1.2.*

Then for every nonzero $\alpha \in \Delta$ the Hermitian form

$$\langle \cdot, \cdot \rangle_\alpha : \mathfrak{g}_{\bar{1}}^{\mathbb{C},\alpha} \times \mathfrak{g}_{\bar{1}}^{\mathbb{C},\alpha} \to \mathbb{C}$$

defined by $\langle X, Y \rangle_\alpha = \mu([X, \overline{Y}])$ is positive definite.

Proof. Let

$$\Omega_\mu : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathbb{R}$$

be the symmetric bilinear form defined by

$$\Omega_\mu(X, Y) = \mu([X, Y]).$$

By Proposition 6.1.2(ii) the form Ω_μ is positive definite. If $X \in \mathfrak{g}_{\bar{1}}^{\mathbb{C},\alpha}$ then $\overline{X} \in \mathfrak{g}_{\bar{1}}^{\mathbb{C},-\alpha}$ and

$$\begin{aligned}\Omega_\mu(X + \overline{X}, X + \overline{X}) &= \mu([X + \overline{X}, X + \overline{X}]) \\ &= \mu([X, X]) + \mu([X, \overline{X}]) + \mu([\overline{X}, X]) + \mu([\overline{X}, \overline{X}]).\end{aligned}$$

But $[X, X] \in \mathfrak{g}_{\bar{0}}^{\mathbb{C},2\alpha}$ and $[\overline{X}, \overline{X}] \in \mathfrak{g}_{\bar{0}}^{\mathbb{C},-2\alpha}$, and from $\mu \in \mathbf{p}^*(\mathfrak{h}_{\bar{0}}^*)$ and $\alpha \neq 0$ it follows that

$$\mu([X, X]) = \mu([\overline{X}, \overline{X}]) = 0.$$

Consequently

$$\mu([X, \overline{X}]) = \frac{1}{2}\Omega_\mu(X + \overline{X}, X + \overline{X}) \geq 0,$$

and $\mu(X, \overline{X}]) = 0$ implies that $X = 0$.

Moreover, if $\mu([X, \overline{X}]) = 0$ then $\Omega_\mu(X + \overline{X}, X + \overline{X})$ from which it follows that $X + \overline{X} = 0$. This means that $iX \in \mathfrak{g}$, hence $[\mathfrak{h}_{\bar{0}}, iX] \subseteq \mathfrak{g}$. However, if $H \in \mathfrak{h}_{\bar{0}}$ is chosen such that $\alpha(H) \neq 0$, then

$$[H, iX] = i[H, X] = i^2\alpha(H)X = -\alpha(H)X$$

and this yields a contradiction because clearly $-\alpha(H)X \notin \mathfrak{g}$. □

6.2 Application to Real Simple Lie Superalgebras

Let $\mathcal{G} = (G, \mathfrak{g})$ be a Lie supergroup such that G is connected and $\mathfrak{g}$ is a real simple Lie superalgebra with nontrivial odd part. Assume that $\mathcal{G}$ has nontrivial unitary representations. The goal of this section is use the necessary conditions obtained in Sect. 6.1 to obtain strong conditions on $\mathfrak{g}$.

Since $\mathfrak{g}$ is simple, $\mathcal{G}$ will be $\star$-reduced and Proposition 6.1.2(iv) implies that $\mathfrak{g}_{\overline{0}}$ contains a compactly embedded Cartan subalgebra. In particular, since complex simple Lie algebras do not have compactly embedded Cartan subalgebras, $\mathfrak{g}$ should be a real form of a complex simple Lie superalgebra. However, as Theorem 6.2.1 below shows, for a large class of these real forms there are no nontrivial unitary representations. For simplicity, we exclude the real forms of exceptional cases $\mathbf{G}(3)$, $\mathbf{F}(4)$ and $\mathbf{D}(2|1,\alpha)$.

Theorem 6.2.1. *If $\mathfrak{g}$ is one of the following Lie superalgebras then $\mathcal{G}$ does not have any nontrivial unitary representations.*

(i) $\mathfrak{sl}(m|n,\mathbb{R})$ *where* $m > 2$ *or* $n > 2$.
(ii) $\mathfrak{su}(p,q|r,s)$ *where* $p,q,r,s > 0$.
(iii) $\mathfrak{su}^*(2p,2q)$ *where* $p,q > 0$ *and* $p+q > 2$.
(iv) $\mathfrak{p}\overline{\mathfrak{q}}(m)$ *where* $m > 1$.
(v) $\mathfrak{usp}(m)$ *where* $m > 1$.
(vi) $\mathfrak{osp}^*(m|p,q)$ *where* $p,q,m > 0$.
(vii) $\mathfrak{osp}(p,q|2n)$ *where* $p,q,n > 0$.
(viii) *Real forms of* $\mathbf{P}(n)$, $n > 1$.
(ix) $\mathfrak{psq}(n,\mathbb{R})$ *where* $n > 2$, $\mathfrak{psq}^*(n)$ *where* $n > 2$, *and* $\mathfrak{psq}(p,q)$, *where* $p,q > 0$.
(x) *Real forms of* $\mathbf{W}(n)$, $\mathbf{S}(n)$, *and* $\widetilde{\mathbf{S}}(n)$.
(xi) $\mathbf{H}(p,q)$ *where* $p+q > 4$.

Proof. Throughout the proof, for every n we denote the $n \times n$ identity matrix by I_n, and set

$$\mathrm{I}_{p,q} = \begin{bmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{bmatrix} \quad \text{and} \quad \mathrm{J}_n = \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix}.$$

(i) Since $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \cong \mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R})$ has no compactly embedded Cartan subalgebra, this follows from Proposition 6.1.2(iv).

(ii) In the standard realization of $\mathfrak{sl}(p+q|r+s,\mathbb{C})$ as quadratic matrices of size $p+q+r+s$, $\mathfrak{su}(p,q|r,s)$ can be described as

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(p+q|r+s,\mathbb{C}) \,\middle|\, \begin{bmatrix} -\mathrm{I}_{p,q} A^* \mathrm{I}_{p,q} & i\mathrm{I}_{p,q} C^* \mathrm{I}_{r,s} \\ i\mathrm{I}_{r,s} B^* \mathrm{I}_{p,q} & -\mathrm{I}_{r,s} D^* \mathrm{I}_{r,s} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

Suppose, on the contrary, that $\mathcal{G}$ is $\star$-reduced. Proposition 6.1.2(iii) implies that the diagonal matrices in $\mathfrak{su}(p,q|r,s)$ constitute a Cartan subalgebra of $\mathfrak{su}(p,q|r,s)_{\overline{0}}$ which is compactly embedded in $\mathfrak{su}(p,q|r,s)$. Let μ be chosen

as in Proposition 6.1.3. For every $a \leq r$ and $b \leq p$, the matrix

$$X_{a,b} = \begin{bmatrix} 0 & 0 \\ \mathrm{E}_{a,b} & 0 \end{bmatrix}$$

is a root vector. Let τ denote the complex conjugation corresponding to the above realization of $\mathfrak{su}(p,q|r,s)$. One can easily check that

$$\tau(X_{a,b}) = \begin{bmatrix} 0 & i\mathrm{E}_{b,a} \\ 0 & 0 \end{bmatrix}.$$

Set $H_{a,b} = [X_{a,b}, \tau(X_{a,b})]$. It is easily checked that

$$H_{a,b} = \begin{bmatrix} i\mathrm{E}_{b,b} & 0 \\ 0 & i\mathrm{E}_{a,a} \end{bmatrix}.$$

For a and b there are three other possibilities to consider. If $a \leq r$ and $b > p$, or if $a > r$ and $b \leq p$, then

$$H_{a,b} = \begin{bmatrix} -i\mathrm{E}_{b,b} & 0 \\ 0 & -i\mathrm{E}_{a,a} \end{bmatrix},$$

and if $a > r$ and $b > p$ then

$$H_{a,b} = \begin{bmatrix} i\mathrm{E}_{b,b} & 0 \\ 0 & i\mathrm{E}_{a,a} \end{bmatrix}.$$

Proposition 6.1.3 implies that $\mu(H_{a,b}) > 0$ for every $1 \leq a \leq p+q$ and every $1 \leq b \leq r+s$. However, from the assumption that p, q, r, and s are all positive, it follows that the zero matrix lies in the convex hull of the $H_{a,b}$'s, which is a contradiction. Therefore $\mathcal{G}$ cannot be $\star$-reduced.

(iii) Note that $\mathfrak{su}^*(2p|2q)_{\overline{0}} \simeq \mathfrak{su}^*(2p) \oplus \mathfrak{su}^*(2q)$. The maximal compact subalgebra of $\mathfrak{su}^*(2n)$ is $\mathfrak{sp}(n)$, which has rank n. The rank of the complexification of $\mathfrak{su}^*(2n)$, which is $\mathfrak{sl}(2n,\mathbb{C})$, is $2n-1$. If $n > 1$, then $2n-1 > n$ implies that $\mathfrak{su}^*(2n)$ does not have a compactly embedded Cartan subalgebra. Now use Proposition 6.1.2(i) and Lemma 5.2.1.

(iv) This Lie superalgebra is a quotient of $\overline{\mathfrak{q}}(m)$ by its center, where $\overline{\mathfrak{q}}(m)$ is defined in the standard realization of $\mathfrak{sl}(m|m,\mathbb{C})$ by

$$\overline{\mathfrak{q}}(m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(m|m,\mathbb{C}) \;\middle|\; \begin{bmatrix} \overline{D} & \overline{C} \\ \overline{B} & \overline{A} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

One can now use Proposition 6.1.2(iv) because $\overline{\mathfrak{q}}(m)_{\overline{0}} \cong \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{R}$ contains no compactly embedded Cartan subalgebra.

(v) This Lie superalgebra is a quotient of $\mathfrak{up}(m)$ by its center, where $\mathfrak{up}(m)$ is defined in the standard realization of $\mathfrak{sl}(m|m,\mathbb{C})$ by

$$\mathfrak{up}(m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{sl}(m|m,\mathbb{C}) \;\middle|\; \begin{bmatrix} -D^* & B^* \\ -C^* & -A^* \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

This implies that $\mathfrak{up}(m)_{\bar{0}} \cong \mathrm{sl}(m,\mathbb{C}) \oplus \mathbb{R}$. Since this Lie algebra has no compactly embedded Cartan subalgebra, the assertion follows from Proposition 6.1.2(iv).

(vi) From Sect. 5.3 it follows that $\mathfrak{osp}^*(m|p,q)_{\bar{0}} \simeq \mathfrak{so}^*(m) \oplus \mathfrak{sp}(p,q)$ has pointed generating invariant cones if and only if $p = 0$ or $q = 0$. One can now use Proposition 6.1.2(i).

(vii) The argument for this case is quite similar to the one given for $\mathfrak{su}(p,q|r,s)$, i.e., the idea is to find root vectors $X_\alpha \in \mathfrak{g}_{\bar{1}}^{\mathbb{C},\alpha}$ such that the convex hull of the $[X_\alpha, \tau(X_\alpha)]$'s contains the origin. The details are left to the reader, but it may be helpful to illustrate how one can find the root vectors. The complex simple Lie superalgebra $\mathfrak{osp}(m|2n,\mathbb{C})$ can be realized inside $\mathfrak{sl}(m|2n,\mathbb{C})$ as

$$\mathfrak{osp}(m|2n,\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;\middle|\; \begin{bmatrix} -A^t & -C^t \mathrm{J}_n \\ -\mathrm{J}_n B^t & \mathrm{J}_n D^t \mathrm{J}_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}.$$

If p and q are nonnegative integers satisfying $p+q=m$ then $\mathfrak{osp}(p,q|2n)$ is the set of fixed points of the map

$$\tau : \mathfrak{osp}(m|2n,\mathbb{C}) \to \mathfrak{osp}(m|2n,\mathbb{C})$$

defined by

$$\tau\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} \mathrm{I}_{p,q}\overline{A}\mathrm{I}_{p,q} & \mathrm{I}_{p,q}\overline{B} \\ \overline{C}\mathrm{I}_{p,q} & \overline{D} \end{bmatrix}.$$

Moreover, $\mathfrak{osp}(p,q|2n)_{\bar{0}} \simeq \mathfrak{so}(p,q) \oplus \mathfrak{sp}(2n,\mathbb{R})$ consists of block diagonal matrices, i.e., matrices for which B and C are zero.

Assume that $\mathfrak{osp}(p,q|2n)$ is $\star$-reduced. Then the span of

$$\left\{ \mathrm{E}_{j,p+1-j} - \mathrm{E}_{p+1-j,j} \mid 1 \le j \le \lfloor \frac{p}{2} \rfloor \right\}$$

and

$$\left\{ \mathrm{E}_{p+j,p+q+1-j} - \mathrm{E}_{p+q+1-j,p+j} \mid 1 \le j \le \lfloor \frac{q}{2} \rfloor \right\}$$

is a compactly embedded Cartan subalgebra of $\mathfrak{so}(p,q)$, and the span of

$$\{ \mathrm{E}_{p+q+j,p+q+n+j} - \mathrm{E}_{p+q+n+j,p+q+j} \mid 1 \le j \le n \}$$

is a compactly embedded Cartan subalgebra of $\mathfrak{sp}(2n,\mathbb{R})$.

Fix $1 \leq b \leq n$. For every $a \leq p$ we can obtain two root vectors as follows. If we set

$$B_{a,b} = \mathrm{E}_{a,b} + i\mathrm{E}_{a,b+n} + i\mathrm{E}_{p+1-a,b} - \mathrm{E}_{p+1-a,b+n}$$

and

$$C_{a,b} = -i\mathrm{E}_{b,a} + \mathrm{E}_{b,p+1-a} + \mathrm{E}_{b+n,a} + i\mathrm{E}_{b+n,p+1-a},$$

then the matrix

$$X_{a,b} = \begin{bmatrix} 0 & B_{a,b} \\ C_{a,b} & 0 \end{bmatrix}$$

is a root vector, and $H_{a,b} = [X_{a,b}, \tau(X_{a,b})]$ is given by

$$H_{a,b} = \begin{bmatrix} A_{a,b} & 0 \\ 0 & D_{a,b} \end{bmatrix},$$

where $A_{a,b} = 2\mathrm{E}_{a,p+1-a} - 2\mathrm{E}_{p+1-a,a}$ and $D_{a,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}$. Similarly, setting

$$B_{a,b} = \mathrm{E}_{a,b} - i\mathrm{E}_{a,b+n} + i\mathrm{E}_{p+1-a,b} + \mathrm{E}_{p+1-a,b+n}$$

and

$$C_{a,b} = i\mathrm{E}_{b,a} - \mathrm{E}_{b,p+1-a} + \mathrm{E}_{b+n,a} + i\mathrm{E}_{b+n,p+1-a}$$

yields another root vector $X_{a,b}$, and in this case for the corresponding $H_{a,b}$ we have

$$A_{a,b} = -2\mathrm{E}_{a,p+1-a} + 2\mathrm{E}_{p+1-a,a}$$

and

$$D_{a,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}.$$

Moreover, when p is odd, setting

$$B_{\lceil \frac{p+1}{2} \rceil,b} = \mathrm{E}_{\lceil \frac{p+1}{2} \rceil,b} + i\mathrm{E}_{\lceil \frac{p+1}{2} \rceil,b+n}$$

and

$$C_{\lceil \frac{p+1}{2} \rceil,b} = -i\mathrm{E}_{b,\lceil \frac{p+1}{2} \rceil} + \mathrm{E}_{b+n,\lceil \frac{p+1}{n} \rceil}$$

yields a root vector $X_{\lceil \frac{p+1}{2} \rceil,b}$, and $H_{\lceil \frac{p+1}{2} \rceil,b}$ is given by

$$A_{\lceil \frac{p+1}{2} \rceil,b} = 0$$

and

$$D_{\lceil \frac{p+1}{2} \rceil,b} = -2\mathrm{E}_{b,b+n} + 2\mathrm{E}_{b+n,b}.$$

The case $p < a \leq p + q$ is similar.

(viii) Follows from Proposition 2.4.2, as the root system of $\mathbf{P}(n)$ is not symmetric.

(ix) For $\mathfrak{psq}(n,\mathbb{R})$ and $\mathfrak{psq}^*(n)$, use Proposition 6.1.2(iv) and the fact that

$$\mathfrak{psq}(n,\mathbb{R})_{\overline{0}} \simeq \mathfrak{sl}(n,\mathbb{R}) \quad \text{and} \quad \mathfrak{psq}^*(n)_{\overline{0}} \simeq \mathfrak{su}^*(n).$$

For $\mathfrak{psq}(p,q)$ and $p,q > 0$, we observe that it is a quotient of the subsuperalgebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{sl}(p+q|p+q,\mathbb{C})$ given by

$$\widetilde{\mathfrak{g}} = \mathfrak{sq}(p,q) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \;\middle|\; \begin{bmatrix} -\mathrm{I}_{p,q} A^* \mathrm{I}_{p,q} & i\mathrm{I}_{p,q} B^* \mathrm{I}_{p,q} \\ i\mathrm{I}_{p,q} B^* \mathrm{I}_{p,q} & -\mathrm{I}_{p,q} A^* \mathrm{I}_{p,q} \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right\}.$$

Let $\zeta \in \mathbb{C}$ be a squareroot of i. Then the maps

$$\mathfrak{u}(p,q) \to \widetilde{\mathfrak{g}}_{\overline{0}}, \quad A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

and

$$\mathfrak{u}(p,q) \to \widetilde{\mathfrak{g}}_{\overline{1}}, \quad B \mapsto \begin{bmatrix} 0 & \zeta^{-1}B \\ \zeta^{-1}B & 0 \end{bmatrix}$$

are linear isomorphisms. Note that $\mathfrak{k}_{\overline{0}} = \mathfrak{u}(p) \oplus \mathfrak{u}(q)$ is a maximal compactly embedded subalgebra of $\widetilde{\mathfrak{g}}_{\overline{0}}$. Its center is

$$\mathscr{Z}(\mathfrak{k}_{\overline{0}}) = \mathbb{R} i \mathrm{I}_p \oplus \mathbb{R} i \mathrm{I}_q$$

and $\widetilde{\mathfrak{g}}_{\overline{0}}$ is quasihermitian. The projection $p_{\mathfrak{z}} \colon \mathfrak{u}(p,q) \to \mathscr{Z}(\mathfrak{k}_{\overline{0}})$ is simply given by

$$p_{\mathfrak{z}} \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} = \begin{bmatrix} \frac{1}{p}\operatorname{tr}(a)\mathrm{I}_p & 0 \\ 0 & \frac{1}{q}\operatorname{tr}(d)\mathrm{I}_q \end{bmatrix}.$$

Let $C \subseteq \widetilde{\mathfrak{g}}_{\overline{0}}$ be the closed convex cone generated by $[X,X]$, $X \in \widetilde{\mathfrak{g}}_{\overline{1}}$. Since $\widetilde{\mathfrak{g}}_{\overline{0}}$ is quasihermitian, Lemma 5.3.2 implies that $p_{\mathfrak{z}}(C) = C \cap \mathscr{Z}(\mathfrak{k}_{\overline{0}})$.

Next we observe that

$$\begin{bmatrix} 0 & \zeta^{-1}B \\ \zeta^{-1}B & 0 \end{bmatrix}^2 = \begin{bmatrix} -iB^2 & 0 \\ 0 & -iB^2 \end{bmatrix} \quad \text{for every} \quad B \in \mathfrak{u}(p,q).$$

For $B = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$ we have

$$B^2 = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bb^* & ab + bd \\ b^*a + ab^* & b^*b + d^2 \end{bmatrix},$$

so that

$$p_{\mathfrak{z}}(-iB^2) = -i \begin{bmatrix} \frac{1}{p}\operatorname{tr}(a^2 + bb^*) & 0 \\ 0 & \frac{1}{q}\operatorname{tr}(b^*b + d^2) \end{bmatrix}.$$

Applying this to positive multiples of matrices where only the a, b or d-component is non-zero, we see that the closed convex cone $p_{\mathfrak{z}}(C)$ contains the elements

$$Z_1 = \begin{bmatrix} i\mathrm{I}_p & 0 \\ 0 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0 & 0 \\ 0 & i\mathrm{I}_q \end{bmatrix} \quad \text{and} \quad Z_3 = -\begin{bmatrix} \frac{1}{p} i\mathrm{I}_p & 0 \\ 0 & \frac{1}{q} i\mathrm{I}_q \end{bmatrix}.$$

This implies that $p_{\mathfrak{z}}(C) = \mathscr{Z}(\mathfrak{k}_{\overline{0}}) \subseteq C$.

We conclude that $\mathscr{Z}(\widetilde{\mathfrak{g}_{\overline{0}}}) = i\mathbb{R}\mathrm{I}_{p+q} \subseteq C$, so that $C = (\widetilde{\mathfrak{g}_{\overline{0}}}) \oplus C_1$, where $C_1 = C \cap [\widetilde{\mathfrak{g}_{\overline{0}}}, \widetilde{\mathfrak{g}_{\overline{0}}}]$ is a non-pointed non-zero invariant closed convex cone in a simple Lie algebra isomorphic to $\mathfrak{su}(p,q)$. This leads to $C_1 = [\widetilde{\mathfrak{g}_{\overline{0}}}, \widetilde{\mathfrak{g}_{\overline{0}}}]$. We conclude that $C = \widetilde{\mathfrak{g}_{\overline{0}}}$ and the same holds also for the quotient $\mathfrak{psq}(p,q)$.

(x) Follows from Proposition 2.4.2, as the root systems of these complex simple Lie superalgebras are not symmetric (see [23, Appendix A]).

(xi) Suppose, on the contrary, that $\mathcal{G}$ is $\star$-reduced. Proposition 6.1.2(i) and Lemma 5.1.1 imply that every abelian ideal of $\mathfrak{g} = \mathbf{H}(p,q)$ lies in its center. The standard $\mathbb{Z}$-grading of $\mathbf{H}(p+q)$ (see [13, Proposition 3.3.6]) yields a grading of $\mathbf{H}(p+q)_{\overline{0}}$, i.e.,

$$\mathbf{H}(p+q)_{\overline{0}} = \mathbf{H}(p+q)_{\overline{0}}^{(0)} \oplus \mathbf{H}(p+q)_{\overline{0}}^{(2)} \oplus \cdots \oplus \mathbf{H}(p+q)_{\overline{0}}^{(k)},$$

where $k = p+q-3$ if $p+q$ is odd and $k = p+q-4$ otherwise. This grading is consistent with the real form $\mathbf{H}(p,q)_{\overline{0}}$. Since $\mathbf{H}(p,q)_{\overline{0}}^{(k)}$ is an abelian ideal of $\mathbf{H}(p,q)_{\overline{0}}$, it should lie in the center of $\mathbf{H}(p,q)_{\overline{0}}$. It follows that $\mathbf{H}(p+q)_{\overline{0}}^{(k)}$ lies in the center of $\mathbf{H}(p+q)_{\overline{0}}$. However, this is impossible because it is known (see [13, Proposition 3.3.6]) that $\mathbf{H}(p+q)_{\overline{0}}^{(0)} \simeq \mathfrak{so}(p+q,\mathbb{C})$ and the representation of $\mathbf{H}(p+q)_{\overline{0}}^{(0)}$ on $\mathbf{H}(p+q)_{\overline{0}}^{(k)}$ is isomorphic to $\wedge^{k+2}\mathbb{C}^{p+q}$, from which it follows that

$$[\mathbf{H}(p+q)_{\overline{0}}^{(0)}, \mathbf{H}(p+q)_{\overline{0}}^{(k)}] \neq \{0\}. \qquad \square$$

Remark 6.2.1. In classical cases, Theorem 6.2.1 can be viewed as a converse to the classification of highest weight modules obtained in [12]. From Theorem 6.2.1 it also follows that for the nonclassical cases, unitary representations are rare.

Remark 6.2.2. The results of [12] imply that real forms of $\mathbf{A}(m|m)$ do not have any unitarizable highest weight modules. However, $\mathbf{A}(m|m)$ is a quotient of $\mathfrak{sl}(m|m,\mathbb{C})$, and there exist unitarizable modules of $\mathfrak{su}(p, m-p|m, 0)$ which do not factor to the simple quotient. For instance, the standard representation is a finite dimensional unitarizable module of $\mathfrak{su}(m, 0|m, 0)$ with this property.

6.3 Application to Real Semisimple Lie Superalgebras

Although real semisimple Lie superalgebras may have a complicated structure, those which have faithful unitary representations are relatively easy to describe.

Given a finite dimensional real Lie superalgebra $\mathfrak{g}$, let us call it $\star$-reduced if there exists a $\star$-reduced Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$.

Theorem 6.3.1. *Let $\mathcal{G} = (G, \mathfrak{g})$ be a $\star$-reduced Lie supergroup. If $\mathfrak{g}$ is a real semisimple Lie superalgebra then there exist $\star$-reduced real simple Lie superalgebras $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ such that*

$$\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \subseteq \mathfrak{g} \subseteq \mathrm{Der}_{\mathbb{R}}(\mathfrak{s}_1) \oplus \cdots \oplus \mathrm{Der}_{\mathbb{R}}(\mathfrak{s}_k).$$

Proof. We use the description of $\mathfrak{g}$ given in Theorem 2.5.1. First note that for every i we have $n_i = 0$. To see this, suppose on the contrary that $n_i > 0$ for some i, and let $\xi_1, \dots, \xi_{n_i}$ be the standard generators of $\mathbf{\Lambda}_{\mathbb{K}_i}(n_i)$. For every nonzero $X \in (\mathfrak{s}_i)_{\bar{0}}$ have $X \otimes \xi_1 \in (\mathfrak{s}_i)_{\bar{1}}$ and

$$[X \otimes \xi_1, X \otimes \xi_1] = 0.$$

Proposition 6.1.1 implies that $X \otimes \xi_1$ lies in the kernel of every unitary representation of $\mathcal{G}$, which is a contradiction.

From the fact that all of the n_i, $1 \leq i \leq k$, are zero it follows that

$$\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \subseteq \mathfrak{g} \subseteq \mathrm{Der}_{\mathbb{K}_1}(\mathfrak{s}_1) \oplus \cdots \oplus \mathrm{Der}_{\mathbb{K}_k}(\mathfrak{s}_k)$$

and from $\mathfrak{s}_i \subseteq \mathfrak{g}$ it follows that every $\mathfrak{s}_i$ is $\star$-reduced. □

6.4 Application to Nilpotent Lie Supergroups

Another interesting by-product of the results of Sect. 6.1 is the following statement about unitary representations of nilpotent Lie supergroups. (A Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called *nilpotent* if $\mathfrak{g}$ is nilpotent.)

Theorem 6.4.1. *If $(\pi, \rho^\pi, \mathscr{H})$ is a unitary representation of a nilpotent Lie supergroup $(G, \mathfrak{g})$ then $\rho^\pi([\mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]]) = \{0\}$.*

Proof. By passing to a quotient one can see that it suffices to show that if $(G, \mathfrak{g})$ is nilpotent and $\star$-reduced then $[\mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]] = \{0\}$. Without loss of generality one can assume that $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$. By Proposition 6.1.2(iv) there exists a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ which is compactly embedded in $\mathfrak{g}$. As $\mathfrak{g}_{\bar{0}}$ is nilpotent, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{h}_{\bar{0}}$. Proposition 2.4.1 implies that $\mathfrak{g}_{\bar{0}}$ acts semisimply on $\mathfrak{g}$. Nevertheless, since $\mathfrak{g}$ is nilpotent, for every $X \in \mathfrak{g}_{\bar{0}}$ the linear map

$$\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$$

is nilpotent. It follows that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}] = \{0\}$. In particular, $[\mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]] = \{0\}$. □

7 Highest Weight Theory

In this section, we analyse the structure of the irreducible unitary representations of Lie supergroups whose Lie algebra $\mathfrak{g}$ is generated by its odd part. The main result is Theorem 7.3.2 which asserts that this structure is quite similar to the structure of highest weight modules of Lie algebras. The crucial difference is that the highest weight space is an irreducible representation of a Clifford Lie superalgebra and therefore it can have dimension higher than one.

7.1 A Fréchet Space of Analytic Vectors

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan subalgebra, and $T = \exp(\mathfrak{t})$ be the corresponding subgroup of G. Then $\mathfrak{g}^{\mathbb{C}}$ carries a norm $\|\cdot\|$ which is invariant under $\mathrm{Ad}(T)$. In particular, for each $r > 0$, the open ball $B_r = \{X \in \mathfrak{g}^{\mathbb{C}} : \|X\| < r\}$ is an open subset which is invariant under $\mathrm{Ad}(T)$.

Let $(\pi, \mathscr{H})$ be a unitary representation of G. A smooth vector $v \in \mathscr{H}^\infty$ is *analytic* if and only if there exists an $r > 0$ such that the power series

$$f_v : B_r \to \mathscr{H}, \quad f_v(X) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{d}\pi(X)^n v \tag{8}$$

defines a holomorphic function on B_r. In fact, if the series (8) converges on some B_r, then it defines a holomorphic function, and the theory of analytic vectors for unitary one-parameter groups implies that $f_v(X) = \pi(\exp(X))v$ for every $X \in B_r \cap \mathfrak{g}$. Therefore the orbit map of v is analytic.

If the series (8) converges on B_r, it converges uniformly on B_s for every $s < r$ ([2, Proposition 4.1]). This means that the seminorms

$$q_n(v) = \sup\{\|\mathrm{d}\pi(X)^n v\| \;\Big|\; \|X\| \leq 1\ ,\ X \in \mathfrak{g}^{\mathbb{C}}\}$$

satisfy

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(v) < \infty \text{ for every } s < r.$$

Note that the seminorms q_n define the topology of $\mathscr{H}^\infty$ (cf. [21, Proposition 4.6]).

For every $r > 0$, let $\mathscr{H}^{\omega,r}$ denote the set of all analytic vectors for which (8) converges on B_r, so that

$$\mathscr{H}^\omega = \bigcup_{r>0} \mathscr{H}^{\omega,r}.$$

If $v \in \mathscr{H}^{\omega,r}$ and $s < r$, set

$$p_s(v) = \sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(v)$$

and note that this is a norm on $\mathscr{H}^{\omega,r}$.

Lemma 7.1.1. The norms p_s, $s < r$, turn $\mathscr{H}^{\omega,r}$ into a Fréchet space.

Proof. Since $p_s < p_t$ for $s < t < r$, the topology on $\mathscr{H}^{\omega,r}$ is defined by the sequence of seminorms $(p_{s_n})_{n\in\mathbb{N}}$ for any sequence (s_n) with $s_n \to r$. Therefore $\mathscr{H}^{\omega,r}$ is metrizable and we have to show that it is complete.

If (v_n) is a Cauchy sequence in $\mathscr{H}^{\omega,r}$ then for every $s < r$ the sequence $f_{v_n} : B_r \to \mathscr{H}$ of holomorphic functions converges uniformly on each B_s to some function $f : B_r \to \mathscr{H}$, which implies that f is holomorphic.

Fix $X \in \mathfrak{g}^{\mathbb{C}}$ and let $v = f(0)$. For each $k \in \mathbb{N}$, $\mathrm{d}\pi(X)^k v_n$ is a Cauchy sequence in $\mathscr{H}$. This implies that $v \in \mathscr{H}^\infty$. Observe that for every $k \in \mathbb{N}$, the operator $\mathrm{d}\pi(X)^k$ has a densely defined adjoint and therefore it is closable. Consequently, $\mathrm{d}\pi(X)^k v_n \to \mathrm{d}\pi(X)^k v$ for every $k \in \mathbb{N}$. Therefore $f = f_v$ on B_r ([2, Proposition 3.1]), and this means that $v \in \mathscr{H}^{\omega,r}$ with $v_n \to v$ in the topology of $\mathscr{H}^{\omega,r}$.

Lemma 7.1.2. If $K \subseteq G$ is a subgroup leaving the norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$ invariant, then the norms p_s, $s < r$, on $\mathscr{H}^{\omega,r}$ are K-invariant and the action of K on $\mathscr{H}^{\omega,r}$ is continuous. In particular, the action of K on $\mathscr{H}^{\omega,r}$ integrates to a representation of the convolution algebra $L^1(K)$ on $\mathscr{H}^{\omega,r}$.

Proof. Since K preserves the defining family of norms, continuity of the K-action on $\mathscr{H}^{\omega,r}$ follows if we show that all orbit maps are continuous at $\mathbf{1}_K$, where $\mathbf{1}_K$ denotes the identity element of K. Let $v \in \mathscr{H}^{\omega,r}$ and suppose that $k_m \to \mathbf{1}_K$ in K. Then

$$p_s(\pi(k_m)v - v) = \sum_{n=0}^{\infty} \frac{s^n}{n!} q_n(\pi(k_m)v - v)$$

and

$$q_n(\pi(k_m)v - v) \le q_n(\pi(k_m)v) + q_n(v) = 2q_n(v).$$

Since K acts continuously on $\mathscr{H}^\infty$, $q_n(\pi(k_m)v - v) \to 0$ for every $n \in \mathbb{N}$, and since $p_s(v) < \infty$, the Dominated Convergence Theorem implies that $p_s(\pi(k_n)v - v) \to 0$.

The fact that $\mathscr{H}^{\omega,r}$ is complete implies that it can be considered as a subspace of the product space $\prod_{s<r} \mathscr{V}_s$, where $\mathscr{V}_s$ denotes the completion of $\mathscr{H}^{\omega,r}$ with respect to the norm p_s. We thus obtain continuous isometric representations of K on the Banach spaces $\mathscr{V}_s$, which leads by integration to representations of $L^1(K)$ on these spaces (see [10, (40.26)]). Finally, since $\mathscr{H}^{\omega,r} \subseteq \prod_{s<r} \mathscr{V}_s$ is closed by completeness (Lemma 7.1.1) and K-invariant, it is also invariant under $L^1(K)$. □

From now on assume that r is small enough such that the exponential function of the simply connected Lie group $\widetilde{G}^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ maps B_r diffeomorphically

onto an open subset of $\widetilde{G}^{\mathbb{C}}$. For every $X \in \mathfrak{g}^{\mathbb{C}}$ the corresponding left and right invariant vector fields define differential operators on $\exp(B_r)$ by

$$(L_X f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)) \quad \text{and} \quad (R_X f)(g) = \frac{d}{dt}\Big|_{t=0} f(\exp(tX)g).$$

Define similar operators L_X^* and R_X^* on B_r by

$$L_X^*(f \circ \exp|_{B_r}) = (L_X f) \circ \exp|_{B_r} \quad \text{and} \quad R_X^*(f \circ \exp|_{B_r}) = (R_X f) \circ \exp|_{B_r}.$$

One can see that

$$L_X^* f_v = f_{\mathsf{d}\pi(X)v} \quad \text{and} \quad R_X^* f_v = \mathsf{d}\pi(X) \circ f_v. \tag{9}$$

If $\mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})$ denotes the Fréchet space of holomorphic $\mathscr{H}$-valued functions on B_r, then the subspace $\mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})^{\mathfrak{g}}$ defined by

$$\mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})^{\mathfrak{g}} = \{f \in \mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H}) \mid R_X^* f = \mathsf{d}\pi(X) \circ f \text{ for every } X \in \mathfrak{g}\},$$

is a closed subspace, hence a Fréchet space. Therefore the map

$$\mathrm{ev}_0 : \mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})^{\mathfrak{g}} \to \mathscr{H}, \quad f \mapsto f(0) \tag{10}$$

is a continuous linear isomorphism onto $\mathscr{H}^{\omega,r}$, hence a topological isomorphism by the Open Mapping Theorem (see [26, Theorem 2.11]).

This implies in particular that

Lemma 7.1.3. The subspace $\mathscr{H}^{\omega,r} \subseteq \mathscr{H}$ is invariant under $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

7.2 *Spectral Theory for Analytic Vectors*

We have already seen in Lemma 7.1.2 that if $(\pi, \mathscr{H})$ is a unitary representation of G then the subspaces $\mathscr{H}^{\omega,r}$ are invariant under the action of the convolution algebras of certain subgroups $K \subseteq G$. As a consequence, we shall now derive that elements of spectral subspaces of certain unitary one-parameter groups can be approximated by analytic vectors.

We begin by a lemma about the relation between one-parameter groups and spectral measures. Let $\mathfrak{B}(\mathbb{R})$ denote the space of Borel measurable functions on $\mathbb{R}$ and $\mathscr{S}(\mathbb{R})$ denote the Schwartz space of $\mathbb{R}$.

Lemma 7.2.1. Let $\gamma : \mathbb{R} \to \mathbf{U}(\mathscr{H})$ be a unitary representation of the additive group of $\mathbb{R}$ and $A = A^* = -i\gamma'(0)$ be its self-adjoint generator, so that $\gamma(t) = e^{itA}$ in terms of measurable functional calculus. Then the following assertions hold.

(i) For each $f \in L^1(\mathbb{R}, \mathbb{C})$, we have $\gamma(f) = \widehat{f}(A)$, where

$$\widehat{f}(x) = \int_{\mathbb{R}} e^{ixy} f(y)\, dy$$

is the Fourier transform of f.

(ii) Let $P : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ be the unique spectral measure with $A = P(\mathrm{id}_{\mathbb{R}})$. Then for every closed subset $E \subseteq \mathbb{R}$ the condition $v \in P(E)\mathscr{H}$ is equivalent to $\gamma(f)v = 0$ for every $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}\big|_E = 0$.

Proof. Since the unitary representation $(\gamma, \mathscr{H})$ is a direct sum of cyclic representations, it suffices to prove the assertions for cyclic representations. Every cyclic representation of $\mathbb{R}$ is equivalent to the representation on some space $\mathscr{H} = L^2(\mathbb{R}, \mu)$, where μ is a Borel probability measure on $\mathbb{R}$ and $(\gamma(t)\xi)(x) = e^{itx}\xi(x)$ (see [20, Theorem VI.1.11]).

(i) This means that $(A\xi)(x) = x\xi(x)$, so that $\widehat{f}(A)\xi(x) = \widehat{f}(x)\xi(x)$. For every $f \in L^1(\mathbb{R}, \mathbb{C})$ the equalities

$$(\gamma(f)\xi)(x) = \int_{\mathbb{R}} f(t)e^{itx}\xi(x)\, dt = \widehat{f}(x)\xi(x)$$

hold in the space $\mathscr{H} = L^2(\mathbb{R}, \mu)$.

(ii) In terms of functional calculus, we have $P(E) = \chi_E(A)$, where χ_E is the characteristic function of E. If $\widehat{f}\big|_E = 0$, then (i) and the fact that $\widehat{f\chi_E} = 0$ imply that

$$0 = (\widehat{f} \cdot \chi_E)(A) = \widehat{f}(A)\chi_E(A) = \gamma(f)P(E).$$

Conversely, suppose that $v \in \mathscr{H}$ satisfies $\gamma(f)v = 0$ for every $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}\big|_E = 0$. If $v \notin P(E)\mathscr{H}$, then $P(E^c)v \neq 0$, and since E^c is open and a countable union of compact subsets, there exists a compact subset $B \subseteq E^c$ with $P(B)v \neq 0$. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that $\psi\big|_B = 1$ and $\mathrm{supp}(\psi) \subseteq E^c$. Then

$$0 \neq P(B)v = \chi_B(A)v = (\chi_B \cdot \psi)(A)v = \chi_B(A)\psi(A)v$$

implies that $\psi(A)v \neq 0$. Since the Fourier transform defines a bijection $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ ([26]), there exists an $f \in \mathscr{S}(\mathbb{R})$ with $\widehat{f} = \psi$. Then $\gamma(f)v = \widehat{f}(A)v = \psi(A)v \neq 0$, contradicting our assumption. This implies that $v \in P(E)\mathscr{H}$.

Proposition 7.2.1. *Let $(\pi, \mathscr{H})$ be a unitary representation of the Lie group G and $X \in \mathfrak{g}$ such that the group $e^{\mathbb{R}\mathrm{ad}(X)}$ preserves a norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$. If $P : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ is the spectral measure of the unitary one-parameter group $\pi_X(t) = \pi(\exp(tX))$ then for every open subset $E \subseteq \mathbb{R}$ the subspace $(P(E)\mathscr{H}) \cap \mathscr{H}^{\omega,r}$ is dense in $P(E)\mathscr{H}^{\omega,r}$.*

Proof. On $\mathscr{H}^{\omega,r}$ we consider the Fréchet topology defined by the seminorms $(p_s)_{s<r}$ in Lemma 7.1.1. Applying Lemma 7.1.2 to $K = \exp(\mathbb{R}X)$ implies that

all of these seminorms are invariant under $\pi_X(\mathbb{R})$ and π_X defines a continuous representation of $\mathbb{R}$ on $\mathscr{H}^{\omega,r}$ which integrates to a representation

$$\widetilde{\pi}_X : (L^1(\mathbb{R},\mathbb{C}),*) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^{\omega,r})$$

of the convolution algebra that is given by

$$\widetilde{\pi}_X(f) = \int_{\mathbb{R}} f(t)\pi_X(t)\,dt.$$

This essentially means that the operators $\widetilde{\pi}_X(f)$ of the integrated representation $L^1(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ preserve the subspace $\mathscr{H}^{\omega,r}$.

Next we write the open set E as the union of the compact subsets

$$E_n := \left\{ t \in E \,\middle|\, |t| \leq n, \mathrm{dist}(t, E^c) \geq \frac{1}{n} \right\}$$

and observe that $\bigcup_n P(E_n)\mathscr{H}$ is dense in $P(E)\mathscr{H}$. For every n, there exists a compactly supported function $h_n \in C_c^\infty(\mathbb{R},\mathbb{R})$ such that $\mathrm{supp}(h_n) \subseteq E$, $0 \leq h_n \leq 1$, and $h_n\big|_{E_n} = 1$. Let $f_n \in \mathscr{S}(\mathbb{R})$ with $\widehat{f}_n = h_n$. Then

$$\widetilde{\pi}_X(f_n) = \widehat{f}_n(-i\,\mathsf{d}\pi(X)) = h_n(-i\,\mathsf{d}\pi(X))$$

and consequently

$$P(E_n)\mathscr{H} \subseteq \widetilde{\pi}_X(f_n)\mathscr{H} \subseteq P(E)\mathscr{H}.$$

If $w = P(E)v$ for some $v \in \mathscr{H}^{\omega,r}$ then

$$\widetilde{\pi}_X(f_n)w = \widetilde{\pi}_X(f_n)P(E)v = \widetilde{\pi}_X(f_n)v \in \mathscr{H}^{\omega,r}$$

and

$$\|\widetilde{\pi}_X(f_n)w - w\|^2 = \|h_n(-i\,\mathsf{d}\pi(X))w - w\|^2 \leq \|P(E\setminus E_n)w\|^2 \to 0$$

from which it follows that $\widetilde{\pi}_X(f_n)w \to w$. □

Proposition 7.2.2. *If $Y \in \mathfrak{g}^{\mathbb{C}}$ satisfies $[X,Y] = i\mu Y$ then for every open subset $E \subseteq \mathbb{R}$ the spectral measure of π_X satisfies*

$$\mathsf{d}\pi(Y)\big(P(E)\mathscr{H} \cap \mathscr{H}^\infty\big) \subseteq P(E+\mu)\mathscr{H}. \tag{11}$$

Proof. To verify this relation, we first observe that

$$\pi_X(t)\mathsf{d}\pi(Y)v = \mathsf{d}\pi(e^{t\,\mathrm{ad}X}(Y))\pi_X(t)v = e^{it\mu}\mathsf{d}\pi(Y)\pi_X(t)v$$

for every $v \in \mathscr{H}^\infty$. For $f \in \mathscr{S}(\mathbb{R})$, the continuity of the map

$$\mathscr{S}(\mathbb{R}) \to \mathscr{H}^\infty, \quad f \mapsto \widetilde{\pi}_X(f)v$$

leads to

$$\widetilde{\pi}_X(f)\mathrm{d}\pi(Y)v = \mathrm{d}\pi(Y)\int_{\mathbb{R}} f(t)e^{it\mu}\pi_X(t)v = \mathrm{d}\pi(Y)\widetilde{\pi}_X(f\cdot e_\mu)v,$$

where $e_\mu(t) = e^{it\mu}$. If $v \in P(E)\mathscr{H}$ and $\widehat{f}$ vanishes on $E+\mu$ then the function $(e_\mu f)\widehat{\ } = \widehat{f}(\mu + \cdot)$ vanishes on E, and Lemma 7.2.1(ii) implies that $\widetilde{\pi}_X(f\cdot e_\mu)v = 0$. By applying Lemma 7.2.1(ii) one more time, we obtain that $\mathrm{d}\pi(Y)v \in P(E+\mu)\mathscr{H}$. □

7.3 Application to Irreducible Unitary Representations of Lie Supergroups

Let $(\pi, \rho^\pi, \mathscr{H})$ be an irreducible unitary representation of the Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$. Before we turn to the fine structure of such a representation, we verify that Lemma 7.1.3 generalizes to the super context.

Lemma 7.3.1. The subspace $\mathscr{H}^{\omega,r} \subseteq \mathscr{H}$ is invariant under $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

Proof. In view of Lemma 7.1.3, it only remains to show that, for every $Y \in \mathfrak{g}_{\overline{1}}$ and $v \in \mathscr{H}^{\omega,r}$, we have $\rho^\pi(Y)v \in \mathscr{H}^{\omega,r}$. For every $X \in \mathfrak{g}_{\overline{0}} \cap B_r$, we have the relation

$$\pi(\exp X)\rho^\pi(Y)v = \rho^\pi(e^{\mathrm{ad}\,X}Y)\pi(\exp X)v = \rho^\pi(e^{\mathrm{ad}\,X}Y)f_v(X). \tag{12}$$

The complex bilinear map

$$\mathfrak{g}_{\overline{1}}^{\mathbb{C}} \times \mathscr{H}^\infty \to \mathscr{H}^\infty, \quad (Z, v) \mapsto \rho^\pi(Z)v$$

is continuous by Lemma 4.2.4 and therefore holomorphic. Moreover, the map

$$\mathfrak{g}_{\overline{0}}^{\mathbb{C}} \to \mathfrak{g}_{\overline{1}}^{\mathbb{C}}, \quad X \mapsto e^{\mathrm{ad}\,X}Y$$

is holomorphic. Since compositions of holomorphic maps are holomorphic, it therefore suffices to show that $f_v(B_r) \subseteq \mathscr{H}^\infty$ and that the map $f_v\colon B_r \to \mathscr{H}^\infty$ is holomorphic. In fact, this implies that the map

$$\mathfrak{g}_{\overline{0}} \cap B_r \to \mathscr{H}, \quad X \mapsto \pi(\exp X)\rho^\pi(Y)v$$

extends holomorphically to B_r, i.e., $\rho^\pi(Y)v \in \mathscr{H}^{\omega,r}$.

We recall the topological isomorphism

$$\mathrm{ev}_0 \colon \mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})^{\mathfrak{g}} \to \mathscr{H}^{\omega,r}, \quad f \mapsto f(0).$$

By definition of $\mathcal{H}\mathcal{O}\ell(B_r, \mathscr{H})^{\mathfrak{g}}$, we have for each $X \in \mathfrak{g}_{\overline{0}}$ the relation

$$\mathsf{d}\pi(X) \circ f_v = R_X^* f_v,$$

showing in particular that $\mathsf{d}\pi(X) \circ f_v \colon B_r \to \mathscr{H}$ is a holomorphic function. From the definition of the topology on $\mathscr{H}^\infty$, it therefore follows that f_v is holomorphic as a map $B_r \to \mathscr{H}^\infty$. □

The following theorem clarifies the key features of the $\mathfrak{g}$-representation on $\mathscr{H}^\infty$.

Theorem 7.3.2. *Let $(\pi, \rho^\pi, \mathscr{H})$ be an irreducible unitary representation of the Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ which is $\star$-reduced and satisfies*

$$\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}].$$

Pick a regular element $X_0 \in Int(Cone(\mathcal{G}))$ and let $\mathfrak{t} = \mathfrak{t}_{\overline{0}} \oplus \mathfrak{t}_{\overline{1}}$ be the corresponding Cartan subsuperalgebra of $\mathfrak{g}$ (see Lemma 5.2.1 and Proposition 2.3.1). Suppose that no root vanishes on X_0. Then the following assertions hold.

(i) *$\mathfrak{t}_{\overline{0}}$ is compactly embedded and $\Delta^+ = \{\, \alpha \in \Delta \mid \alpha(X_0) > 0 \,\}$ satisfies $\Delta \backslash \{0\} = \Delta^+ \dot{\cup} -\Delta^+$.*

(ii) *The space $\mathscr{H}^{\mathfrak{t}}$ of $\mathfrak{t}$-finite elements in $\mathscr{H}^\infty$ is an irreducible $\mathfrak{g}$-module which is a $\mathfrak{t}_{\overline{0}}$-weight module and dense in $\mathscr{H}$.*

(iii) *The maximal eigenspace $\mathscr{V}$ of $i\rho^\pi(X_0)$ is an irreducible finite dimensional $\mathfrak{t}$-module on which $\mathfrak{t}_{\overline{0}}$ acts by some weight $\lambda \in \mathfrak{t}_{\overline{0}}^*$. It generates the $\mathfrak{g}$-module $\mathscr{H}^{\mathfrak{t}}$ and all other $\mathfrak{t}_{\overline{0}}$-weights in this space are of the form*

$$\lambda - m_1\alpha_1 - \cdots - m_k\alpha_k, \quad \alpha_j \in \Delta^+, \ k \in \mathbb{N}, m_1, \ldots, m_k \in \mathbb{N} \cup \{0\}.$$

(iv) *Two representations $(\pi, \rho^\pi, \mathscr{H})$ and $(\pi', \rho^{\pi'}, \mathscr{H}')$ of $\mathcal{G}$ are isomorphic if and only if the corresponding $\mathfrak{t}$-representations on $\mathscr{V}$ and $\mathscr{V}'$ are isomorphic.*

Proof. (i) Proposition 6.1.2 implies that $Cone(\mathcal{G})$ is a pointed generating invariant cone and $\mathfrak{g}_{\overline{0}}$ has a Cartan subalgebra $\mathfrak{t}_{\overline{0}}$ which is compactly embedded in $\mathfrak{g}$. Then the corresponding Cartan supersubalgebra is given by its centralizer $\mathfrak{t} = \mathscr{Z}_{\mathfrak{g}}(\mathfrak{t}_{\overline{0}})$. Pick a regular element $X_0 \in \mathfrak{t}_{\overline{0}} \cap Int(Cone(\mathcal{G}))$, so that Δ^+ satisfies $\Delta \backslash \{0\} = \Delta^+ \dot{\cup} -\Delta^+$.

(ii) Recall from (7) that $i\rho^\pi(X_0) \leq 0$. We want to prove the existence of an eigenvector of maximal eigenvalue for $i\rho^\pi(X_0)$. Let

$$\delta = \min\{\alpha(X_0) | \alpha \in \Delta^+\}$$

and note that $\delta > 0$. Let $P([a,b])$, $a \leq b \in \mathbb{R}$, denote the spectral projections of the self-adjoint operator $i\overline{\rho^\pi(X_0)}$ and put

$$\lambda = \sup(\mathrm{Spec}(i\overline{\rho^\pi(X_0)})) \leq 0.$$

Since $(\pi, \rho^\pi, \mathscr{H})$ is irreducible and the space $\mathscr{H}^\omega$ of analytic vectors is dense, there exists an $r > 0$ with $\mathscr{H}^{\omega,r} \neq \{0\}$. Then the invariance of $\mathscr{H}^{\omega,r}$ under $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$ (Lemma 7.3.1) implies that $\mathscr{H}^{\omega,r}$ is dense in $\mathscr{H}$. Hence Proposition 7.2.1 implies that, for every $\varepsilon > 0$, the intersection

$$P(]\mu - \varepsilon, \mu])\mathscr{H} \cap \mathscr{H}^{\omega,r}$$

is dense in $P(]\mu - \varepsilon, \mu])\mathscr{H}$. In particular, it contains a non-zero vector v_0. With Proposition 7.2.2 for $\varepsilon < \delta$ and $\alpha \in \Delta^+$ we obtain

$$\rho(\mathfrak{g}^{\mathbb{C},\alpha})v_0 \subseteq P(]\mu, \infty[)\mathscr{H} = \{0\}.$$

In view of the Poincaré–Birkhoff–Witt Theorem, this leads to

$$\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_0 = \mathscr{U}(\mathfrak{g}^- \rtimes \mathfrak{t}^{\mathbb{C}})v_0.$$

Since $\mathfrak{t}^{\mathbb{C}}$ commutes with $\mathfrak{t}_{\overline{0}}$, the subspace $\mathscr{U}(\mathfrak{t}^{\mathbb{C}})v_0$ is contained in $P([\mu-\varepsilon, \mu])\mathscr{H}$, so that Proposition 7.2.2 yields

$$\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_0 \subseteq P(]-\infty, \mu - \delta])\mathscr{H} + P([\mu - \varepsilon, \mu])\mathscr{H}$$

for every $\varepsilon > 0$. As $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_0$ is dense in $\mathscr{H}$, for every sufficiently small $\varepsilon > 0$ we have $P([\mu - \varepsilon, \mu]) = P(\{\mu\})$. Hence $i\rho(X_0)v_0 = \mu v_0$. Since $\mathfrak{g}^{\mathbb{C}}$ is spanned by $\mathrm{ad}(X_0)$-eigenvectors, the same holds for $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$, and hence for $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_0$. Since $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_0$ is dense in $\mathscr{H}$, this means that $\mathscr{H}$ is a direct sum of weight spaces for $\exp(\mathbb{R}X_0)$. Repeating the same argument for other regular elements in $\mathfrak{t}_{\overline{0}} \cap \mathit{Int}(\mathit{Cone}(\mathcal{G}))$ forming a basis of $\mathfrak{t}_{\overline{0}}$, we conclude that $\mathscr{H}$ is an orthogonal direct sum of weight spaces for the group $T = \exp(\mathfrak{t}_{\overline{0}})$.

Let $\mathscr{V} = P(\{\mu\})\mathscr{H}$ be the maximal eigenspace of $i\rho^\pi(X_0)$. Then Proposition 7.2.1 applied to sets of the form $E =]\mu - \varepsilon, \mu + \varepsilon[$ implies that $\mathscr{H}^{\omega,r} \cap \mathscr{V}$ is dense in $\mathscr{V}$. Further $\mathscr{V}$ is T-invariant, hence an orthogonal direct sum of T-weight spaces. From Lemma 7.1.2, applied to $K = T$, we now derive that in each T-weight space $\mathscr{V}^\alpha(T)$, the intersection with $\mathscr{H}^{\omega,r}$ is dense.

Let $v_\alpha \in \mathscr{V}^\alpha \cap \mathscr{H}^{\omega,r}$ be a T-eigenvector. From the density of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})v_\alpha = \mathscr{U}(\mathfrak{g}^-)\mathscr{U}(\mathfrak{t}^{\mathbb{C}})v_\alpha$ in $\mathscr{H}$ we then derive as above that

$$\mathscr{U}(\mathfrak{t}^{\mathbb{C}})v_\alpha \subseteq \mathscr{V}^\alpha$$

is dense in $\mathscr{V}$. The left hand side of the latter inclusion is finite dimensional, and therefore $\mathscr{V} = \mathscr{V}^\alpha$ is also finite dimensional and contained in $\mathscr{H}^{\omega,r}$.

Since all $\mathfrak{t}_{\overline{0}}$-weight spaces in $\mathscr{U}(\mathfrak{g}^-)$ are finite dimensional, we conclude that $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$ is a locally finite $\mathfrak{t}$-module with finite $\mathfrak{t}_{\overline{0}}$-multiplicities. In view of the finiteness of multiplicities, density of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$ in $\mathscr{H}$ leads to the equality $\mathscr{H}^{\mathfrak{t}} = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathscr{V}$. As this $\mathfrak{g}$-module consists of analytic vectors, its irreducibility follows from the irreducibility of the $\mathcal{G}$-representation on $\mathscr{H}$.

(iii) If $\mathscr{V}' \subseteq \mathscr{V}$ is a non-zero $\mathfrak{t}$-submodule, then $\mathscr{U}(\mathfrak{t})\mathscr{V}'$ is dense in $\mathscr{V}$ and orthogonal to the subspace $\mathscr{V}'' = (\mathscr{V}')^{\perp}$, which leads to $\mathscr{V}'' = \{0\}$. Therefore the $\mathfrak{t}$-module $\mathscr{V}$ is irreducible. All other assertions have already been verified above.

(iv) Clearly, the equivalence of the $\mathcal{G}$-representations implies equivalence of the $\mathfrak{t}$-representations on $\mathscr{V}$ and $\mathscr{V}'$.

Suppose, conversely, that there exists a $\mathfrak{t}$-isomorphism $\phi\colon \mathscr{V} \to \mathscr{V}'$. We consider the direct sum representation $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}'$ of $\mathcal{G}$, for which

$$\mathscr{K}^{\mathfrak{t}} = \mathscr{H}^{\mathfrak{t}} \oplus (\mathscr{H}')^{\mathfrak{t}}$$

as $\mathfrak{g}$-modules. Consider the $\mathfrak{g}$-submodule $W \subseteq \mathscr{K}^{\mathfrak{t}}$ generated by the $\mathfrak{t}$-submodule

$$\Gamma(\phi) = \{(v, \phi(v))\colon v \in \mathscr{V}\} \subseteq \mathscr{V} \oplus \mathscr{V}'.$$

Since $\Gamma(\phi)$ is annihilated by $\mathfrak{g}^+$, the PBW Theorem implies that

$$W = \mathscr{U}(\mathfrak{g})\Gamma(\phi) = \mathscr{U}(\mathfrak{g}^-)\mathscr{U}(\mathfrak{t})\mathscr{U}(\mathfrak{g}^+)\Gamma(\phi) = \mathscr{U}(\mathfrak{g}^-)\Gamma(\phi).$$

It follows that

$$W \cap (\mathscr{V} \oplus \mathscr{V}') = \Gamma(\phi)$$

is the maximal eigenspace for iX_0 on W.

As W consists of analytic vectors, its closure $\overline{W}$ is a proper G-invariant subspace of $\mathscr{K}$, so that we obtain a unitary $\mathcal{G}$-representation on this space.

If the two $\mathcal{G}$ representations $(\pi, \rho^{\pi}, \mathscr{H})$ and $(\pi', \rho^{\pi'}, \mathscr{H}')$ are not equivalent, then Schur's Lemma implies that $\mathscr{H}$ and $\mathscr{H}'$ are the only non-trivial $\mathcal{G}$-invariant subspaces of $\mathscr{K}$, contradicting the existence of $\overline{W}$. □

Remark 7.3.3. (a) The preceding theorem suggests to call the $\mathfrak{g}$-representation on $\mathscr{H}^{\mathfrak{t}}$ a *highest weight representation* because it is generated by a weight space of $\mathfrak{t}_{\overline{0}}$ which is an irreducible $\mathfrak{t}$-module, hence a (finite dimensional) irreducible module of the Clifford Lie superalgebra $\mathfrak{t}_{\overline{1}} + [\mathfrak{t}_{\overline{1}}, \mathfrak{t}_{\overline{1}}]$.

(b) Suppose that $\mathfrak{g}$ is $\star$-reduced with $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{D} \subseteq \mathscr{H}$ a dense subspace on which we have a unitary representation $(\rho, \mathscr{D})$ of $\mathfrak{g}$ in the sense that (i), (iii), (v) in Definition 4.2.1 are satisfied.

Suppose further that the action of $\mathfrak{t}_{\overline{0}}$ on $\mathscr{D}$ is diagonalizable with finite dimensional weight spaces. Then the $\mathfrak{g}$-module $\mathscr{D}$ is semisimple, hence irreducible if it is generated by a $\mathfrak{t}_{\overline{0}}$-weight space V on which $\mathfrak{t}$ acts irreducibly.

The finite dimensionality of the $\mathfrak{t}_{\overline{0}}$-weight spaces on $\mathscr{D}$ also implies the semisimplicity of $\mathscr{D}$ as a $\mathfrak{g}_{\overline{0}}$-module. Hence, as $i\rho(X_0) \leq 0$, an argument as in the proof of Theorem 7.3.2 implies that each simple submodule of $\mathscr{D}$ is a unitary highest weight

module, hence integrable by [20, Corollary XII.2.7]. We conclude in particular that the $\mathfrak{g}_0$-representation on $\mathscr{D}$ is integrable with $\mathscr{D}$ consisting of analytic vectors.

8 The Orbit Method and Nilpotent Lie Supergroups

One of the most elegant and powerful ideas in the theory of unitary representations of Lie groups since the early stages of its development is the *orbit method*. The basic idea of the orbit method is to attach unitary representations to special homogeneous symplectic manifolds, such as the coadjoint orbits, in a natural way. One of the goals of the orbit method is to obtain a concrete realization of the representation and to extract information about the representation (e.g., its distribution character) from this realization.

Recall that a Lie supergroup $\mathcal{G} = (G, \mathfrak{g})$ is called *nilpotent* when the Lie superalgebra $\mathfrak{g}$ is nilpotent. In this article the orbit method is only studied for nilpotent Lie supergroups. It is known that among Lie groups, the orbit method works best for the class of nilpotent ones. For further reading on the subject of the orbit method, the reader is referred to [15] and [35].

8.1 Quantization and Polarizing Subalgebras

All of the irreducible unitary representations of nilpotent Lie groups can be classified by the orbit method. Let G be a nilpotent real Lie group and $\mathfrak{g}$ be its Lie algebra. For simplicity, G is assumed to be simply connected. In this case, there exists a bijective correspondence between coadjoint orbits (i.e., G-orbits in $\mathfrak{g}^*$) and irreducible unitary representations of G. In some sense the correspondence is surprisingly simple. To construct a representation $\pi_{\mathcal{O}}$ of G which corresponds to a coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, one first chooses an element $\lambda \in \mathcal{O}$ and considers the skew symmetric form

$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \text{ defined by } \Omega_\lambda(X, Y) = \lambda([X, Y]). \tag{13}$$

It can be shown that there exist maximal isotropic subspaces of Ω_λ which are also subalgebras of $\mathfrak{g}$. Such subalgebras are called *polarizing subalgebras*. For a given polarizing subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, one can consider the one dimensional representation of the subgroup $M = \exp(\mathfrak{m})$ of G given by

$$\chi_\lambda(m) = e^{i\lambda(\log(m))} \quad \text{for } m \in M.$$

The unitary representation of G corresponding to $\mathcal{O}$ is $\pi_{\mathcal{O}} = \mathrm{Ind}_M^G \chi_\lambda$. Of course one needs to prove that the construction is independent of the choices of λ and

$\mathfrak{m}$, the representation $\pi_{\mathcal{O}}$ is irreducible, and the correspondence is bijective. These statements are proved in [14]. Many other proofs have been found as well.

8.2 *Heisenberg–Clifford Lie Supergroups*

Heisenberg groups play a distinguished role in the harmonic analysis of nilpotent Lie groups. Therefore it is natural to expect that the analogues of Heisenberg groups in the category of Lie supergroups play a similar role in the representation theory of nilpotent Lie supergroups. These analogues, which deserve to be called *Heisenberg–Clifford* Lie supergroups, can be described as follows. Let (W, Ω) be a finite dimensional real *super symplectic* vector space. This means that $\mathsf{W} = \mathsf{W}_{\bar{0}} \oplus \mathsf{W}_{\bar{1}}$ is endowed with a bilinear form

$$\Omega : \mathsf{W} \times \mathsf{W} \to \mathbb{R}$$

that satisfies the following properties.

(i) $\Omega(\mathsf{W}_{\bar{0}}, \mathsf{W}_{\bar{1}}) = \Omega(\mathsf{W}_{\bar{1}}, \mathsf{W}_{\bar{0}}) = \{0\}$.
(ii) The restriction of Ω to $\mathsf{W}_{\bar{0}}$ is a symplectic form.
(iii) The restriction of Ω to $\mathsf{W}_{\bar{1}}$ is a nondegenerate symmetric form.

The Heisenberg–Clifford Lie supergroup corresponding to (W, Ω) is the super Harish–Chandra pair $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ where

(i) $\mathfrak{h}^{\mathsf{W}}_{\bar{0}} = \mathsf{W}_{\bar{0}} \oplus \mathbb{R}$ and $\mathfrak{h}^{\mathsf{W}}_{\bar{1}} = \mathsf{W}_{\bar{1}}$ (as vector spaces).
(ii) For every $X, Y \in \mathsf{W}$ and every $a, b \in \mathbb{R}$, the superbracket of $\mathfrak{h}^{\mathsf{W}}$ is defined by

$$[(X, a), (Y, b)] = (0, \Omega(X, Y)).$$

(iii) H^{W} is the simply connected Lie group with Lie algebra $\mathfrak{h}^{\mathsf{W}}_{\bar{0}}$.

When $\dim \mathsf{W}_{\bar{1}} = 0$ the Lie supergroup $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ is purely even, i.e., it is a Lie group. In this case, it is usually called a *Heisenberg Lie group*. When $\dim \mathsf{W}_{\bar{0}} = 0$ the Lie supergroup $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ is called a *Clifford Lie supergroup*.

Irreducible unitary representations of Heisenberg Lie groups are quite easy to classify. One can use the orbit method of Sect. 8.1 to classify them, but their classification was known as a consequence of the Stone–von Neumann Theorem long before the orbit method was developed. The Stone–von Neumann Theorem implies that there exists a bijective correspondence between infinite dimensional irreducible unitary representations of a Heisenberg Lie group and nontrivial characters (i.e., one dimensional unitary representations) of its center.

For Heisenberg–Clifford Lie supergroups there is a similar classification of representations. Let $(\pi, \rho^{\pi}, \mathscr{H})$ be an irreducible unitary representation of $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$. By a super version of Schur's Lemma, for every $Z \in \mathscr{Z}(\mathfrak{h}^{\mathsf{W}})$ the action of $\rho^{\pi}(Z)$

is via multiplication by a scalar $c_{\rho^\pi}(Z)$. If $c_{\rho^\pi}(Z) = 0$ for every $Z \in \mathscr{Z}(\mathfrak{h}^{\mathsf{W}})$, then $\mathscr{H}$ is one dimensional, and essentially obtained from a unitary character of $\mathsf{W}_{\bar{0}}$. The irreducible unitary representations for which $\rho^\pi(\mathscr{Z}(\mathfrak{h}^{\mathsf{W}})) \neq \{0\}$ are classified by the following statement (see [27, Theorem 5.2.1]).

Theorem 8.2.1. *Let* S *be the set of unitary equivalence classes of irreducible unitary representations* $(\pi, \rho^\pi, \mathscr{H})$ *of* $(H^{\mathsf{W}}, \mathfrak{h}^{\mathsf{W}})$ *for which* $\rho^\pi(\mathscr{Z}(\mathfrak{h}^{\mathsf{W}})) \neq \{0\}$. *Then* S *is nonempty if and only if the restriction of* Ω *to* $\mathsf{W}_{\bar{1}}$ *is (positive or negative) definite. Moreover, the map*

$$[(\pi, \rho^\pi, \mathscr{H})] \mapsto c_{\rho^\pi}$$

yields a surjection from S *onto the set of* $\mathbb{R}$*-linear functionals* $\gamma : \mathscr{Z}(\mathfrak{h}^{\mathsf{W}}) \to \mathbb{R}$ *which satisfy*

$$i\gamma([X, X]) < 0 \text{ for every } 0 \neq X \in \mathsf{W}_{\bar{1}}.$$

When $\dim \mathsf{W}_{\bar{1}}$ *is odd the latter map is a bijection, and when* $\dim \mathsf{W}_{\bar{1}}$ *is even it is two-to-one, and the two representations in the fiber are isomorphic via parity change.*

Every irreducible unitary representation of a Clifford Lie supergroup is finite dimensional (see [27, Sect. 4.5]). In fact the theory of Clifford modules implies that the only possible values for the dimension of such a representation are one or

$$2^{\left(\dim \mathsf{W}_{\bar{1}} - \lfloor \frac{\dim \mathsf{W}_{\bar{1}}}{2} \rfloor\right)}.$$

It will be seen below that Clifford Lie supergroups are used to define analogues of polarizing subalgebras for Lie supergroups.

8.3 Polarizing Systems and a Construction

In order to construct the irreducible unitary representations of a nilpotent Lie supergroup using the orbit method, first we need to generalize the notion of polarizing subalgebras. What makes the case of Lie supergroups more complicated than the case of Lie groups is the fact that irreducible unitary representations of nilpotent Lie supergroups are not necessarily induced from one dimensional representations. However, it will be seen that they are induced from certain finite dimensional representations which are obtained from representations of Clifford Lie supergroups.

Let $(G, \mathfrak{g})$ be a Lie supergroup. Associated to every $\lambda \in \mathfrak{g}_{\bar{0}}^*$ there exists a skew symmetric bilinear form Ω_λ on $\mathfrak{g}_{\bar{0}}$ which is defined in (13). There is also a symmetric bilinear form

$$\Omega_\lambda : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathbb{R} \tag{14}$$

associated to λ, which is defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.

Definition 8.3.1. Let $\mathcal{G} = (G, \mathfrak{g})$ be a nilpotent Lie supergroup. A polarizing system in $(G, \mathfrak{g})$ is a pair $(\mathcal{M}, \lambda)$ satisfying the following properties.

(i) $\lambda \in \mathfrak{g}_{\overline{0}}^*$ and Ω_λ is a positive semidefinite form.
(ii) $\mathcal{M} = (M, \mathfrak{m})$ is a Lie subsupergroup of $\mathcal{G}$ and $\dim \mathfrak{m}_{\overline{1}} = \dim \mathfrak{g}_{\overline{1}}$.
(iii) $\mathfrak{m}_{\overline{0}}$ is a polarizing subalgebra of $\mathfrak{g}_{\overline{0}}$ with respect to λ, i.e., a subalgebra of $\mathfrak{g}_{\overline{0}}$ which is also a maximal isotropic subspace with respect to Ω_λ.

Given a polarizing system $(\mathcal{M}, \lambda)$, one can construct a unitary representation of $\mathcal{G}$ as follows. Let

$$\mathfrak{j} = \ker \lambda \oplus \operatorname{rad}(\Omega_\lambda), \tag{15}$$

where $\operatorname{rad}(\Omega_\lambda)$ denotes the radical of the symmetric form Ω_λ. One can show that $\mathfrak{j}$ is an ideal of $\mathfrak{m}$ that corresponds to a Lie subsupergroup $\mathcal{J} = (J, \mathfrak{j})$ of $\mathcal{M}$, and the quotient $\mathcal{M}/\mathcal{J}$ is a Clifford Lie supergroup. Let $\mathscr{Z}(\mathfrak{m}/\mathfrak{j})$ denote the center of $\mathfrak{m}/\mathfrak{j}$. Since Ω_λ is positive semidefinite, from Theorem 8.2.1 it follows that up to parity and unitary equivalence there exists a unique unitary representation $(\sigma, \rho^\sigma, \mathscr{K})$ of $\mathcal{M}/\mathcal{J}$ such that for every $Z \in \mathscr{Z}(\mathfrak{m}/\mathfrak{j})$, the operator $\rho^\sigma(Z)$ acts via multiplication by $i\lambda(Z)$. Clearly $(\sigma, \rho^\sigma, \mathscr{K})$ can also be thought of as a representation of $\mathcal{M}$, and one can consider the induced representation

$$(\pi, \rho^\pi, \mathscr{H}) = \operatorname{Ind}_{\mathcal{M}}^{\mathcal{G}}(\sigma, \rho^\sigma, \mathscr{K}). \tag{16}$$

8.4 Existence of Polarizing Systems

Throughout this section $\mathcal{G} = (G, \mathfrak{g})$ will be a nilpotent Lie supergroup such that G is simply connected.

It is natrual to ask for which $\lambda \in \mathfrak{g}_{\overline{0}}^*$ such that Ω_λ is positive semidefinite there exists a polarizing system $(\mathcal{M}, \lambda)$ in the sense of Definition 8.3.1. It turns out that for all such λ the answer is affirmative. The latter statement can be proved as follows. Fix such a $\lambda \in \mathfrak{g}_{\overline{0}}^*$. Proving the existence of a polarizing system $(\mathcal{M}, \geq)$ amounts to showing that there exists a polarizing subalgebra $\mathfrak{m}_{\overline{0}}$ of $\mathfrak{g}_{\overline{0}}$ such that $\mathfrak{m}_{\overline{0}} \supseteq [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. Since $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ is an ideal of the Lie algebra $\mathfrak{g}_{\overline{0}}$ and $\mathfrak{g}_{\overline{0}}$ is nilpotent, one can find a sequence of ideal of $\mathfrak{g}_{\overline{0}}$ such as

$$\{0\} = \mathfrak{i}^{(0)} \subseteq \mathfrak{i}^{(1)} \subseteq \cdots \subseteq \mathfrak{i}^{(k-1)} \subseteq \mathfrak{i}^{(k)} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \subseteq \mathfrak{i}^{k+1} \subseteq \cdots \subseteq \mathfrak{i}^{(r)} = \mathfrak{g}_{\overline{0}}$$

where for every $0 \leq s \leq r-1$, the codimension of $\mathfrak{i}^{(s)}$ in $\mathfrak{i}^{(s+1)}$ is equal to one. For every $0 \leq s \leq r-1$, let

$$\Omega_\lambda^{(s)} : \mathfrak{i}^{(s)} \times \mathfrak{i}^{(s)} \to \mathbb{R}$$

be the skew symmetric form defined by

$$\Omega_\lambda^{(s)}(X, Y) = \lambda([X, Y])$$

and let $\mathrm{rad}(\Omega_\lambda^{(s)})$ denote the radical of $\Omega_\lambda^{(s)}$. It is known that the subspace of $\mathfrak{g}_{\overline{0}}$ defined by

$$\mathrm{rad}(\Omega_\lambda^{(1)}) + \cdots + \mathrm{rad}(\Omega_\lambda^{(s)})$$

is a polarizing subspace of $\mathfrak{g}_{\overline{0}}$ corresponding to λ (see [8, Theorem 1.3.5]). To prove that

$$[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \subseteq \mathrm{rad}(\Omega_\lambda^{(1)}) + \cdots + \mathrm{rad}(\Omega_\lambda^{(s)})$$

it suffices to show that $\mathrm{rad}(\Omega_\lambda^{(k)}) = \mathfrak{i}^{(k)}$. This is where one needs the fact that Ω_λ is positive semidefinite. The proof is by a backward induction on the dimension of $\mathcal{G}$. Details of the argument appear in [27, Sect. 6.3].

8.5 A Bijective Correspondence

Throughout this section $\mathcal{G} = (G, \mathfrak{g})$ will be a nilpotent Lie supergroup such that G is simply connected.

One can check easily that the set

$$\mathscr{P}(\mathcal{G}) = \left\{ \lambda \in \mathfrak{g}_{\overline{0}}^* \;\middle|\; \Omega_\lambda \text{ is positive semidefinite} \right\}$$

is an invariant cone in $\mathfrak{g}_{\overline{0}}^*$. Section 8.4 shows that for every $\lambda \in \mathscr{P}(\mathcal{G})$ one can find a polarizing system $(\mathcal{M}, \lambda)$. Therefore the construction of Sect. 8.3 yields a unitary representation $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ of $\mathcal{G}$ which is given by (16). The main result of [27] can be stated as follows.

Theorem 8.5.1. *The map which takes a $\lambda \in \mathscr{P}(\mathcal{G})$ to the representation $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ results in a bijective correspondence between G-orbits in $\mathscr{P}(\mathcal{G})$ and irreducible unitary representations of $\mathcal{G}$ up to unitary equivalence and parity change.*

To prove Theorem 8.5.1 one needs to show that the construction given in Sect. 8.3 yields an irreducible representation and is independent of the choice of λ in a G-orbit or the polarizing system. One also has to show that if λ and λ' are not in the same G-orbit then inducing from polarizing systems $(\mathcal{M}, \lambda)$ and $(\mathcal{M}', \lambda')$ does not lead to representations which are identical up to parity or unitary equivalence. The proofs of all of these facts are given in [27, Sect. 6]. To some extent, the method of proof is similar to the original proof of the Lie group case in [14], where induction on the dimension is used. In the Lie group case, what makes the inductive argument work is the existence of three dimensional Heisenberg subgroups in any nilpotent Lie group of dimension bigger than one with one dimensional center. For Lie supergroups a similar statement only holds under extra assumptions. The next proposition shows that it suffices to assume that the corresponding Lie superalgebra has no self-commuting odd elements.

Proposition 8.5.2. *Let $\mathcal{G} = (G, \mathfrak{g})$ be as above. Assume that there are no nonzero $X \in \mathfrak{g}_{\overline{1}}$ such that $[X, X] = 0$. If $\dim \mathcal{Z}(\mathfrak{g}) = 1$ then either $\mathcal{G}$ is a Clifford Lie supergroup, or it has a Heisenberg Lie subsupergroup of dimension* (3|0).

Using Proposition 6.1.1 one can pass to a quotient and reduce the analysis of the general case to the case where the assumptions of Proposition 8.5.2 are satisfied. Proposition 8.5.2 makes induction on the dimension of $\mathfrak{g}$ possible.

Although the proof of Theorem 8.5.1 is inspired by the methods and arguments in [14] and [8], one must tackle numerous additional analytic technical difficulties which emerge in the case of Lie supergroups. This is because many facts in the theory of unitary representations of Lie supergroups are generally not as powerful as their analogues for Lie groups. For instance to prove that $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ is irreducible one cannot use Mackey theory and needs new ideas.

8.6 Branching to the Even Part

Let $\mathcal{G} = (G, \mathfrak{g})$ be as in Sect. 8.5. For every $\lambda \in \mathscr{P}(\mathcal{G})$ let $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ be the representation of $\mathcal{G}$ associated to λ in Sect. 8.5. As an application of Theorem 8.5.1 one can obtain a simple decomposition formula for the restriction of $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ to G.

Recall that $(\pi_\lambda, \rho^{\pi_\lambda}, \mathscr{H}_\lambda)$ is induced from a polarizing system $(\mathcal{M}, \lambda)$. Let $\mathfrak{m}$ be the Lie superalgebra of $\mathcal{M}$ and $\mathfrak{j}$ be defined as in (15).

Corollary 8.6.1. *The representation $(\pi_\lambda, \mathscr{H}_\lambda)$ of G decomposes into a direct sum of $2^{\dim \mathfrak{m} - \dim \mathfrak{j}}$ copies of the irreducible unitary representation of G which is associated to the coadjoint orbit containing λ (in the sense of Sect. 8.1).*

9 Conclusion

In this article we discussed irreducible unitary representations of Lie supergroups in some detail for the case where $\mathcal{G}$ is either nilpotent or $\mathfrak{g}$ is $\star$-reduced and satisfies $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$. The overlap between these two classes is quite small because for any nilpotent Lie superalgebra satisfying the latter conditions $\mathfrak{g}_{\overline{0}}$ is central, so that it essentially is a Clifford–Lie superalgebra, possibly with a multidimensional center, and in this case the irreducible unitary representations are the well-known spin representations. Precisely these representations occur as the $\mathfrak{k}$-modules on the highest weight space $\mathscr{V}$ in the other case.

Clearly, the condition of being $\star$-reduced is natural if one is interested in unitary representations. The requirement that $\mathfrak{g}_{\overline{0}} = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ is more serious, as we have seen in the nilpotent case. In general one can consider the ideal $\mathfrak{g}_c = [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \oplus \mathfrak{g}_{\overline{1}}$ and our results show that the irreducible unitary representations of this ideal are highest weight representations. For nilpotent Lie supergroups, how to use them to

parametrize the irreducible unitary representations of $\mathcal{G}$ was explained in Sect. 8. It is conceivable that other larger classes of groups could be studied by combining tools from the Orbit Method, induction procedures and highest weight theory.

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