Combinatorial Identities and Six Vertex Models

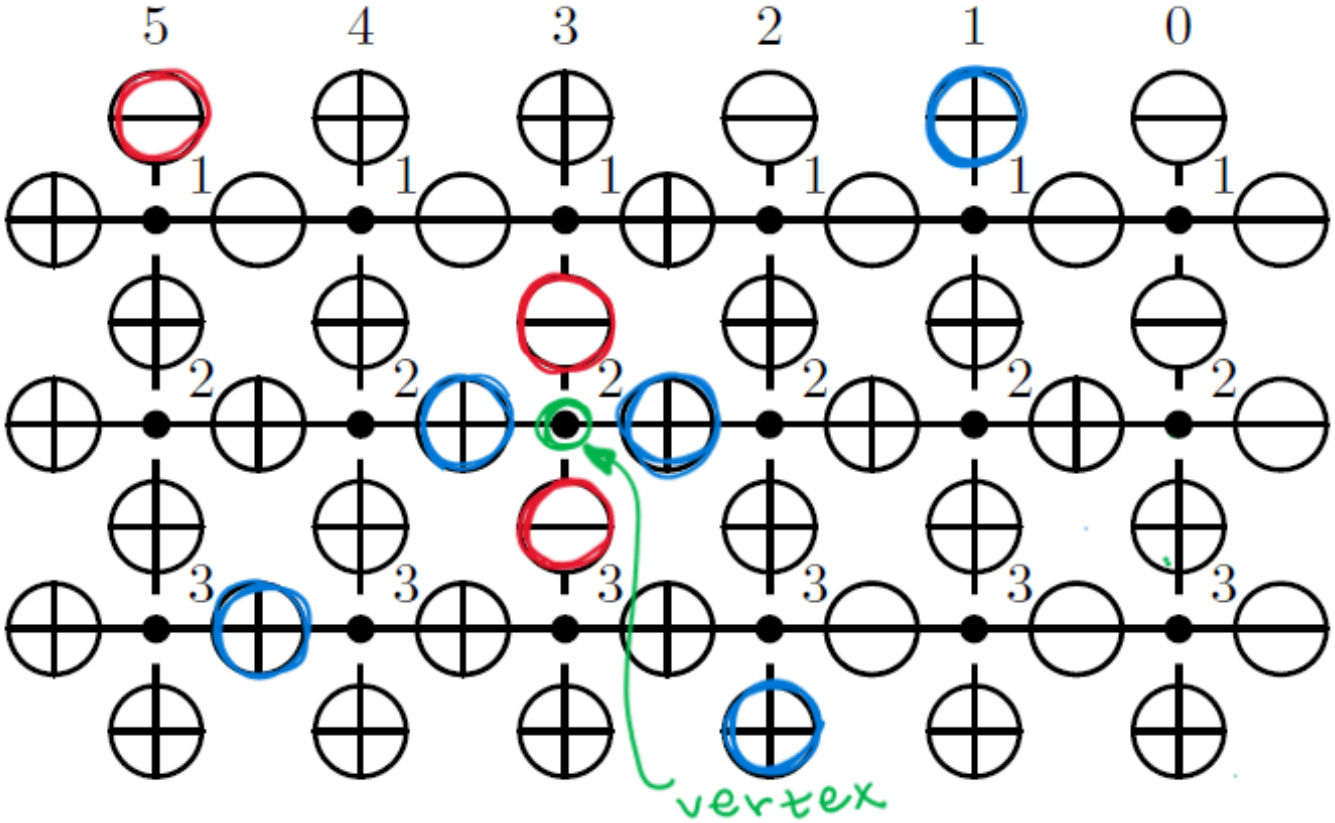
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Work under the guidance of Prof. Hadi Salmasian

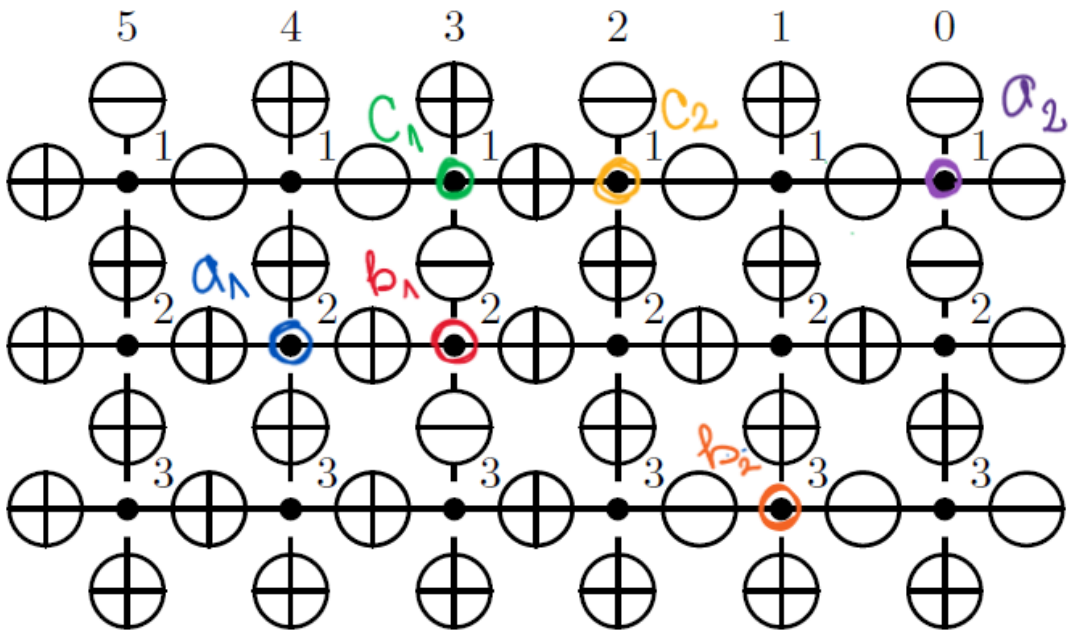
Fields Undergraduate Summer Research Program, Final Presentation
08-August-2024

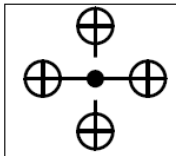
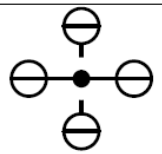
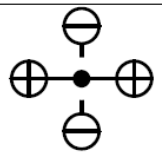
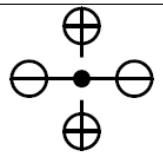
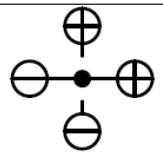
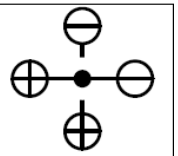
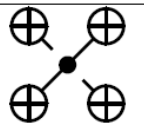
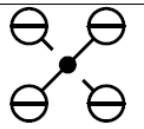
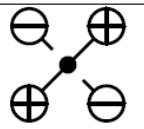
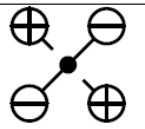
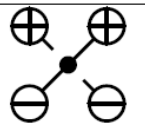
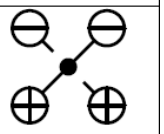
The Six-Vertex model



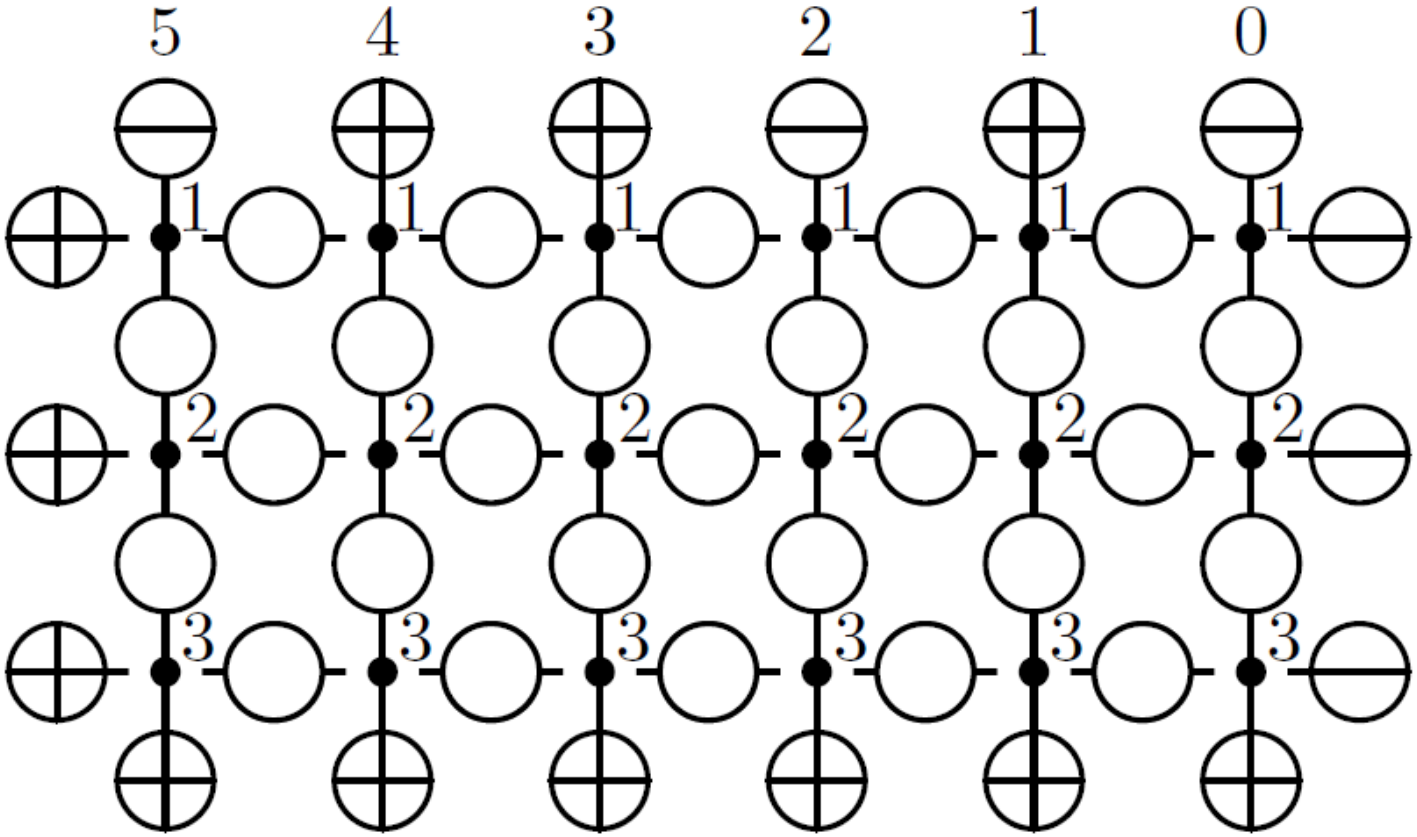
Example of a lattice

Boltzmann weights



					
					
a_1	a_2	b_1	b_2	c_1	c_2

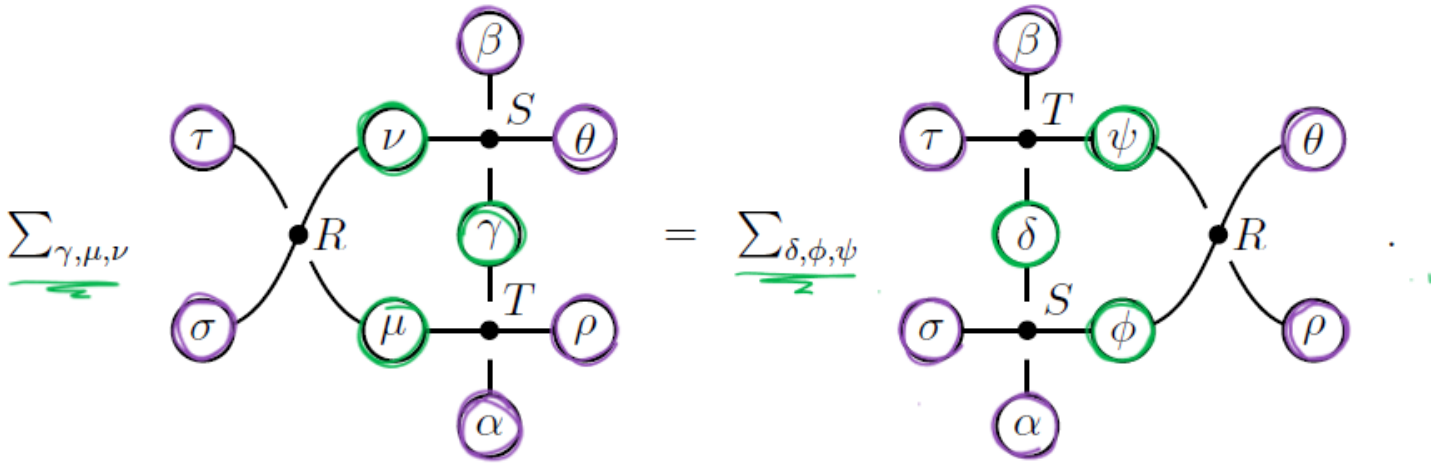
The Six-Vertex model



Boundary conditions

The star-triangle equality and Yang-Baxter equation

We can show that, for any fixed combination of spins σ, τ, α, β, ρ, θ:



The star-triangle equality

So, a system containing the configuration on the left would be described by the same partition function as one on the right, which will prove very useful later.

This is equivalent to the Yang-Baxter equation

$$[R, S, T] = 0.$$

Schur Polynomials

Definition

The *Schur Polynomials* are a family of symmetric polynomials $s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}} x^T = \sum_{T \in \text{SSYT}} x_1^{t_1} \cdots x_n^{t_n}$.

Example of an SSYT:

1	3	3
2	4	

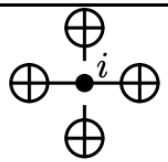
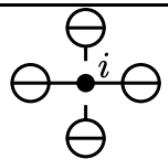
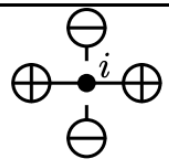
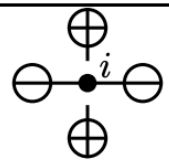
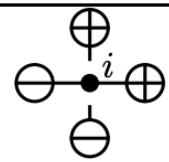
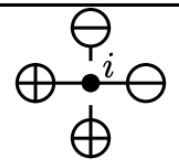
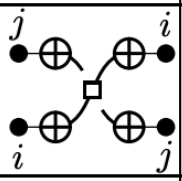
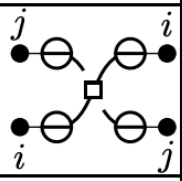
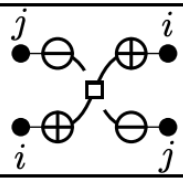
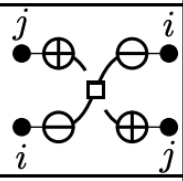
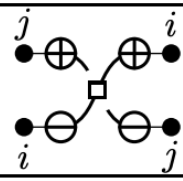
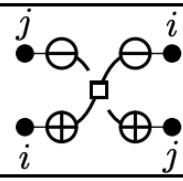
- Fact: The Schur Polynomials Form a Basis for the Graded Ring of Symmetric Polynomials $\Lambda = \bigoplus_{n \in \mathbb{N}} \Lambda_n$.
- The Schur Polynomials are *characters* for finite irreducible representations of GL_n.

Factorial Schur are a "deformation" of the Schur Polynomials:

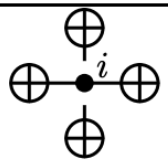
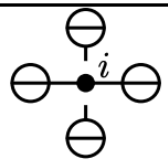
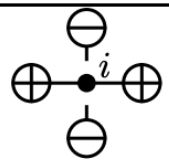
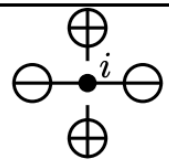
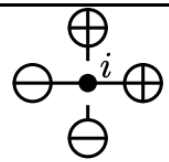
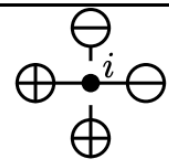
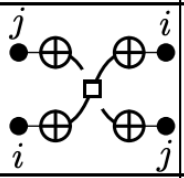
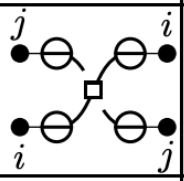
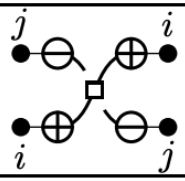
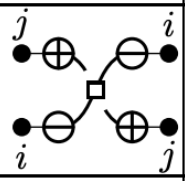
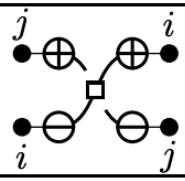
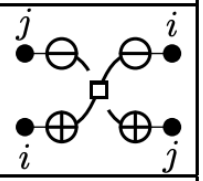
$$s_\lambda(z|\alpha) = \sum_{T \in \text{SSYT}} (z_{T(i,j)} + \alpha_{T^*(i,j)})$$

Six Vertex Model For the Schur Polynomials

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Gamma Ice						
Boltzmann weight	1	z_i	t_i	z_i	$z_i(t_i+1)$	1
Gamma-Gamma R-ice						
Boltzmann weight	$t_jz_i+z_j$	$t_iz_j+z_i$	$t_iz_j-t_jz_i$	z_i-z_j	$(t_i+1)z_i$	$(t_j+1)z_j$

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So that $\Gamma(i) = \begin{pmatrix} 1 & & \\ & t_i & z_i(t_i+1) \\ & 1 & z_i \\ & & & z_i \end{pmatrix}$ and $[\![R_{\Gamma\Gamma}(i,j), \Gamma(i), \Gamma(j)]\!] = 0.$

Six Vertex Model For the Schur Polynomials Continued

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- This set of weights satisfies $a_1 a_2 + b_1 b_2 = c_1 c_2$, so Yang-Baxter Equation holds.

Six Vertex Model For the Schur Polynomials Continued

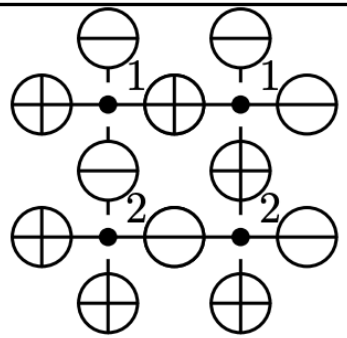
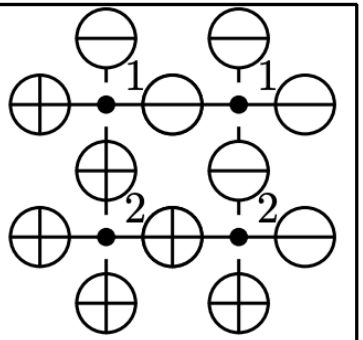
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Example!

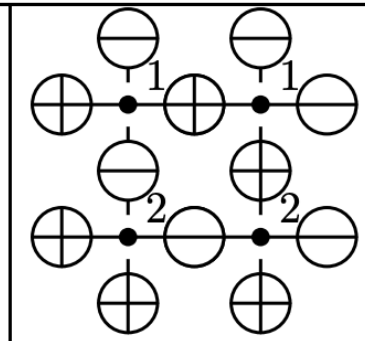
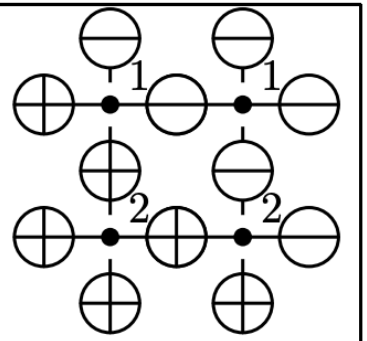
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$$\lambda = (0,0), \rho = (1,0) \rightarrow \lambda + \rho = (1,0)$$

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Partition Function:

$$t_1 z_2 + z_1$$

First Parameter Tweaking

Using earlier observations, we obtain

$$\Gamma(i) = \begin{pmatrix} s_i & 0 & 0 & 0 \\ 0 & t_i & (t_i+1)z_i & 0 \\ 0 & 1 & \frac{\alpha_j s_i t_i + \beta_i s_i t_i - s_i z_i + t_i z_i + z_i}{t_i} & 0 \\ 0 & 0 & 0 & -\alpha_j t_i - \beta_i t_i + z_i \end{pmatrix}$$

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Satisfies all observations, so Yang-Baxter holds and symmetry holds.

For symmetry only in z, need

$$\{s_i = s, t_i = t, \beta = 0\} \to \Gamma(i) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & (t+1)z_i & 0 \\ 0 & 1 & \frac{(1+t-s)z_i + st\alpha_j}{t} & 0 \\ 0 & 0 & 0 & -\alpha_j t + z_i \end{pmatrix}$$

First Parameter Tweaking

The partition function for a given partition λ is of the form

$$\prod_{i<j} \left(-sz_i + tz_i + sz_j + z_i\right) P_\lambda$$

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Theorem

P_λ is a constant multiple of a substitution of the factorial Schur polynomials, namely $P_\lambda(z|\alpha) = const \times s_\lambda(z|\alpha * (s - t + 1)/st)$

A rigidity theorem for our model

Theorem

For any six-vertex model with boundary conditions described as above, such that all weights are linear functions of z_i and all other parameters are constants and such that the model is free-fermionic, the partition function $Z(\Gamma)$ can be written as

$$const \times \prod_{i<j} a_2(\Gamma\Gamma(i,j)) \prod_i c_2(\Gamma(i))\, \mathbf{z}^{\mu} P(\mathbf{z})$$

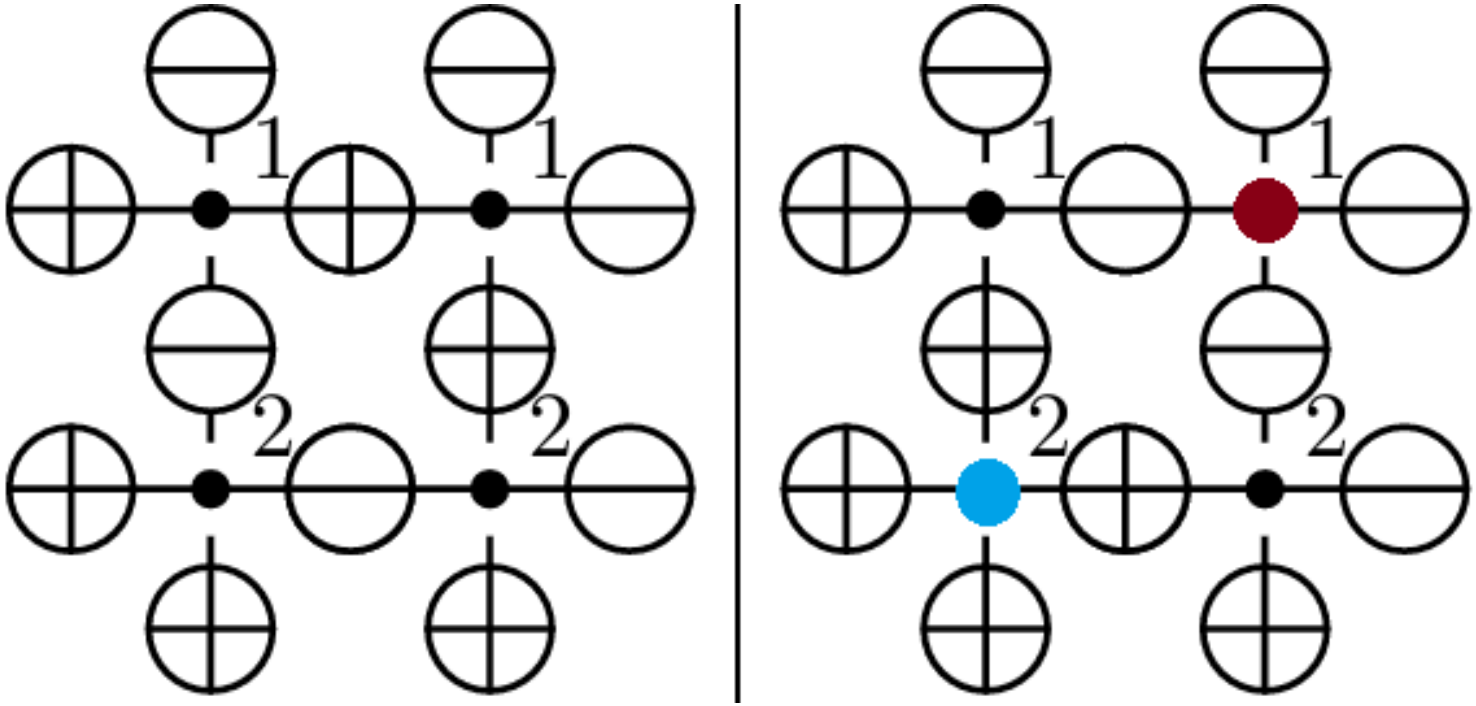
where $P(\mathbf{z})$ is either the Schur polynomial $s_\lambda(\mathbf{z}+\beta)$, a constant or $s_\lambda(\frac{1}{\mathbf{z}+\gamma}+\beta)$ for some constants β, γ and some tuple of integers μ.

This tells us that no truly interesting polynomial different from Schur can come out of linear choices.

And now some combinatorics

Lemma

In every state of the six-vertex model, the number of a_1-vertices and a_2-vertices differs by a constant. The same also holds for b and c.



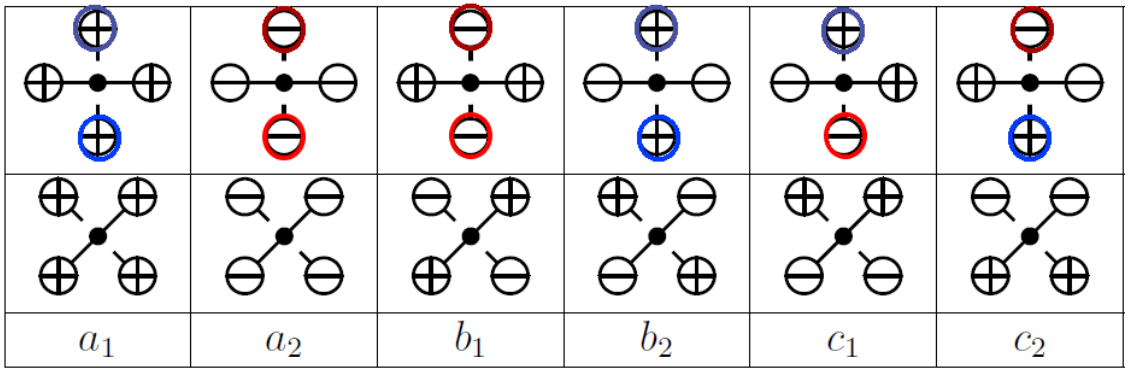
$|a_1| - |a_2|$ =const, in this case 0

And now some combinatorics

Lemma

In every state row of every state, $|a_1| + |b_2| + |c_1|$ is the same. This also holds for $|a_2| + |b_1| + |c_2|$.

$$\begin{pmatrix} \underline{a_1(R)} & & \\ & \underline{b_1(R)} \quad \underline{c_1(R)} & \\ & \underline{c_2(R)} \quad \underline{b_2(R)} & \\ & & \underline{a_2(R)} \end{pmatrix}$$

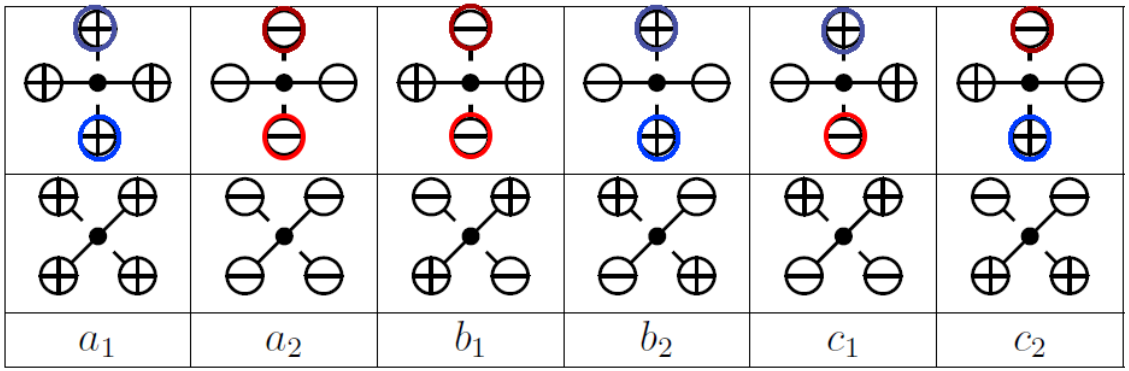


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$$\begin{pmatrix} \underline{a_1(R)} & & \\ & \underline{b_1(R)} \ \underline{c_2(R)} & \underline{c_1(R)} \ \underline{b_2(R)} \\ & & \underline{a_2(R)} \end{pmatrix}$$



The sum of occurrences of the red set and blue set in states is the same and also the same in every row. This is because they count the number of pluses and minuses in rows

Algebraic applications

Lemma

In every state of the six-vertex model, the number of a_1-vertices and a_2-vertices differs by a constant. The same holds for b and c as well

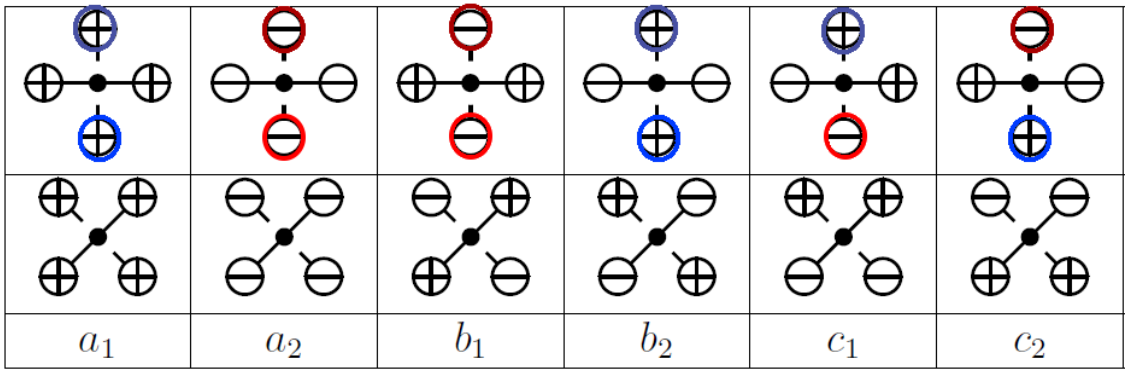
Corollary

Given a fixed partition and let $Z(M)$ denote the partition function of the six-vertex model. Then

$$Z\begin{pmatrix} k\cdot a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & a_2 \end{pmatrix} = k^{const}\cdot Z\begin{pmatrix} a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & k\cdot a_2 \end{pmatrix}$$

Algebraic applications

$$\begin{pmatrix} \underline{a_1(R)} & & & \\ & \underline{b_1(R)} & \underline{c_1(R)} & \\ & \underline{c_2(R)} & \underline{b_2(R)} & \\ & & & \underline{a_2(R)} \end{pmatrix}$$



Corollary

Let i be the row index and let $\rho = (n-1, n-2, .., 0)$. Then

$$Z\begin{pmatrix} a_1 & & & \\ & z_i b_1 & z_i c_1 & \\ & c_2 & b_2 & \\ & & & z_i a_2 \end{pmatrix} = \mathbf{z}^{\rho} \cdot Z\begin{pmatrix} a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & a_2 \end{pmatrix}$$

Prelimiaries: Jack Polynomials

Jack polynomials $J_\lambda(z, \alpha)$ are symmetric polynomials indexed by a partition λ having coefficients from $\mathbb{Q}(\alpha)$.

- They form an orthogonal basis of the vector space of all symmetric functions over the field $\mathbb{Q}(\alpha)$, $\Lambda \otimes \mathbb{Q}(\alpha)$.
- They have various degenerations depending on the parameter α.
  - $(\alpha = \infty)$: Monomial symmetric functions
  - $(\alpha = 0)$: Elementary symmetric functions
  - $(\alpha = 1)$: Schur symmetric functions
  - $(\alpha = 1/2, 2)$: Two classes of Zonal symmetric functions

Generalization: Colored Lattice Models

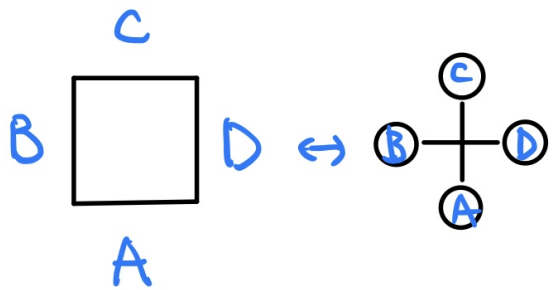
We can generalize the classical lattice model to a colored lattice model by labeling the edges with vectors.

Model Name	**Edge Labelings**	**Examples**
Classical Lattice Model	Signs $((-),(+))$	Schur Polyn.
Colored Lattice Model	Vectors $(v \in \{0,1\}^n)$	Grothendieck Polyn.

Each unit vectors represents a 'color' and the model is called monochromatic when $n = 1$, degenerating to the case where we label with signs $(-),(+)$.

Generalization 1: Colored Lattice Models (Fermionic)

Let A, B, C, D be the labels of edges incident to a vertex branching in the downward, leftward, upward, rightward direction, respectively.



In the most general case, the only restrictions are the following:

$$A, B, C, D \in \{0, 1\}^n,\ A + B = C + D$$

For fermionic models, we pose the following condition:

$$\max(A + B) \leq 1$$

where max is taken over all coordinates of the vector.

Cylinderic Colored Fermionic Model for Jack Polynomials

The Jack polynomials for a partition λ can be expressed as the partition function of a cylinderic colored ($n = l(\lambda)$) fermionic model with the following restrictions:

- **Boundary Condition:**
  - **Top/Bottom Boundary:** The 'almost-periodic boundary condition'
  - **Right/Left Boundary:** Set all edges with 0's.

Vertex Weights:

The weights can be expressed as follows using the notion we used:

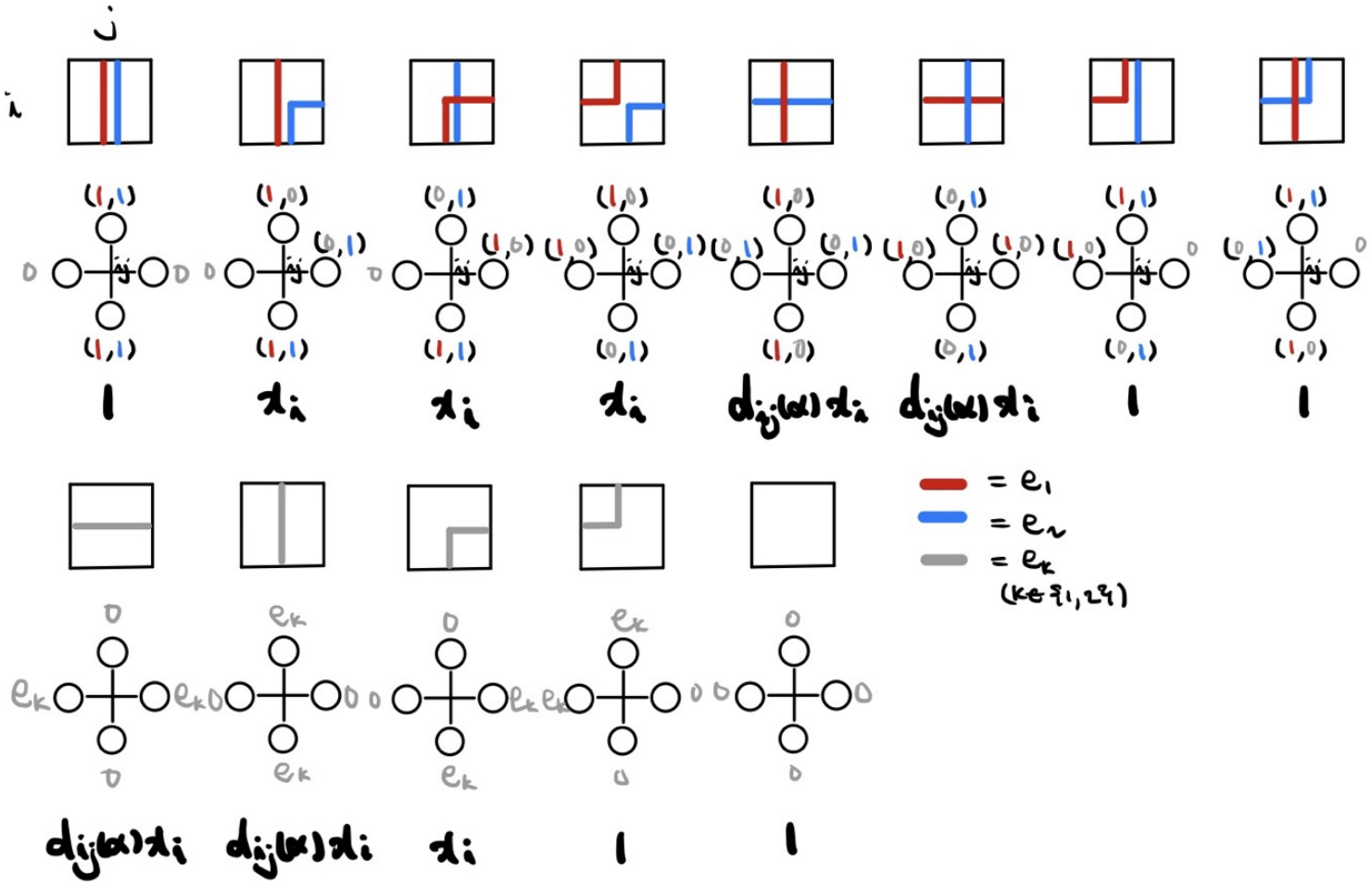
$$\begin{aligned} w(A,B,C,D;i,j) := \mathbb{1}_{A,B,C,D\in\{0,1\}} \cdot \mathbb{1}_{A+B=C+D} \cdot \mathbb{1}_{\max(A+B)\leq 1} \\ \cdot \mathbb{1}_{\varphi(D,C+D)\cdot\varphi(A,B+A)=0} \\ \cdot [d(n-idx(B)+1,j;\alpha)]^{B\cdot D} \cdot x_i^{|D|} \end{aligned}$$

Where,

- i, j denotes the row and column index of vertices, respectively.
- $\varphi(X,Y) := \sum_{1\leq i<j\leq n} X_i Y_j$
- $idx(e_k) := k$
- $l_\lambda(i,j) = |\{k \in [i-1] \mid j \leq \lambda_k + 1 \leq \lambda_i\}|$
  $+|\{k \in [n]/[i] \mid j \leq \lambda_k \leq \lambda_i\}|$
- $a_\lambda(i,j) := \lambda_i - j$
- $d_\lambda(i,j,\alpha) := \alpha(a_\lambda(i,j)+1) + (l_\lambda(i,j)+1)$

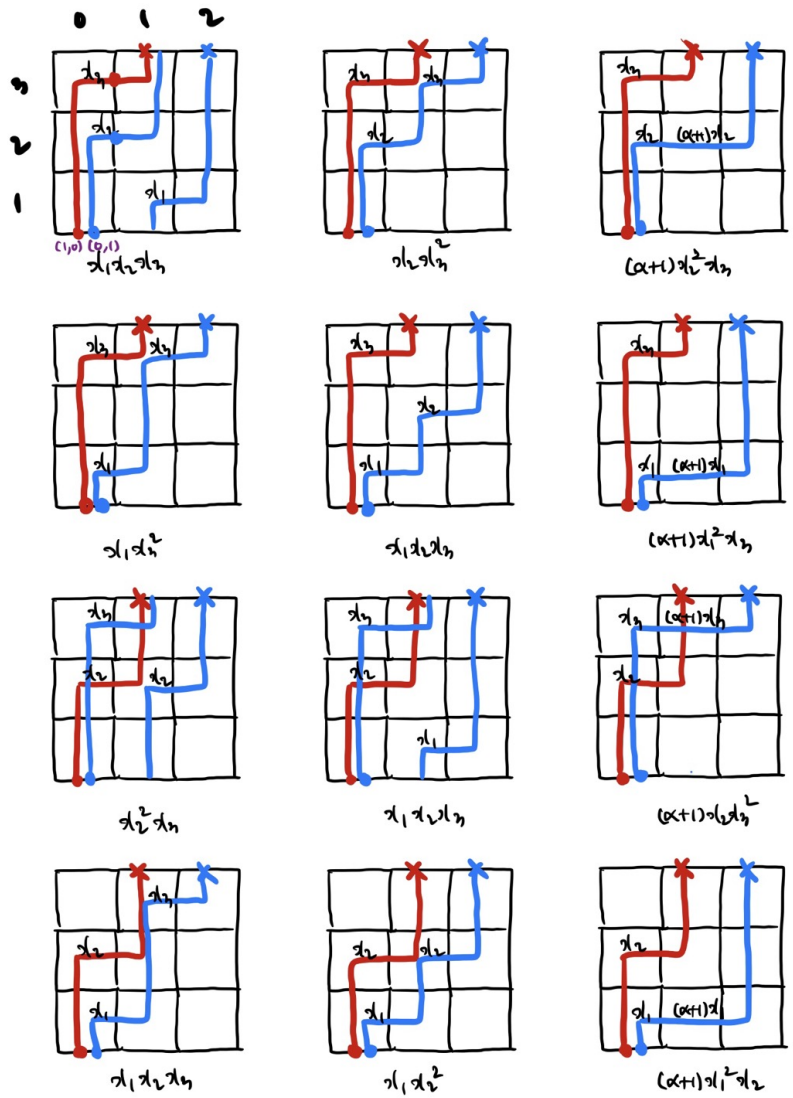
Cylinderic Colored Fermionic Model for Jack Polynomials

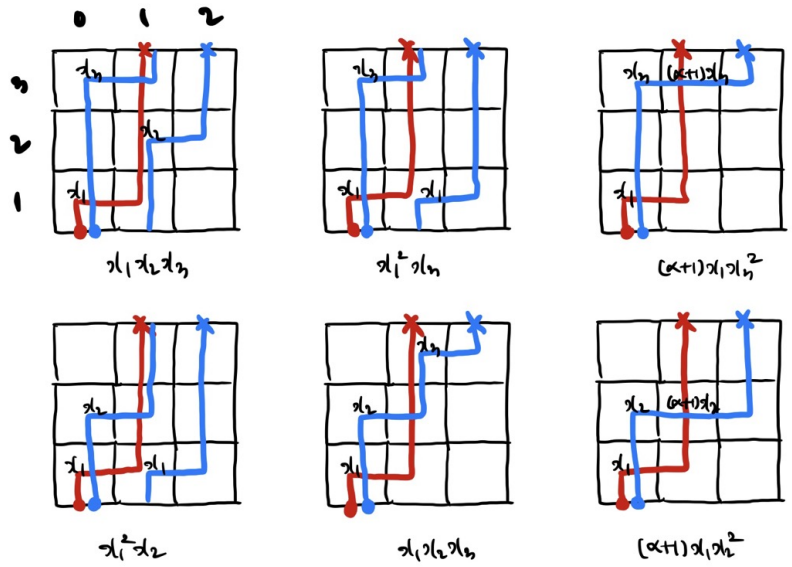
The previous weight description generates following vertices ($l(\lambda) = 2$):



Cylinderic Colored Fermionic Model for Jack Polynomials

Computational example showing that the partition function of the lattice model is indeed the Jack polynomial for the case $\lambda = (2, 1)$.





we get $(\alpha+2)\, m_{(2,1)} + 6 m_{(1,1,1)} = J^{\alpha}_{(2,1)}(x_1, x_2, x_3)$ nice !!

Concluding Remarks, Future Directions

- Adding extra parameters depending on z_i;
- Different boundary conditions: partitions on bottom and right side;
- Proving combinatorial identities of Jack polynomials via lattice models
- Factorial Schur-Q in Schur basis and Sterling numbers.

Thank You!

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[1] B. Brubaker, D. Bump, and S. Friedberg, "Schur polynomials and the yang-baxter equation,", Oct. 23, 2018.

[2] D. Bump, P. J. McNamara, and M. Nakasuji, "Factorial schur functions and the yang-baxter equation,", Jan. 13, 2018.

[3] F. Knop and S. Sahi, "A recursion and a combinatorial formula for Jack polynomials," *Invent. Math.*, vol. 128, no. 1, pp. 9–22, 1997, ISSN: 0020-9910,1432-1297. DOI: 10.1007/s002220050134.

[4] R. P. Stanley, "Some combinatorial properties of Jack symmetric functions," *Adv. Math.*, vol. 77, no. 1, pp. 76–115, 1989, ISSN: 0001-8708,1090-2082. DOI: 10.1016/0001-8708(89)90015-7.