WEIL REPRESENTATION AND CENTRAL EXTENSION OF LOOP SYMPLECTIC GROUPS

By
Gabriel Bergeron-Legros
September 2014

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Science in Mathematics[1]

© Copyright 2014
by Gabriel Bergeron-Legros, Ottawa, Canada

[1]The M.Sc. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

In this thesis, we present the Weil representation over loop symplectic groups. Then we study the question of whether or not the Schrödinger representation and the Weil representation are continuous. Finally, we define a cocycle of $SL_2(\mathbb{R}((t)))$, adapt Kubota's theorem to this case and verify that this cocycle splits over $SL_2(\mathbb{R}[[t]])$.

Acknowledgements

Above all else, I want to thank my supervisor Hadi Salmasian for the continuous support and help he has provided, even years before the writing of this thesis. In addition to the knowledge I have received through our meetings and four different courses, his help towards scholarship and PhD applications, presentations, conference preparation and writing style has been invaluable.

I want to thank Erhard Neher for his never-ending source of wisdom that he is always willing to share, his organizing of the Kac-Moody seminar and the reading of this thesis. Without Erhard, I would have missed much of the beauty of Lie Theory and Algebra in general.

I would like to thank Paul Mezo for taking the time to read this thesis and the knowledge he has shared in Galois Theory, two things I could not graduate without.

This thesis would not have been the same without the funding and support of the National Sciences and Engineering Research Council of Canada. I have received funding for two different Undergraduate Student Research Awards and a Canada Graduate Scholarship, each of which helped immensely.

I would like to thank the Centre de Recherche Mathématiques and the Fields Institute for the workshops and accommodation they have provided. I have learned a lot from the time I have spent at these two places.

Lastly, I want to thank the University of Ottawa, the faculty of the department of mathematics and statistics and more particularly professors Benoit Dionne, David Handelman, Daniel Daigle, Barry Jessup and Monica Nevins, for their support and teaching throughout my stay.

Contents

Abstract **ii**

Acknowledgements **iii**

1 Introduction **1**

2 Preliminaries **5**

2.1 Local fields 5

2.2 The Schwartz space 7

2.3 Central extensions of groups 10

3 The Weil Representation **18**

3.1 The symplectic group 18

3.2 The Weil representation 23

4 Continuity of the representations **43**

4.1 Finite dimensional case 43

4.2 Infinite-dimensional case 48

5 A specific central extension **56**

5.1 Kubota's theorem 56

5.2 Splitting of the cocycle over a subgroup 69

Chapter 1

Introduction

In 1959, Segal[9] defines and proves the existence of a certain projective representation of infinite dimensional symplectic groups. Inspired by Shale's[11] study of this representation, Weil [8] introduces a generalization of this representation to symplectic groups over an arbitrary local field. For this reason, this distinguished representation of the symplectic group, or rather of the metaplectic group is called the Segal-Shale-Weil representation, or simply Weil representation.

The usual starting point is to consider two different representations $\tilde{U}$ and $\tilde{U}^s$ of the Heisenberg group $H = V \oplus V^* \oplus \mathbb{F}$ on $L^2(V)$ where V is a finite-dimensional vector space over $\mathbb{F}$, a local field of characteristic not 2, and s is a symplectic matrix. Then by the Stone-Von Neumann theorem, these two representations will turn out to be unitarily equivalent [7, Section 3.2]. In other words, there exists a unique up to scalar unitary operator ξ_s such that

$$\xi_s^{-1}\tilde{U}\xi_s = \tilde{U}^s.$$

Furthermore, by the uniqueness up to scalar of this intertwining operator, ξ_s is in fact a projective representation of the symplectic group over the space of Schwartz functions. An explicit formula for ξ_s can be found in [7, Lemma 3.2 (3)].

In 2008, Zhu [14] writes that the same construction can be done over loop symplectic groups, namely by replacing the field $\mathbb{F}$ with a field of Laurent series with coefficients in $\mathbb{F}$. However, instead of using the Stone Von-Neumann theorem, one carries over the explicit formula for the Weil representation to a specific subgroup and checks that the expected properties are indeed satisfied[14, Proposition 2.8].

In a different, but related topic, Kubota [12] describes a cocycle of $SL_2(\mathbb{F})$, for $\mathbb{F}$ again a local field, which depends on so-called Hilbert symbols. He also shows that the cocycle induces a non-trivial central extension.

The main contribution of this thesis is three-fold. Firstly, we rewrite Section 2 of [14] in detail in Chapter 3, and by doing so we fill some loose ends in the paper. Secondly, we give in Chapter 4 explicit, elementary proofs of the continuity of some of the representations defined in Chapter 3, including the Weil representation for loop symplectic groups. Finally, we define a cocycle for $SL_2(\mathbb{F}((t)))$ similar to Kubota's cocycle and prove that it splits over $SL_2(\mathbb{F}[[t]])$. The remainder of this introduction serves to give an outline of each of the sections.

In Section 2.1, we define what a local field is without delving into the theory, and we also describe the field $\mathbb{F}((t))$ as a topological field in Proposition 2.1.5, or in other words that the operations of addition, multiplication and inversion are continuous. Furthermore, we give an explicit basis for this topology.

In Section 2.2, we define and discuss some properties of the Schwartz space $S(V)$. We do so first over a finite dimensional vector space for both cases where $\mathbb{F}$ is an archimedean local field or non archimedean. For an infinite dimensional space, we will define a Schwartz function to be a function whose restriction to every finite dimensional space is also Schwartz. If $\mathbb{F} = \mathbb{R}$, then by inducing a topology on the Schwartz space with the use of seminorms, it will turn out that $S(V)$ inherits the Fréchet space structure of the finite case.

In Section 2.3, we introduce the study of central extensions through the use of cocycles, and in fact prove that there is a bijection between isomorphism classes of central extensions and equivalence classes of cocycles in Proposition 2.3.10.

These preliminary sections allow us to define the Weil representation over loop symplectic groups in Chapter 3, Proposition 3.2.4. We will first consider a symplectic form $\langle ., . \rangle$ on an infinite dimensional vector space $X = V \oplus V^*$, where V^* is the vector space dual of V, and its isometry group $Sp(X)$. However, $Sp(X)$ will be too large to allow a definition of the Weil representation which is compatible with its formula, hence we will concern ourselves with a subgroup $Sp(X, V^*)$, which is the biggest subgroup over which the formula for the Weil representation can be carried over. The good thing is that $Sp_{2n}(\mathbb{F}((t)))$ will be a subgroup of $Sp(X, V^*)$ (Lemma 3.2.19) for a

suitable choice of V, and this is a group which is well known and understood, hence it is always possible to restrict the Weil representation to it.

The Stone Von-Neumann theorem in the finite dimensional case gives a certain relationship between two representations of the Heisenberg group and the Weil representation. We should expect the same relationship to appear in the infinite dimensional case, and in fact, we will give a direct proof of this statement in Proposition 3.2.4.

To prove that the Weil representation we define is indeed a projective representation, we will have to reduce to the standard Weil representation. The trick will be to write every element of $Sp(X, V^*)$ as a product $g = pg'$ where p and g' lie in two subgroups P and $Sp_{\text{fin}}(X)$ which are much more manageable. Indeed, the Weil representation for P will have a very simple formula where no integration is happening, and the one for g' will turn out to be very similar to the standard Weil representation, which we already know to be a projective representation. Lemmas 3.2.14, 3.2.15 and Corollary 3.2.9 will work out the technical details so that we obtain our main result in Theorem 3.2.16. Finally we will end the chapter with a short study of $V = (t^{-1}\mathbb{R}[t^{-1}])^{2n}$. We will prove that its vector space dual can be thought to be $\mathbb{R}[[t]]^{2n}$ in Lemma 3.2.17, and this will induce a symplectic form on $\mathbb{R}((t))^{2n}$.

In Chapter 4, we will study whether or not the Heisenberg group representation and the Weil representation are continuous in the case $\mathbb{F} = \mathbb{R}$. In fact, we will start with an explicit proof in the finite dimensional case for the Heisenberg group, which will turn out to be much more technical than one would initially expect. The main strategy will be to consider the action of three subgroups of the Heisenberg group and how they interact with each other, which will be done in the numerous technical Lemmas 4.1.1, 4.1.2, 4.1.34.1.5, 4.1.6, 4.1.7 and 4.1.8 to obtain the main result in Proposition 4.1.9. The proofs for the infinite dimensional case $V = t^{-1}\mathbb{R}[t^{-1}]$ will be in appearance easier, because the topologies involved are much more easier to work with than the Euclidean topology. Thus, we will not need any technical Lemmas and we will directly proceed to the proofs in Proposition 4.2.2 and Theorem 4.2.4.

In Chapter 5, we define a cocycle very similar to Kubota's cocycle, but for $SL_2(\mathbb{R}((t)))$. Then we carry over Kubota's results[5] to $SL_2(\mathbb{R}((t)))$ in Chapter 5 in full detail. The definition of the cocycle is very difficult to work with, and a proof

by brute force would involve over a hundred of cases. Therefore, we spend time obtaining Lemmas 5.1.3, 5.1.4, 5.1.5, 5.1.6, 5.1.7, 5.1.8 and 5.1.10 to significantly reduce the number of cases needed for Theorem 5.1.11. Finally, we also prove that the cocycle splits over the subgroup $SL_2(\mathbb{R}[[t]])$ in Theorem 5.2.3 with a case by case argument, or more precisely that it is equal to a coboundary.

Chapter 2

Preliminaries

2.1 Local fields

Suppose $\mathbb{F}$ is a field. We say that $\mathbb{F}$ is a local field if $(\mathbb{F}, +)$ is a locally compact topological group with a non-discrete topology. Denote by $\mathbb{F}((t))$ the vector space of all Laurent series over t, i.e. $\mathbb{F}((t)) = \mathbb{F}[[t]] \oplus t^{-1}\mathbb{F}[t^{-1}]$. Then $\mathbb{F}((t))$ with the obvious operations of addition and multiplication, is a field. [1, p. 238, Exercise 4]

The following theorem is a combination of Theorem 5 and Theorem 8 in [12].

Theorem 2.1.1. *Suppose $\mathbb{F}$ is a local field. Then $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or $F((t))$ where F is a finite field.*

We will refer to $\mathbb{R}$ and $\mathbb{C}$ as archimedean, and to the finite extensions of $\mathbb{Q}_p$ or $F((t))$ for F a finite field as non-archimedean. We will often be interested in $\mathbb{F}((t))$ where $\mathbb{F}$ is a local field. Suppose $x = \sum_{i=-n}^{\infty} a_i t^i$. Then we set $v(x) = \min\{i : a_i \neq 0\}$ for $x \neq 0$, and $v(0) = +\infty$. Let

$$U_n = \{x \in \mathbb{F}((t)) : v(x) \geq n\}.$$

We equip $\mathbb{F}((t))$ with the topology generated by all sets of the form $x + U_n$ where $x \in \mathbb{F}((t))$. Remark that if $m \geq n$, then $U_m \subseteq U_n$. Furthermore, $U_n + U_n \subseteq U_n$. The following proposition will be very useful :

Proposition 2.1.2. *The collection of open sets of the form $x + U_n$ is a basis for the topology on $\mathbb{F}((t))$.*

Proof. Suppose U is an open set. Then it can be written as an infinite union of finite intersections of sets of the form $x+U_n$. We need to show that for all $y \in U$, there exist a set of the form $y' + U_n$ containing y and contained in U. Without loss of generality, we can assume $U = (x_1+U_{n_1})\cap...\cap(x_m+U_{n_m})$ is a non-empty intersection of elements of the form $x + U_n$. Let $N = \max\{n_1, ..., n_m\}$ and $y \in U$. We claim that $y + U_N$ is contained in $x_i + U_{n_i}$ for all i. Indeed, we know that $x_i - y \in U_{n_i}$. If $y + v_N \in y + U_N$, then $x_i-(y+v_N) \in U_{n_i}-U_N \subseteq U_{n_i}$. Therefore, $y+v_N = x_i-(x_i-(y+v_N)) \in x_i+U_{n_i}$, hence $y + U_N \subseteq x_i + U_{n_i}$ for all i and then $y + U_N \subseteq U$. □

The main application of this proposition is the following :

Lemma 2.1.3. *Suppose $U \subseteq \mathbb{F}((t))$ is open and $x \in U$. Then there exists $m \in \mathbb{Z}$ such that $x + U_m \subseteq U$.*

Proof. Choose m and $x' \in \mathbb{F}((t))$ such that $x \in x' + U_m \subseteq U$. Then $x - x' \in U_m$. But then $x + U_m = x' + U_m \subseteq U$. □

Remark 2.1.4. Often, we will choose m to be greater than or equal to some other arbitrary number. This is not a problem because $m \geq n$ implies that $U_m \subseteq U_n$.

Proposition 2.1.5. *The operations of addition, multiplication and inversion of $\mathbb{F}((t))$ are continuous.*

Proof. Let us write $x = \sum_{i\geq -M} a_i t^i$ and $y = \sum_{i\geq -N} b_i t^i$ for $M, N \geq 0$ such that $a_{-M} \neq 0$, $b_{-N} \neq 0$. Then we have $x+y = \sum_{i\geq -\max\{M,N\}} (a_i+b_i)t^i$ and $xy = \sum_{i\geq -(M+N)} \sum_{k=-M}^{i+N} a_k b_{i-k} t^i$. We see from these formulas that $v(x + y) \geq \min\{v(x), v(y)\}$ and $v(xy) = v(x) + v(y)$ for x and $y \neq 0$.

Now suppose $U \subseteq \mathbb{F}((t))$ is open and (x, y) is in the pre-image of U under the map $(x, y) \mapsto x + y$. Then there exists $m \in \mathbb{Z}$ such that $x + y + U_m \subseteq U$. But now, $(x + U_m) \times (y + U_m)$ is an open set of $\mathbb{F}((t)) \times \mathbb{F}((t))$ containing (x, y). Furthermore, if $x' \in x + U_m$ and $y' \in y + U_m$, we have $x' + y' \in x + y + U_m$. We conclude that $(x + U_m) \times (y + U_m)$ is contained in the pre-image of U and therefore the pre-image of U is open.

If $U \subseteq \mathbb{F}((t))$ is open and (x, y) is in the pre-image of U under the map $(x, y) \mapsto xy$, then once again there exists $m \geq 0$ such that $xy + U_m \subseteq U$. Let

$$R = \max\{m - v(x), m - v(y), m\}.$$

Clearly, $(x + U_R) \times (y + U_R)$ is an open set of $\mathbb{F}((t))$ containing (x, y). But now, if $x' \in x + U_R$ and $y' \in y + U_R$, we have

$$x'y' \in xy + xU_R + yU_R + U_RU_R \subseteq xy + U_m + U_m + U_{2m} \subseteq xy + U_m.$$

We conclude that $(x+U_R)\times(y+U_R)$ is contained in the pre-image of U and therefore the pre-image of U is open.

Finally, suppose $U \subseteq \mathbb{F}((t))$ and that $x \neq 0$ is in the pre-image of U under the map $x \mapsto x^{-1}$. Because $0 = v(xx^{-1}) = v(x) + v(x^{-1})$, we know that $v(x^{-1}) = M$. Write $x^{-1} = \sum_{i \geq M} c_i t^i$. Then

$$xx^{-1} = \sum_{i \geq 0} \sum_{k=-M}^{i-M} a_k c_{i-k} t^i = 1.$$

Therefore, we obtain that $c_M = a_{-M}^{-1}$ and then recursively that

$$c_{i+M} = -a_{-M}^{-1} \sum_{k=-M+1}^{i-M} a_k c_{i-k}.$$

In particular, c_{i+M} depends on a_k for $-M \leq k \leq -M + i$. Now, choose $m \geq 1$ such that $x^{-1} + U_m \subseteq U$. Then c_i for $M \leq i < m$ is determined uniquely by a_{-M}, ... a_{-M+m-1}. We claim that $x+U_{-M+m} \subseteq (x^{-1}+U_m)^{-1}$. Indeed, if $y \in x+U_{-M+m}$, then y^{-1} (which exists because $a_{-M} \neq 0$ and $m \geq 1$, hence $y \neq 0$) will share the same first m coefficients with x^{-1}, because they are uniquely determined by a_{-M},...,a_{-M+m-1}, which are the same for x and y. Therefore, $y^{-1} \in x^{-1} + U_m \subseteq U$, and thus the pre-image of U is open. □

2.2 The Schwartz space

Suppose $\mathbb{F}$ is a non-archimedean local field. If V is a finite-dimensional vector space over $\mathbb{F}$, we say that f is a Schwartz function if f is locally constant with compact support. If V is not finite-dimensional, we say that f is a Schwartz function if its restriction to every finite-dimensional subspace of V is Schwartz. We will denote the space of Schwartz functions by $S(V)$. For the rest of this section, we deal with $\mathbb{F} = \mathbb{R}$.

Remark 2.2.1. For $\mathbb{F} = \mathbb{C}$, we simply think of a vector space over $\mathbb{F}$ as a vector space over $\mathbb{R}$ with twice the dimension and the Schwartz space is then defined in the same way as in the real case.

Suppose $V \cong \mathbb{R}^n$, $f \in C^\infty(V)$ and let $\alpha = (\alpha_1, ..., \alpha_m)$ and $\beta = (\beta_1, ...\beta_m) \in \mathbb{N}^m$. Write

$$p_{\alpha,\beta}(f) = \sup_{x \in V} \left| x^\alpha \partial^\beta f(x) \right|,$$

where $x = (x_1, ..., x_n)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} ... \partial_n^{\beta_n}$. Then let

$$S(V) = \{f \in C^\infty(V) : p_{\alpha,\beta}(f) < \infty \text{ for all multi-indices } \alpha, \beta\}.$$

Proposition 2.2.2. *Each $p_{\alpha,\beta}$ is a semi-norm.*

Proof. This proof is an easy exercise. □

Consider the sets $U_{\alpha,\beta,\epsilon} = \{f \in S(V) : p_{\alpha,\beta}(f) < \epsilon\}$. We equip $S(V)$ with the topology generated by the collection of all sets of the form $f + U_{\alpha,\beta,\epsilon}$.

Definition 2.2.3. Suppose X is a Hausdorff topological vector space. Then we say that X is a Fréchet space if

1. Its topology is induced by a countable family of seminorms $\|.\|_k$ in the following way : we say that $U \subseteq X$ is open if and only if for all $v \in U$, there exists finitely many seminorms $\|.\|_{k_1}, ..., \|.\|_{k_n}$ and $\epsilon > 0$ such that

$$\bigcap_{i=1}^{n} \{w \in X : \|w - v\|_{k_i} < \epsilon\} \subseteq U$$

2. It is complete with respect to the seminorms, i.e., every sequence which is Cauchy with respect to every seminorm $\|.\|_k$ has a limit in X.

Proposition 2.2.4. *$S(V)$ with the topology defined above, is a Fréchet space.*

Proof. This is Proposition 8.2 in [4]. □

Next, suppose V is an arbitrary vector space. Then we define $S(V)$ to be the space of all functions $f : V \to \mathbb{C}$ such that for any finite-dimensional subspace $W \subseteq V$, we have that $f|_W \in S(W)$.

We will be particularly interested in the case where $V = (t^{-1}\mathbb{R}[t^{-1}])^{2n}$. We can write an element v of V as

$$v = \sum_{i=1}^{2n} t^{-1} p_i(t^{-1}) e_i$$

for polynomials $p_i \in \mathbb{R}[t]$, and the e_i's is the standard basis of V as a $t^{-1}\mathbb{R}[t^{-1}]$-module. Next, suppose $m \in \mathbb{N}$. Then let $\alpha = (\alpha_{i,j})_{1 \leq i \leq m, 1 \leq j \leq 2n}$ and $\beta = (\beta_{i,j})_{1 \leq i \leq m, 1 \leq j \leq 2n}$ and

$$V_m = \text{span}\{t^{-j} e_i : 1 \leq j \leq m, 1 \leq i \leq 2n\}.$$

If $f \in S(V)$, we know that $f|_{V_m}$ is a Schwartz function in the ordinary sense, hence we can define the maps

$$p_{\alpha,\beta,m}(f) = \sup_{x \in V_m} \left\| x_{0,0}^{\alpha_{0,0}} ... x_{m,2n}^{\alpha_{m,2n}} \frac{\partial^{\beta_{0,0}}}{\partial x_{0,0}^{\beta_{0,0}}} ... \frac{\partial^{\beta_{m,2n}}}{\partial x_{m,2n}^{\beta_{m,2n}}} f(x) \right\|$$

for $x = \sum_{i \leq m, j \leq 2n} x_{i,j} t_j^{-i} \in V_m$.

Proposition 2.2.5. *Each $p_{\alpha,\beta,m}$ is a seminorm.*

Proof. Follows immediately from Proposition 2.2.2. □

We equip $S(V)$ with a topology in the same way that we did in the finite-dimensional case with the seminorms $p_{\alpha,\beta}$, but $U_{\alpha,\beta,\epsilon}$ is replaced by $U_{\alpha,\beta,m,\epsilon}$. Namely,

$$U_{\alpha,\beta,m,\epsilon} = \{f \in S(V) : p_{\alpha,\beta,m}(f) < \epsilon\}$$

We also obtain

Proposition 2.2.6. *The space $S(V)$ with the topology induced by the seminorms $p_{\alpha,\beta,m}$ is a Fréchet space.*

Proof. Suppose $f_1, f_2 \in S(V)$ and that $f_1 \neq f_2$. Then there exists $v \in V$ such that $f_1(v) \neq f_2(v)$. In particular, v is contained in V_m for some m. Write $c = f_1(v) - f_2(v)$. Then the open sets $U_{0,0,m,c/2} + f_1$ and $U_{0,0,m,c/2} + f_2$ are disjoint open sets separating f_1 from f_2. We conclude that $S(V)$ is Hausdorff.

Now suppose f_n is a sequence of functions in $S(V)$ which is Cauchy with respect to every seminorm $p_{\alpha,\beta,m}$. Suppose $v \in V$. Then $v \in V_m$ for some m. In particular, $f_n|_{V_m}$ is a Cauchy sequence with respect to $p_{0,0,m}$ and therefore $f_n(v)$ is also a Cauchy

sequence, hence has a point-wise limit. We let $f(v)$ be the point-wise limit of $f_n(v)$. Now suppose $W \subseteq V$ is finite-dimensional. Then $f_n|_W$ are Schwartz in the ordinary sense. Because $W \subseteq V_{m'}$ for some m', we see that $f_n|_W$ form a Cauchy sequence of Schwartz functions on W and thus by Proposition 2.2.4 converges to a Schwartz function g. However, it is not hard to see that g agrees with $f|_W$. Therefore, f is a Schwartz function. □

Notation 2.2.7. *We will sometimes replace $p_{\alpha,\beta}(f)$ or $p_{\alpha,\beta,m}(f)$ by the notation $\|f\|_{\alpha,\beta}$ or $\|f\|_{\alpha,\beta,m}$. Furthermore, we will also sometime replace $p_{0,0}(f)$ by $\|f\|_\infty$.*

2.3 Central extensions of groups

In Chapter 5, we will be interested in studying central extensions of loop symplectic groups. We recall some of the standard facts about central extensions in this section.

Definition 2.3.1. Let K, G, H be groups. An extension is a short exact sequence

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1\,.$$

The extension is said to be central if $i(H)$ is contained in the center of G.

Example 2.3.2. *Suppose $\widetilde{G} = K \ltimes H$. Define $i' : H \to \widetilde{G}$ by $i'(h) = (1, h)$ and $j' : \widetilde{G} \to K$ by $j'(k, h) = k$. Then,*

$$1 \longrightarrow H \xrightarrow{i'} \widetilde{G} \xrightarrow{j'} K \longrightarrow 1 \tag{1}$$

is an extension. If $\widetilde{G} = K \times H$ and H is abelian, then (1) is a central extension.

Proof. The proof is an easy, straightforward computation. □

By an isomorphism of extensions (or of central extensions), we mean a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{j} & K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathbf{1}_H} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathbf{1}_K} & & \\ 1 & \longrightarrow & H & \xrightarrow[i']{} & G' & \xrightarrow[j']{} & K & \longrightarrow & 1 \end{array} \tag{2}$$

where ϕ is an isomorphism between G and G' and each row is an extension (or a central extension).

Proposition 2.3.3. *Suppose*

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1 \tag{3}$$

is an extension. Then the following conditions are equivalent :

1. *There exists a group homomorphism $q : K \to G$ such that $jq = \mathbf{1}_K$.*
2. *(3) is isomorphic to (1) where $\widetilde{G} = K \ltimes H$ for some semidirect product.*

Proof. Suppose that there exist a group homomorphism $q : K \to G$ such that $jq = \mathbf{1}_K$. Note that the image of H in G is normal, because $i(H) = \ker j$. Define a map $\alpha : K \to \mathrm{Aut}(H)$ by $\alpha(k)h = khk^{-1}$. For each k, $\alpha(k)$ is a well-defined group homomorphism because H is normal, and is clearly injective and surjective, hence an automorphism. Consider the subgroups $q(K) \subseteq G$ and $i(H) \subseteq G$. Suppose $x \in q(K) \cap i(H)$. Then $x \in \ker j$ by exactness, hence $j(x) = 1$. But then $x = q(y)$ for some $y \in K$, hence $j(x) = jq(y) = y = 1$. Therefore, $x = q(1) = 1$. We conclude that $q(K) \cap i(H) = 1$.

Now suppose $x \in G$. Then $qj(x^{-1})x \in \ker j$. Therefore, $qj(x^{-1})x = i(h)$ for some $h \in H$. But now, $x = qj(x)i(h) \in q(K)i(H)$. Therefore, $G = q(K)i(H)$. We conclude from [1, p. 180, Theorem 12] that $G \cong q(K) \ltimes i(H) \cong K \ltimes H$ with the isomorphism $\phi : q(k)i(h) \mapsto (k, h)$, and it is easy to check from the definition of ϕ that Diagram (2) commutes.

Next suppose $G \cong K \ltimes H$ as in Example 2.3.2. Then in $K \ltimes H$, we have $i'(H) = \{1\} \ltimes H$ and $j'(k, h) = j(k, 1)$. Define $q : K \to K \ltimes H$ by $q'(k) = (k, 1)$. Clearly $j'q' = \mathbf{1}_K$ and q' is a group homomorphism. But then, by the commutativity of the diagram it is easy to check that $q := \phi \circ q'$ is a group homomorphism $K \to G$ satisfying $jq = \mathbf{1}_K$. □

In the previous proof, the assumption that H is contained in the center of G was not needed. For central extensions, we obtain the following :

Corollary 2.3.4. *Suppose that*

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1 \tag{4}$$

is a central extension. Then the following conditions are equivalent :

1. *There exists a group homomorphism* $q : K \to G$ *such that* $jq = \mathbf{1}_K$.

2. (4) *is isomorphic to* (1) *where* $\widetilde{G} = K \times H$.

Proof. Suppose that there exists a group homomorphism $q : K \to G$ such that $jq = \mathbf{1}_K$. Then by Proposition 2.3.3, (4) is isomorphic to (1). Therefore, there exists $\phi : K \to Aut(H)$ a homomorphism such that $G \cong K \ltimes_\phi H$ with operation

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, h_1\phi_{k_1}(h_2))$$

However,

$$(1, h) = (k, 1)(1, h)(k^{-1}, 1) = (k, \phi_k(h))(k^{-1}, 1) = (1, \phi_k(h)\phi_k(1)) = (1, \phi_k(h))$$

and therefore $\phi_k(h) = h$ for all k and the semidirect product is in fact a direct product.

The converse follows immediately from the previous proposition because a direct product is also a semidirect product. □

A central extension satisfying the conditions of the previous corollary is also called split. Our next objective is to establish a bijection between isomorphism classes of central extensions and equivalence classes of maps which we call cocycle.

Definition 2.3.5. Suppose H and K are groups where H is abelian. Then a map $\alpha : K \times K \to H$ is called a cocycle if

1. $\alpha(k, 1) = \alpha(1, k) = 1$ for all $k \in K$

2. $\alpha(k_1k_2, k_3)\alpha(k_1, k_2) = \alpha(k_1, k_2k_3)\alpha(k_2, k_3)$ for all $k_1, k_2, k_3 \in K$

Proposition 2.3.6. *Suppose* H, K *and* α *are as in definition 2.3.5. Then the set* $G = K \times H$ *with operation*

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, \alpha(k_1, k_2)h_1h_2)$$

is a group. Furthermore, we obtain a central extension

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1$$

with $i(h) = (1, h)$ *and* $j(k, h) = k$.

Proof. First, note that

$$\alpha(kk^{-1}, k)\alpha(k, k^{-1}) = \alpha(k, k^{-1}k)\alpha(k^{-1}, k)$$

hence

$$\alpha(k, k^{-1}) = \alpha(k^{-1}, k).$$

Second, we verify the axioms of a group.

1. $(1,1)(k,h) = (k, \alpha(1,k)h) = (k,h) = (k, \alpha(k,1)h) = (k,h)(1,1)$

2. For all $k_1, k_2, k_3 \in K$ and $h_1, h_2, h_3 \in H$,

$$\begin{aligned}((k_1,h_1)(k_2,h_2))(k_3,h_3) &= (k_1k_2, \alpha(k_1,k_2)h_1h_2)(k_3,h_3)\\ &= (k_1k_2k_3, \alpha(k_1k_2,k_3)\alpha(k_1,k_2)h_1h_2h_3)\\ &= (k_1k_2k_3, \alpha(k_1,k_2k_3)\alpha(k_2,k_3)h_1h_2h_3)\\ &= (k_1,h_1)(k_2k_3, \alpha(k_2,k_3)h_2h_3)\\ &= (k_1,h_1)((k_2,h_2)(k_3,h_3))\end{aligned}$$

3. $(k,h)(k^{-1}, \alpha(k,k^{-1})^{-1}h^{-1}) = (kk^{-1}, \alpha(k,k^{-1})\alpha(k,k^{-1})^{-1}hh^{-1}) = (1,1)$. Furthermore, we have $(k^{-1}, \alpha(k,k^{-1})^{-1}h^{-1}) = (k^{-1}, \alpha(k^{-1},k)^{-1}h^{-1})$. Therefore,

$$(k^{-1}, \alpha(k,k^{-1})^{-1}h^{-1})(k,h) = (1, \alpha(k^{-1},k)\alpha(k^{-1},k)^{-1}h^{-1}h) = (1,1)$$

We conclude that G is a group. It is easy to see that i injective group homomorphism. Similarly, it is easy to see that j is a surjective group homomorphism. Furthermore, $(k,h) \in \ker j$ if and only if $k = 1$ and it is clear that $(1,h) \in \operatorname{Im} i$. Finally, for all $h, h' \in H$ and $k \in K$,

$$(1,h')(k,h) = (k, \alpha(1,k)h'h) = (k,h'h) = (k, \alpha(k,1)hh') = (k,h)(1,h').$$

Therefore, $i(H) \subseteq Z(G)$ and the extension is a central extension. This completes the proof. □

Definition 2.3.7. Suppose K, H are as above. Then a coboundary α is a map $\alpha : K \times K \to H$ satisfying

$$\alpha(x,y) = s(xy)s(x)^{-1}s(y)^{-1}$$

for some map $s : K \to H$.

Remark 2.3.8. It is easy to see that every coboundary is a cocycle.

The next two results allow us to classify all central extensions in terms of cocycles.

Lemma 2.3.9. *For two cocycles α and β, define*

$$\alpha \sim \beta \text{ if and only if } \alpha\beta^{-1} \text{ is a coboundary.}$$

Then $\sim$ is an equivalence relation.

Proof. 1. Clearly, $\alpha\alpha^{-1} = 1$, and 1 is obtained from $s(x) = 1$.

2. Suppose $\alpha \sim \beta$. Then $\alpha\beta^{-1}(x, y) = s(xy)s(x)^{-1}s(y)^{-1}$. Therefore, by taking the inverse, we have $\beta\alpha^{-1}(x, y) = (s(xy))^{-1}s(x)s(y)$. Defining $s^{-1}(x) = (s(x))^{-1}$, we obtain that $\beta \sim \alpha$.

3. Suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. Then $\alpha\beta^{-1}(x, y) = s(xy)s(x)^{-1}s(y)^{-1}$ and $\beta\gamma^{-1}(x, y) = r(xy)r(x)^{-1}r(y)^{-1}$. But then,

$$\begin{aligned}\alpha\gamma^{-1}(x, y) &= s(xy)r(xy)s(x)^{-1}r(x)^{-1}s(y)^{-1}r(y)^{-1} \\ &= sr(xy)(sr)^{-1}(x)(sr)^{-1}(y).\end{aligned}$$

□

The following justifies why we study central extensions using cocycles and coboundaries.

Proposition 2.3.10. *There is a bijection between isomorphism classes of central extensions and equivalence classes of cocycles.*

Proof. Firstly, suppose that

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1$$

is a central extension. We prove that it is obtained from a cocycle.

The map j is surjective, hence using the axiom of choice we can choose a set-theoretic map $s : K \to G$ such that $js = \mathbf{1}_K$. Setting

$$s'(x) = \begin{cases} 1 & \text{if } x = 1 \\ s(x) & \text{if } x \neq 1 \end{cases}$$

we can assume that $s(1) = 1$. We claim that $G = i(H)s(K)$. Indeed, suppose $g \in G$. Then $j(gs(j(g)^{-1})) = j(g)j(g)^{-1} = 1$, hence $gs(j(g)^{-1}) \in \ker j = \operatorname{Im} i$. Therefore, $gs(j(g)^{-1}) = i(h)$ for some h. Then $g = i(h)s(j(g))$, as claimed.

Now, consider the product $s(k_1)s(k_2)$. Applying j, we obtain k_1k_2. However, we also obtain k_1k_2 by applying j to $s(k_1k_2)$. Therefore,

$$s(k_1)s(k_2)s(k_1k_2)^{-1} \in \ker j = \operatorname{Im} i.$$

We denote by c_{k_1,k_2} elements of H such that $i(c_{k_1,k_2}) = s(k_1)s(k_2)s(k_1k_2)^{-1}$ and define $\alpha(k_1, k_2) = c_{k_1,k_2}$. It is easy to see that $\alpha(1, k) = \alpha(k, 1) = s(1) = 1$. Suppose that $k_1, k_2, k_3 \in K$. Then we have

$$\begin{aligned}
i(c_{k_1,k_2}c_{k_1k_2,k_3}) &= i(c_{k_1,k_2})i(c_{k_1k_2,k_3}) \\
&= i(c_{k_1,k_2})s(k_1k_2)s(k_3)s(k_1k_2k_3)^{-1} \\
&= (s(k_1)s(k_2))s(k_3)s(k_1k_2k_3)^{-1} \\
&= s(k_1)(s(k_2)s(k_3))s(k_1k_2k_3)^{-1} \\
&= s(k_1)i(c_{k_2,k_3})s(k_2k_3)s(k_1k_2k_3)^{-1} \\
&= i(c_{k_2,k_3})i(c_{k_1,k_2k_3}) \\
&= i(c_{k_2,k_3}c_{k_1,k_2k_3})
\end{aligned}$$

or equivalently from the fact that i is injective,

$$\alpha(k_1k_2, k_3)\alpha(k_1, k_2) = \alpha(k_1, k_2k_3)\alpha(k_2, k_3)$$

which is exactly the second cocycle condition.

Now consider $\tilde{G} = K \times H$ with operation given by the cocycle α as in Proposition 2.3.6. Then we claim that the map $\phi : \tilde{G} \to G$, $(k, h) \mapsto i(h)s(k)$ is a group isomorphism. Indeed, $(1, 1)$ goes to 1. Furthermore ϕ is surjective because $G = s(K)i(H)$, as proven earlier. ϕ is also a group homomorphism. Indeed,

$$\begin{aligned}
\phi((k_1, h_1)(k_2, h_2)) &= \phi(k_1k_2, c_{k_1k_2}h_1h_2) = i(c_{k_1,k_2}h_1h_2)s(k_1k_2) \\
&= i(h_1)i(h_2)s(k_1)s(k_2) = \phi(k_1, h_1)\phi(k_2, h_2).
\end{aligned}$$

The map ϕ is injective because if $\phi(k, h) = i(h)s(k) = 1$, we have $j\phi(k, h) = k = 1$ and then $\phi(1, h) = i(h) = 1$ implies that $h = 1$ by the injectivity of i. We conclude

that $G \cong \tilde{G}$ and it is not difficult to check directly that ϕ is in fact an isomorphism of central extensions. Indeed, denoting the maps in Proposition 2.3.6 by i' and j', we have

$$\phi i'(h) = \phi(1,h) = i\mathbf{1}_H(h)$$

and

$$j\phi(k,h) = j(i(h)s(k)) = j(s(k)) = k = \mathbf{1}_K j'(k,h).$$

Secondly, suppose we have two isomorphic central extensions G, G'. Then we can assume that $G = K \times H$ with product given by some cocycle α and $G' = K \times H$ with product given by some cocycle β as in Proposition 2.3.6. We wish to show that these cocycles differ by a coboundary. First, note that from the commutativity of the diagram,

$$j'\phi(k,1) = \mathbf{1}_K j(k,1) = k.$$

Therefore, $\phi(k,1) = (k, \phi'(k))$ for some map $\phi' : K \to H$. Furthermore, again by the commutativity of the diagram,

$$\phi(1,h) = \phi(i(h)) = i'\mathbf{1}_H(h) = (1,h).$$

Therefore, $\phi(1,h) = (1,h)$ and then

$$\phi(k,h) = \phi(k,1)\phi(1,h) = (k,\phi'(k))(1,h) = (k,\phi'(k)h).$$

However,

$$\phi((k_1,h_1)(k_2,h_2)) = \phi(k_1k_2, \alpha(k_1,k_2)h_1h_2) = (k_1k_2, \phi'(k_1k_2)\alpha(k_1,k_2)h_1h_2)$$

and

$$\phi(k_1,h_1)\phi(k_2,h_2) = (k_1,\phi'(k_1)h_1)(k_2,\phi'(k_2)h_2) = (k_1k_2, \phi'(k_1)\phi'(k_2)\beta(k_1,k_2)h_1h_2).$$

Equating the two equations we obtain

$$\alpha(k_1,k_2)\beta(k_1,k_2)^{-1} = \phi'(k_1)\phi'(k_2)\phi'(k_1k_2)^{-1}$$

and therefore $\alpha\beta^{-1}$ is a coboundary.

Finally, we must show that two equivalent cocycles induce an isomorphism between their central extensions. Suppose α, β are cocycles such that $\alpha\beta^{-1}(k_1,k_2) =$

$s^{-1}(k_1k_2)s(k_1)s(k_2)$ for some $s : K \to H$. Let $G_1 = K \times H$ with the group structure induced by $\alpha(k_1, k_2)$ and $G_2 = K \times H$ with the group structure induced by $\beta(k_1, k_2)$. Then it is easy to check that

$$\phi(k, h) = (k, s(k)h)$$

is an isomorphism of central extensions. □

Corollary 2.3.11. *Suppose α is a coboundary. Then the central extension defined by Proposition 2.3.6 is split.*

Proof. If α is a coboundary, then it is equivalent to the trivial cocycle $\beta(x, y) = 1$. Therefore, by Proposition 2.3.10, G is isomorphic (as central extension) to a direct product, hence is split. □

Chapter 3

The Weil Representation

For this chapter, we let $\mathbb{F}$ be a local field of odd or zero characteristic, and $\psi : \mathbb{F} \to \mathbb{C}^\times$ be a non-trivial continuous additive character of $\mathbb{F}$. By an additive character, we mean a group homomorphism $(\mathbb{F}, +) \to \mathbb{C}^\times$. The main reference for this chapter will be [14], from which we rewrite many of the proofs in full detail.

$\mathbb{F}$ is a local field, hence a locally compact group under addition, so there exists a Haar measure on $\mathbb{F}$ [3, Theorem 2.10].

3.1 The symplectic group

Let V be a vector space over $\mathbb{F}$ (possibly infinite dimensional) and V^* its algebraic dual. We denote $(v^*, v) = (v, v^*) = v^*(v)$, for $v \in V$ and $v^* \in V^*$. This pairing defines a symmetric, non-degenerate bilinear form. Furthermore, the space $X = V \oplus V^*$ also has the following bilinear form : $\langle v_1 + v_1^*, v_2 + v_2^* \rangle = (v_1, v_2^*) - (v_2, v_1^*)$. It is easy to see that this form is antisymmetric and non-degenerate.

We denote by $Sp(X)$ the isometry group of X with respect to the form $\langle ., . \rangle$ and by $Sp_{2n}(\mathbb{F})$ the classical symplectic group over $\mathbb{F}$, i.e. the set of $2n \times 2n$ matrices A satisfying $A^T \Omega A = \Omega$ where $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. If V happens to be finite-dimensional, then we have $Sp(X) \cong Sp_{2n}(\mathbb{F})$ where $n = \dim V$.

We will denote the action of an element of $Sp(X)$ on X on the right. If $g \in Sp(X)$, $v \in V$ and $v^* \in V^*$, we have $(v + v^*)g = vg + v^*g = v\alpha + v\beta + v^*\gamma + v^*\delta$, where

$v\alpha \in V, v\beta \in V^*, v^*\gamma \in V, v^*\delta \in V^*$. Then we can think of g as a 2×2 matrix :

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

where $\alpha : V \to V$, $\beta : V \to V^*$, $\gamma : V^* \to V$, $\delta : V^* \to V^*$, and the action of g on $v + v^* \in X$ can be seen as right multiplication of this matrix on the vector

$$\begin{bmatrix} v & v^* \end{bmatrix}.$$

Definition 3.1.1. Suppose $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ as above. Then $\alpha^* : V^* \to V^*$ is a linear map such that $(v_1\alpha, v_2^*) = (v_1, v_2^*\alpha^*)$, $\beta^* : V \to V^*$ is a linear map such that $(v_1\beta, v_2) = (v_1, v_2\beta^*)$, $\gamma^* : V^* \to V$ is a linear map such that $(v_1^*\gamma, v_2^*) = (v_1^*, v_2^*\gamma^*)$, and $\delta^* : V \to V$ is a linear map such that $(v_1^*\delta, v_2^*) = (v_1^*, v_2^*\delta^*)$. It also follows directly from the definition that the same properties apply to the form $\langle ., . \rangle$.

The above definition is valid because of Lemma 3.1.2.

Lemma 3.1.2. *$\alpha^*, \beta^*, \gamma^*, \delta^*$ exist and are unique.*

Proof. We start with uniqueness. The four proofs follow the exact same idea, so we only do the proof for α^*. Suppose $v_1, v_2 \in V$ and that α'^* is another map satisfying $(v_1\alpha, v_2^*) = (v_1, v_2^*\alpha'^*)$. Then

$$\begin{aligned} 0 &= (v_1\alpha, v_2^*) - (v_1\alpha, v_2^*) \\ &= (v_1, v_2^*\alpha^*) - (v_1, v_2^*\alpha'^*) \\ &= (v_1, v_2^*(\alpha^* - \alpha'^*)). \end{aligned}$$

But the form $(.,.)$ is non-degenerate, hence we must have $v_2^*(\alpha^* - \alpha') = 0$ for all v_2^* and therefore $\alpha^* = \alpha'$.

The existence of α and β are well-known facts. By definition of $Sp(X)$, we know that for all $v_1, v_2 \in V$

$$\begin{aligned} 0 = \langle v_1, v_2 \rangle = \langle v_1 g, v_2 g \rangle &= \langle v_1\alpha + v_1\beta, v_2\alpha + v_2\beta \rangle \\ &= \langle v_1\beta, v_2\alpha \rangle + \langle v_1\alpha, v_2\beta \rangle = -(v_1\beta, v_2\alpha) + (v_1\alpha, v_2\beta) \\ &= -(v_1\beta\alpha^*, v_2) + (v_1\alpha\beta^*, v_2) = (v_1(-\beta\alpha^* + \alpha\beta^*), v_2). \end{aligned}$$

By non-degeneracy of $(.,.)$, we obtain

$$\beta\alpha^* = \alpha\beta^* \tag{5}$$

We also have

$$\begin{aligned}(v, v^*) = \langle v, v^*\rangle = \langle vg, v^*g\rangle &= \langle v\alpha + v\beta, v^*\gamma + v^*\delta\rangle = -(v\beta, v^*\gamma) + (v\alpha, v^*\delta) \\ &= (v, -v^*\gamma\beta^*) + (v, v^*\delta\alpha^*) = (v, (v^*)(\delta\alpha^* - \gamma\beta^*)).\end{aligned}$$

Therefore, we obtain

$$\delta\alpha^* - \gamma\beta^* = \mathbf{1}_{V^*}. \tag{6}$$

Where $\mathbf{1}_{V^*}$ is the identity on V^*. In particular, this means that we have,

$$\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}\begin{bmatrix}-\beta^* \\ \alpha^*\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{1}_{V^*}\end{bmatrix}$$

Now write

$$g^{-1} = \begin{bmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{bmatrix}.$$

Then we also have, by replacing the role of g with g^{-1}

$$\begin{bmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{bmatrix}\begin{bmatrix}-\beta'^* \\ \alpha'^*\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{1}_{V^*}\end{bmatrix}.$$

Multiplying this last equation by g on the left yields

$$\begin{bmatrix}-\beta'^* \\ \alpha'^*\end{bmatrix} = \begin{bmatrix}\beta \\ \delta\end{bmatrix}.$$

Then $\delta = \alpha'^*$ and we conclude that $\delta^* = \alpha'$. This shows that δ^* exists. Now note that for all $v_1^*, v_2^* \in V^*$,

$$\begin{aligned}0 = \langle v_1^*, v_2^*\rangle = \langle v_1^*g, v_2^*g\rangle &= \langle v_1^*\gamma + v_1^*\delta, v_2^*\gamma + v_2^*\delta\rangle \\ &= -(v_1^*\delta, v_2^*\gamma) + (v_1^*\gamma, v_2^*\delta) \\ &= -(v_1^*, v_2^*\gamma\delta^*) + (v_1^*\gamma\delta^*, v_2^*).\end{aligned}$$

Therefore $\gamma\delta^*$ is self-adjoint. Also, we know that $\gamma\beta^* = \delta\alpha^* - \mathbf{1}_{V^*}$ from Eq. (6), and we can check that $(\gamma\beta^*)^* = \alpha\delta^* - \mathbf{1}_V$. Indeed,

$$(v^*\gamma\beta^*, v) = (v^*(\delta\alpha^* - \mathbf{1}_V), v) = (v^*\delta\alpha^*, v) - (v^*, v) = (v^*\delta, v\alpha) - (v^*, v)$$
$$= (v^*, v\alpha\delta^*) - (v^*, v) = (v^*, v(\alpha\delta^* - \mathbf{1}_V))$$

as desired.

For each $v^* \in V^*$, note that by matrix multiplication of g^{-1} with g, we have $v^*(\gamma'\beta + \delta'\delta) = v^*$. Now, define $v^*\gamma^* = v^*(\gamma'(\gamma\beta^*)^*) + v^*(\delta'\gamma\delta^*)$. This is obviously linear in V^* and by combining the last few facts, we obtain

$$\begin{aligned}(w^*\gamma, v^*) &= (w^*\gamma, v^*\gamma'\beta + v^*\delta'\delta) \\ &= (w^*\gamma\beta^*, v^*\gamma') + (w^*\gamma\delta^*, v^*\delta') \\ &= (w^*, v^*\gamma'(\gamma\beta^*)^*) + (w^*, v^*\delta'\gamma\delta^*) \\ &= (w^*, v^*(\gamma'(\gamma\beta^*)^* + \delta'\gamma\delta^*)).\end{aligned}$$

Therefore, γ^* exists, which completes the proof. We also obtain the equation

$$\gamma\delta^* = \delta\gamma^* \tag{7}$$

because $\gamma\delta^*$ is self-adjoint. □

Lemma 3.1.3. *The following relations hold :*

$$\alpha\beta^* = \beta\alpha^*, \gamma\delta^* = \delta\gamma^*, \delta\alpha^* - \gamma\beta^* = \mathbf{1}_{V^*}$$

Proof. These relations are Eq. (5), (7) and (6) in Lemma 3.1.2. □

Corollary 3.1.4.

$$g^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{bmatrix}$$

Proof. We multiply the two matrices together

$$gg^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{bmatrix} = \begin{bmatrix} \alpha\delta^* - \beta\gamma^* & -\alpha\beta^* + \beta\alpha^* \\ \gamma\delta^* - \delta\gamma^* & -\gamma\beta^* + \delta\alpha^* \end{bmatrix} = \begin{bmatrix} \mathbf{1}_V & 0 \\ 0 & \mathbf{1}_{V^*} \end{bmatrix}.$$

where $a\delta^* - \beta\gamma^* = \mathbf{1}_V$ from taking the transpose on each side of $\delta\alpha^* - \gamma\beta^* = \mathbf{1}_V^*$. Therefore, g^{-1} is a right inverse for g. However, $Sp(X)$ is a group, hence a right inverse is also a left inverse, which completes the proof. □

Proposition 3.1.5. *$H = X \times \mathbb{F}$ with operations*

$$(x_1, k_1)(x_2, k_2) = (x_1 + x_2, \frac{1}{2}\langle x_1, x_2\rangle + k_1 + k_2)$$

and identity $(0, 0)$ is a group.

Remark 3.1.6. The group H is known as Heisenberg group.

Proof. Associativity can be verified by a direct computation. Furthermore, suppose $(x_1, k_1) \in H$. Then, we have $(0, 0)(x_1, k_1) = (0 + x_1, \frac{1}{2}\langle 0, x_1\rangle + 0 + k_1) = (x_1, k_1)$ and similarly, $(x_1, k_1)(0, 0) = (x_1, k_1)$. We also have $(x_1, k_1)(-x_1, -k_1) = (0, \langle x_1, -x_1\rangle) = (0, 0)$ where the last equality follows from antisymmetry of $\langle ., .\rangle$ and the fact that $\operatorname{char} \mathbb{F} \neq 2$. □

We obtain an action of $Sp(X)$ on H by $(x, k) \cdot g = (xg, k)$. Indeed, suppose $g_1, g_2 \in Sp(X)$. Then

$$(x, k) \cdot (g_1 g_2) = (xg_1 g_2, k) = (xg_2, k) \cdot g_2 = (x, k) \cdot g_1 \cdot g_2$$

as desired.

Proposition 3.1.7. *Any element $(v+v^*, k)$ of H can be written uniquely as a product $(0, k')(v, 0)(v^*, 0)$ where $k' = -\frac{1}{2}\langle v, v^*\rangle + k$. The group H acts on $S(V)$, where $S(V)$ was defined in Section 2.2 for an arbitrary local field, by*

1. $(k \cdot f)(x) = \psi(k)f(x)$
2. $(v \cdot f)(x) = f(x + v)$
3. $(v^* \cdot f)(x) = \psi(\langle x, v^*\rangle)f(x)$

Proof. First, we prove that every element of H can be written as a product of elements of V, V^* and $\mathbb{F}$ in a unique way. Indeed, suppose $(v + v^*, k) \in H$. Then note that

$$(0, -\frac{1}{2}\langle v, v^*\rangle + k)(v, 0)(v^*, 0) = (v, -\frac{1}{2}\langle v, v^*\rangle + k)(v^*, 0) = (v + v^*, k).$$

Now, suppose we can write $(v+v^*, k)$ as a product of $(v', 0)$, $(v'^*, 0)$ and $(0, k')$. Then

$$(0, k')(v', 0)(v'^*, 0) = (v', k')(v'^*, 0) = (v' + v'^*, \frac{1}{2}\langle v', v'^*\rangle + k').$$

Therefore, $v' = v$, $v'^* = v^*$ and $k' = k - \frac{1}{2}\langle v, v^* \rangle$.

Define the map $p : H \to \mathrm{End}(S(V))$ by $p(v + v^*, k)f = k' \cdot (v \cdot (v^* \cdot f))$. It can be seen that $p(h)$ is a composition of three linear maps each preserving $S(V)$ (the only non-obvious part is to prove that $v^* \cdot f \in S(V)$. If $\mathbb{F}$ is non-archimedean, then the restriction of $v^* \cdot f$ to any finite-dimensional space still has compact support and is a product of locally constant functions. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ however, it is an easy exercise), hence $p(h)$ is a linear map preserving $S(V)$. Therefore, p is well-defined. We still need to show that p is a group homomorphism, but this can be done directly by computing the actions of elements $(v_1 + v_1^*, k_1)$, $(v_2 + v_2^*, k_2) \in H$ on an arbitrary function f and then compare it to the action of their product

$$(v_1 + v_2 + v_1^* + v_2^*, \frac{1}{2}\langle v_1 + v_1^*, v_2 + v_2^* \rangle k_1 + k_2)$$

on f again. In both cases, one will obtain as final result

$$\psi(\langle x, v_1^* + v_2^* \rangle + \langle v_1, v_2^* \rangle + \frac{1}{2}\langle v_1, v_1^* \rangle + \frac{1}{2}\langle v_2, v_2^* \rangle + k_1 + k_2) f(x + v_1 + v_2),$$

hence p is a group homomorphism. □

3.2 The Weil representation

One would naturally expect that one can define the Weil representation over $Sp(X)$. However, this appears to be impossible, hence we shall restrict ourselves to defining it over a subgroup.

Let $Sp(X, V^*)$ be the subset of $Sp(X)$ containing the elements $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ satisfying $\dim \mathrm{Im}\, \gamma < \infty$. We will see that $Sp(X, V^*)$ is indeed a subgroup of $Sp(X)$, but we first prove a technical lemma.

Lemma 3.2.1. *Suppose $\gamma : V^* \to V$ and $n = \dim \mathrm{Im}\, \gamma < \infty$. Then $\dim \mathrm{Im}\, \gamma^* < \infty$.*

Proof. Suppose by contradiction that $\dim \mathrm{Im}\, \gamma^* > \dim \mathrm{Im}\, \gamma$. Then let $w_1, ..., w_{n+1}$ be linearly independent elements of $\mathrm{Im}\, \gamma^*$ and let $v_1^*, ..., v_{n+1}^*$ be elements of V^* such that $v_i^* \gamma^* = w_i$. Extend the w_i's to a basis $\{w_i\}_{i \in \mathbb{N}}$ of V and choose $f_1^*, ..., f_n^*, f_{n+1}^*$ to be elements of V^* such that $(f_i^*, w_j) = \delta_{i,j}$. This can be done by simply defining f_i^* to

be 1 on w_i and 0 on w_j for $j \neq i$, and then extending f_i^* by linearity. Now, suppose $a_1\gamma(f_1^*) + ... + a_{n+1}\gamma(f_{n+1}^*) = 0$. Then

$$\begin{aligned} 0 = (a_1 f_1^*\gamma + \cdots + a_{n+1} f_{n+1}^*\gamma, v_i^*) &= (a_1 f_1^* + \cdots + a_n f_{n+1}^*, v_i^*\gamma^*) \\ &= (a_1 f_1^* + \cdots + a_n f_{n+1}^*, w_i) = a_i. \end{aligned}$$

Therefore, the $f_i^*\gamma$'s form a linearly independent set, a contradiction. We conclude that $\dim \operatorname{Im} \gamma^* \leq \dim \operatorname{Im} \gamma < \infty$. □

Proposition 3.2.2. *$Sp(X, V^*)$ is a subgroup of $Sp(X)$.*

Proof. We need to show that $Sp(X, V^*)$ is closed under multiplication and inversion. Write $g_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix} \in Sp(X, V^*)$. Then we have the product

$$g_1 g_2 = \begin{bmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & \alpha_1\beta_2 + \beta_1\delta_2 \\ \gamma_1\alpha_2 + \delta_1\gamma_2 & \gamma_1\beta_2 + \delta_1\delta_2 \end{bmatrix}.$$

But now,

$$\dim \operatorname{Im}(\gamma_1\alpha_2 + \delta_1\gamma_2) \leq \dim \operatorname{Im}(\gamma_1\alpha_2) + \dim \operatorname{Im}(\delta_1\gamma_2) \leq \dim \operatorname{Im} \gamma_1 + \dim \operatorname{Im} \gamma_2 < \infty$$

as desired.

For inversion, write $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Then $g^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{bmatrix}$ by Corollary 3.1.4. But now, by Lemma 3.2.1, we know that $\dim \operatorname{Im} -\gamma^* < \infty$ and therefore $g^{-1} \in Sp(X, V^*)$. Therefore $Sp(X, V^*)$ is indeed a subgroup of $Sp(X)$. □

Lemma 3.2.3. *Suppose $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in Sp(X)$ and $x_1^*, x_2^* \in V^*$. If $x_1^*\gamma = x_2^*\gamma$, then $\langle x_1^*\gamma, x_1^*\delta \rangle = \langle x_2^*\gamma, x_2^*\delta \rangle$.*

Proof. Suppose $x_1^*\gamma = x_2^*\gamma$. Then using Lemma 3.1.3, we obtain

$$\begin{aligned} \langle x_1^*\gamma, x_1^*\delta \rangle = \langle x_2^*\gamma, x_1^*\delta \rangle = \langle x_2^*, x_1^*\delta\gamma^* \rangle = \langle x_2^*, x_1^*\gamma\delta^* \rangle \\ = \langle x_2^*, x_2^*\gamma\delta^* \rangle = \langle x_2^*, x_2^*\delta\gamma^* \rangle = \langle x_2^*\gamma, x_2^*\delta \rangle \end{aligned}$$

□

The following family of operators, parametrized by $g \in Sp(X)$, will yield theWeil representation upon restriction to a suitable subgroup.

Proposition 3.2.4. *Suppose* $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in Sp(X, V^*)$. *Fix a Haar measure for* $\operatorname{Im}\gamma$. *Then define*

$$(T_g f)(x) = \int_{V_g} S_g(x + x^*) f(x\alpha + x^*\gamma) d(x^*\gamma)$$

where $V_g = \operatorname{Im}\gamma$ *and*

$$S_g(x + x^*) = \psi(\frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle x^*\gamma, x^*\delta\rangle + \langle x^*\gamma, x\beta\rangle).$$

Then $h \cdot (T_g f) = T_g((h \cdot g) \cdot f)$ *for all* $f \in S(V)$ *and* $h \in H$, *where* $h \cdot g$ *is the action of* $Sp(X)$ *defined after Remark 3.1.6.*

Remark : We can choose a Haar measure on $\operatorname{Im}\gamma$ because $\dim \operatorname{Im}\gamma < \infty$. Furthermore, by Lemma 3.2.3, $S_g(x + x^*)$ is a function of x and $x^*\gamma$ only, and therefore the integration makes sense. We also have

$$|(T_g f)(x)| \leq \int_{V_g} |f(x\alpha + x^*\gamma)| \, d(x^*\gamma) < \infty$$

because $((x\alpha, 0) \cdot f)|_{V_g} \in S(V_g)$. We will prove in Theorem 3.2.16 that $T_g f \in S(V)$, but for now we will be content with the fact that $T_g f$ is a well-defined function.

Proof. To check that $h \cdot (T_g f) = T_g((h \cdot g) \cdot f)$, it is enough to verify equality in the three cases where $h = (0, k)$ for $k \in \mathbb{F}$, $h = (v, 0)$ for $v \in V$ and $h = (v^*, 0)$ for $v^* \in V^*$. This follows from the fact that $(h_1 \cdot g)(h_2 \cdot g) = (h_1 h_2) \cdot g$ for all $h_1, h_2 \in H$. Indeed, assuming the statement holds in these three cases, we have

$$\begin{aligned}
(0,k) \cdot ((v^*,0) \cdot ((v,0) \cdot T_g f)) &= (0,k) \cdot ((v^*,0) \cdot (T_g(((v,0) \cdot g) \cdot f))) \\
&= (0,k) \cdot T_g(((v^*,0) \cdot g) \cdot ((v,0) \cdot g)) \cdot f)) \\
&= (0,k) \cdot T_g(((v^*,0)(v,0) \cdot g) \cdot f) \\
&= T_g(((0,k) \cdot g) \cdot (((v^*,0)(v,0) \cdot g) \cdot f)) \\
&= T_g(((0,k)(v^*,0)(v,0) \cdot g) \cdot f)
\end{aligned}$$

The case $h = (0, k)$ is easy to check. Suppose that $h = (v, 0)$ for $v \in V$. Then

$$\begin{aligned} h \cdot (T_g f)(x) &= (T_g f)(x + v) \\ &= \int_{V_g} S_g(x + v + x^*) f((x + v)\alpha + x^*\gamma) d(x^*\gamma). \end{aligned}$$

However, we also have

$$\begin{aligned} &T_g(((v, 0) \cdot g) \cdot f)(x) = T_g((v\alpha + v\beta, 0) \cdot f)(x) \\ &= \int_{V_g} S_g(x + x^*)(v\alpha + v\beta \cdot f)(x\alpha + x^*\gamma) d(x^*\gamma) \\ &= \int_{V_g} \psi(\langle (x\alpha + x^*\gamma + \frac{1}{2} v\alpha, v\beta\rangle) S_g(x + x^*) f((x + v)\alpha + x^*\gamma) d(x^*\gamma) \end{aligned}$$

where the last line follows from $(v\alpha + v\beta, 0) = (0, -\frac{1}{2}\langle v\alpha, v\beta\rangle)(v\alpha, 0)(v\beta, 0)$. However,

$$\begin{aligned} &\psi(\langle (x + \frac{1}{2} v)\alpha + x^*\gamma, v\beta\rangle) S_g(x + x^*) \\ &= \psi(\langle (x + \frac{1}{2} v)\alpha + x^*\gamma, v\beta\rangle) \psi(\frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle x^*\gamma, x^*\delta\rangle + \langle x^*\gamma, x\beta\rangle) \\ &= \psi(\langle (x + \frac{1}{2} v)\alpha, v\beta\rangle + \frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle x^*\gamma, x^*\delta\rangle + \langle x^*\gamma, (x + v)\beta\rangle) \\ &= S_g(x + v + x^*) \psi(\frac{1}{2}(\langle x\alpha, v\beta\rangle - \langle v\alpha, x\beta\rangle)) \\ &= S_g(x + v + x^*) \psi(\frac{1}{2}(\langle x\alpha, v\beta\rangle + \langle x\beta, v\alpha\rangle)) \\ &= S_g(x + v + x^*) \psi(\frac{1}{2}(\langle x\alpha + x\beta, v\beta\rangle + \langle x\alpha + x\beta, v\alpha\rangle)) \\ &= S_g(x + v + x^*) \psi(\frac{1}{2}(\langle x\alpha + x\beta, v\alpha + v\beta\rangle)) \\ &= S_g(x + v + x^*) \psi(\frac{1}{2}(\langle xg, vg\rangle)) = S_g(x + v + x^*) \psi(\frac{1}{2}\langle x, v\rangle) = S_g(x + v + x^*). \end{aligned}$$

We conclude that $h \cdot (T_g f)(x) = T_g((h \cdot g) \cdot f)(x)$. Next, suppose $h = (v^*, 0)$ for $v^* \in V^*$. Then

$$h \cdot (T_g f)(x) = \psi(\langle x, v^*\rangle)(T_g f)(x)$$

and $h \cdot g = (v^*\gamma + v^*\delta, 0) = (0, -\frac{1}{2}\langle v^*\gamma, v^*\delta\rangle)(v^*\gamma, 0)(v^*\delta, 0)$, hence

$$\begin{aligned} &T_g((v^*, 0) g \cdot f)(x) \\ &= \int_{V_g} S_g(x + x^*)((v^* g, 0) \cdot f)(x\alpha + x^*\gamma) d(x^*\gamma) \\ &= \int_{V_g} S_g(x + x^*) \psi(\langle x\alpha + x^*\gamma, v^*\delta\rangle + \frac{1}{2}\langle v^*\gamma, v^*\delta\rangle) f(x\alpha + x^*\gamma + v^*\gamma) d(x^*\gamma) \end{aligned}$$

The Haar measure is translation invariant, hence we can replace x^* by $x^* - v^*$ to obtain

$$T_g(((v^*,0)\cdot g)\cdot f)(x) = \int_{V_g} S_g(x+x^*-v^*)\psi(\langle x\alpha + x^*\gamma - \frac{1}{2}v^*\gamma, v^*\delta\rangle)f(x\alpha+x^*\gamma)d(x^*\gamma)$$

However,

$$\begin{aligned}
&S_g(x+x^*-v^*)\psi(\langle x\alpha + x^*\gamma - \frac{1}{2}v^*\gamma, v^*\delta\rangle)\\
&= \psi(\frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle x^*\gamma - v^*\gamma, x^*\delta - v^*\delta\rangle + \langle x^*\gamma - v^*\gamma, x\beta\rangle + \langle x\alpha + x^*\gamma - \frac{1}{2}v^*\gamma, v^*\delta\rangle)\\
&= S_g(x+x^*)\psi(-\frac{1}{2}\langle v^*\gamma, x^*\delta\rangle - \frac{1}{2}\langle x^*\gamma, v^*\delta\rangle - \langle v^*\gamma, x\beta\rangle + \langle x\alpha + x^*\gamma, v^*\delta\rangle)\\
&= S_g(x+x^*)\psi(\langle x\alpha + x\beta, v^*\gamma + v^*\delta\rangle)\psi(\frac{1}{2}(\langle x^*\gamma, v^*\delta\rangle + \langle x^*\delta, v^*\gamma\rangle))\\
&= S_g(x+x^*)\psi(\langle x\alpha + x\beta, v^*\gamma + v^*\delta\rangle)\psi(\frac{1}{2}(\langle x^*\gamma + x^*\delta, v^*\delta\rangle + \langle x^*\gamma + x^*\delta, v^*\gamma\rangle))\\
&= S_g(x+x^*)\psi(\langle x\alpha + x\beta, v^*\gamma + v^*\delta\rangle)\psi(\frac{1}{2}(\langle x^*\gamma + x^*\delta, v^*\gamma + v^*\delta\rangle))\\
&= S_g(x+x^*)\psi(\langle x\alpha + x\beta, v^*\gamma + v^*\delta\rangle)\psi(\frac{1}{2}(\langle x^*g, v^*g\rangle))\\
&= S_g(x+x^*)\psi(\langle x\alpha + x\beta, v^*\gamma + v^*\delta\rangle) = S_g(x+x^*)\psi(\langle x, v^*\rangle)
\end{aligned}$$

which proves that $(v^*,0)\cdot(T_gf)(x) = T_g(((v^*,0)\cdot g)\cdot f)(x)$ and completes the proof. □

Our goal for the next few pages is to show that T_g is in fact a projective representation of $Sp(X,V^*)$ on $S(V)$. We will need many lemmas before we can get there. The general strategy is to reduce to the Weil representation in the finite-dimensional case, which is known to be a projective representation [7, Theorem 4.1 (5)].

Lemma 3.2.5. *For $v = x + x^* \in X$, define $Q(v) = Q(x+x^*) = \psi(\frac{\langle x,x^*\rangle}{2})$. Then we have*

$$S_g(x+x^*) = Q((x+x^*)g)Q(x+x^*)^{-1},$$

and

$$S_{g_1g_2}(x+x^*) = S_{g_1}(x+x^*)S_{g_2}((x+x^*)g_1).$$

Proof. We prove the result with a direct computation.

$$\begin{aligned}
Q((x+x^*)g)Q(x+x^*)^{-1} &= \psi\left(\frac{\langle x\alpha + x^*\gamma, x\beta + x^*\delta\rangle}{2}\right)\psi\left(-\frac{\langle x, x^*\rangle}{2}\right) \\
&= \psi\left(\frac{\langle x\alpha + x^*\gamma, x\beta + x^*\delta\rangle}{2} - \frac{\langle xg, x^*g\rangle}{2}\right) \\
&= \psi\left(\frac{\langle x\alpha + x^*\gamma, x\beta + x^*\delta\rangle}{2} - \frac{\langle x\alpha + x\beta, x^*\gamma + x^*\delta\rangle}{2}\right) \\
&= \psi\left(\frac{\langle x\alpha + x^*\gamma, x\beta + x^*\delta\rangle}{2} - \frac{\langle x\alpha, x^*\delta\rangle}{2} - \frac{\langle x\beta, x^*\gamma\rangle}{2}\right) \\
&= \psi\left(\frac{\langle x\alpha, x\beta\rangle}{2} + \frac{\langle x^*\gamma, x^*\delta\rangle}{2} + \langle x^*\gamma, x\beta\rangle\right) \\
&= S_g(x+x^*).
\end{aligned}$$

For the second claim, we have

$$\begin{aligned}
S_{g_1g_2}(x+x^*) &= Q((x+x^*)g_1g_2)Q(x+x^*)^{-1} \\
&= Q(((x+x^*)g_1)g_2)Q((x+x^*)g_1)^{-1}Q((x+x^*)g_1)Q(x+x^*)^{-1} \\
&= S_{g_2}((x+x^*)g_1)S_{g_1}(x+x^*) = S_{g_1}(x+x^*)S_{g_2}((x+x^*)g_1).
\end{aligned}$$

This completes the proof. □

Set $P = \{g \in Sp(X) : \gamma = 0\}$.

Proposition 3.2.6. *P is a subgroup of $Sp(X, V^*)$.*

Proof. It is clear that P is a subset of $Sp(X, V^*)$. We must show that it is closed under multiplication and inversion. Suppose $p_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{bmatrix}$ and $p_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & \delta_2 \end{bmatrix}$ are in P. Then we have

$$p_1p_2 = \begin{bmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 + \beta_1\delta_2 \\ 0 & \delta_1\delta_2 \end{bmatrix}$$

which is clearly in P. Next, write $p = \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}$. From Corollary 3.1.4, we have

$$p^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{bmatrix}.$$

Since $\gamma = 0$, we see that $\gamma^* = 0$, hence

$$p^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ 0 & \alpha^* \end{bmatrix}$$

which is clearly in P. □

Lemma 3.2.7. *Suppose* $p = \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix} \in P$. *Then* α *and* δ *are invertible.*

Proof. Write

$$p^{-1} = \begin{bmatrix} \delta^* & -\beta^* \\ 0 & \alpha^* \end{bmatrix}$$

Then

$$pp^{-1} = \begin{bmatrix} \alpha\delta^* & -\alpha\beta^* + \beta\alpha^* \\ 0 & \delta\alpha^* \end{bmatrix} = \begin{bmatrix} \mathbf{1}_V & 0 \\ 0 & \mathbf{1}_{V^*} \end{bmatrix}$$

Then α^* is a right inverse of δ. By computing $p^{-1}p$, we can easily find that α^* is also a left inverse of δ. Therefore, δ is invertible. Similarly, α is the inverse of δ^*. □

Lemma 3.2.8. *Suppose*

$$g = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \in Sp(X, V^*) \quad \textit{and} \quad p = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & \delta_2 \end{bmatrix} \in P$$

Then

1. $V_g = V_{pg}$. *Furthermore, if we choose the same Haar measure in* V_g *and* V_{pg} *and the counting measure for* $V_p = \{0\}$, *then* $T_pT_g = T_{pg}$.

2. $V_g\alpha_2 = V_{gp}$. *Furthermore, the Haar measures can be chosen so that* $T_gT_p = T_{gp}$.

Proof. First, note that

$$pg = \begin{bmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 + \beta_2\delta_1 \\ \delta_2\gamma_1 & \delta_2\delta_1 \end{bmatrix}.$$

By Lemma 3.2.7, we know that $V^*\delta_2 = V^*$, hence $V^*\delta_2\gamma_1 = V^*\gamma_1$. Therefore, $V_g = V_{pg}$. Now, we have

$$\begin{aligned}(T_pT_gf)(x) &= S_p(x)(T_gf)(x\alpha_2) \\ &= \int_{V_g} S_p(x)S_g(x\alpha_2 + x^*)f(x\alpha_2\alpha_1 + x^*\gamma_1)d(x^*\gamma_1).\end{aligned}$$

Because the Haar measure is translation invariant, we can replace x^* by $x^* + x\beta_2$ to obtain

$$(T_pT_gf)(x) = \int_{V_g} S_p(x)S_g(x\alpha_2 + x\beta_2 + x^*)f(x\alpha_2\alpha_1 + x^*\gamma_1 + x\beta_2\gamma_1)d(x^*\gamma_1). \quad (8)$$

By Lemma 3.2.5, and Lemma 3.2.7 we have

$$\begin{aligned} S_{pg}(x + x^*\delta_2^{-1}) &= S_g((x + x^*\delta_2^{-1})p)S_p(x + x^*\delta_2^{-1}) \\ &= S_g(x\alpha_2 + x\beta_2 + x^*)S_p(x + x^*\delta_2^{-1}) \\ &= S_g(x\alpha_2 + x\beta_2 + x^*)\psi(\frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle 0x^*, x^*\rangle + \langle 0x^*, x\beta\rangle) \\ &= S_g(x\alpha_2 + x\beta_2 + x^*)S_p(x) = S_p(x)S_g(x\alpha_2 + x\beta_2 + x^*). \end{aligned}$$

Furthermore, $\gamma_3 = \delta_2\gamma_1$ implies that $\gamma_1 = \delta_2^{-1}\gamma_3$. Then equation 8 becomes

$$(T_pT_gf)(x) = \int_{V_{pg}} S_{pg}(x + x^*\delta_2^{-1})f(x\alpha_3 + x^*\delta_2^{-1}\gamma_3)d(x^*\delta_2^{-1}\gamma_3)$$

which is exactly $(T_{pg}f)(x)$ with integration variable $x^*\delta_2^{-1}$ instead of x^*.

The second statement is verified very similarly. However, we would like to make it clear how we choose the Haar measures. If $v_1, ..., v_n$ is a basis for V_g, then $v_1\alpha_2, ..., v_n\alpha_2$ is also a basis for $V_g\alpha_2$ because α_2 is invertible. Then if we take the product measure $dv_1...dv_n$, we obtain a Haar measure for V_g. But then, $d(v_1\alpha_2)...d(v_n\alpha_2)$ is also a Haar measure for V_{pg}. □

The following is an immediate corollary to Lemma 3.2.9 :

Corollary 3.2.9. *Suppose g and p are given as in the previous lemma. Then we have for any choice of Haar measures*

1. $T_pT_g \sim T_{pg}$
2. $T_gT_p \sim T_{gp}$

where $a \sim b$ means that $a = cb$ for some $c \in \mathbb{F}$.

Now, suppose $g \in Sp(X)$. We let $X^g = \{v \in X : vg = v\}$. It is clear that X^g is a subspace of X. Then let $Sp_{\text{fin}}(X) = \{g \in Sp(X) : X^g \text{ has finite co-dimension in } X\}$.

Lemma 3.2.10. *Suppose $g \in Sp(X)$ and that $U \subseteq X$ is a subspace of X of dimension at least k such that for all $v \in U \setminus \{0\}$, $vg \neq v$. Then, X^g has co-dimension in X at least k.*

Proof. We wish to show that $\dim X/X^g \geq k$. Let $v_1, ..., v_k$ be a linearly independent subset of U. Suppose that

$$a_1(v_1 + X^g) + a_2(v_2 + X^g) + ... + a_k(v_k + X^g) = 0.$$

Then $a_1v_1 + a_2v_2 + ... + a_kv_k \in X^g$, hence $a_1v_1 + a_2v_2 + ... + a_kv_k = 0$. Therefore, $a_1 = a_2 = ... = a_k = 0$ by linear independence and $v_1 + X^g, ..., v_k + X^g$ are linearly independent. □

Lemma 3.2.11. *$Sp_{\text{fin}}(X)$ is a subgroup of $Sp(X, V^*)$.*

Proof. First, we must prove $Sp_{\text{fin}}(X)$ is a subset of $Sp(X, V*)$. Suppose $g \in Sp_{\text{fin}}(X)$ and as usual write

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

We need to show that $\dim \operatorname{Im} \gamma < \infty$. Suppose by contradiction that $\dim \operatorname{Im} \gamma = \infty$. Then we can choose infinitely many linearly independent elements $f_1^*, f_2^*, ...$ of V^* such that $f_i^*\gamma$ are linearly independent. In particular, $U = \operatorname{span}\{f_1^*, f_2^*, ...\}$ satisfies the property that if $v \in U \setminus \{0\}$, then $vg \neq v$, hence X^g has infinite co-dimension in X.

Next, we wish to prove that $Sp_{\text{fin}}(X, V^*)$ is a subgroup by the subgroup test. Suppose $g_1, g_2 \in Sp_{\text{fin}}(X)$. Now, it is clear that $X^{g_1} \cap X^{g_2} \subseteq X^{g_1g_2}$. Then

$$\operatorname{codim}(X^{g_1g_2}) \leq \operatorname{codim}(X^{g_1} \cap X^{g_2}) \leq \operatorname{codim}(X^{g_1}) + \operatorname{codim}(X^{g_2}) < \infty.$$

Therefore, $X^{g_1g_2}$ has finite co-dimension as well. Now, suppose $g \in Sp_{\text{fin}}(X)$. We claim that $X^g = X^{g^{-1}}$. Indeed, if $vg = v$, then $v = vg^{-1}$ by applying g^{-1} on each side. But then $X^g \subseteq X^{g^{-1}}$ and the other inclusion follows by substituting g by g^{-1} into $X^g \subseteq X^{g^{-1}}$. □

The following lemma is a standard exercise in linear algebra and will be useful in the next two lemmas.

Lemma 3.2.12. *Suppose $\{f_1, ..., f_n\}$ are linearly independent elements of V^*. Then there exists $x_1, ..., x_n \in V$ such that $\langle f_i, x_j \rangle \delta_{i,j}$.*

Proof. This can be found in [2, p. 53, Exercise 9]. The proof is also available at the end of the book. □

For an arbitrary subspace U of V^*, define $U^\perp = \{v \in V : \langle v, U \rangle = 0$. Similarly, for a subspace U of V, define $U^\perp = \{v \in V^* : \langle v, U \rangle = 0\}$.

Definition 3.2.13. We say that a subspace W of X is a *Lagrangian subspace* if for all $v \in X$, $\langle v, W \rangle = 0$ if and only if $v \in W$.

Lemma 3.2.14. *Suppose $g \in Sp(X, V^*)$. Then $g = pg'$, for $p \in P$ and $g' \in Sp_{\text{fin}}(X)$.*

Proof. Denote by $Gr(X, V^*)$ the set of all Lagrangian subspaces W of X such that $W \cap V^*$ has finite co-dimension in both W and V^*. We will first prove that $Sp_{\text{fin}}(X)$ acts on $Gr(X, V^*)$ transitively. Write $\pi_V : X \to V$, $\pi_{V^*} : X \to V^*$ as the canonical projection operators with respect to the direct sum decomposition $X = V \oplus V^*$ and suppose $W \in Gr(X, V^*)$.

Step 1 : We claim that there exists a subspace $K \subseteq X$ such that $W = W \cap V^* \oplus K$ and

$$(\pi_{V^*} K) \cap (W \cap V^*) = \{0\}. \tag{9}$$

To prove this, first choose finite-dimensional subspaces of $L \subseteq W$ and $J \subseteq V^*$ such that $W = W \cap V^* \oplus L$ and $V^* = W \cap V^* \oplus J$. Denote by $\pi_{W \cap V^*} : V^* \to W \cap V^*$ the canonical projection with respect to the decomposition $V^* = W \cap V^* \oplus J$. Now, set

$$K = \{l - \pi_{W \cap V^*} \pi_{V^*}(l) : l \in L\}.$$

We claim that K satisfies the required properties. First, suppose $v \in W \cap V^*$ and $v \in K$. Then $v = l - \pi_{W \cap V^*} \pi_{V^*}(l)$ for some l. But then, $v \in W \cap V^*$, hence $v = \pi_{V^*}(v)$ and $v = \pi_{W \cap V^*} \pi_{V^*}(v)$ and therefore

$$v = \pi_{W \cap V^*} \pi_{V^*}(v) = \pi_{W \cap V^*} \pi_{V^*}(l) - \pi_{W \cap V^*} \pi_{V^*}(l) = 0.$$

Now, suppose $v \in W$. Then $v = w + l$, $w \in W \cap V^*, l \in L$. But then, we also have

$$v = (w + \pi_{W \cap V^*} \pi_{V^*}(l)) + (l - \pi_{W \cap V^*} \pi_{V^*}(l)) \in W \cap V^* + K.$$

Therefore, $W = W \cap V^* \oplus K$. It remains to verify Eq.(9). To do this, suppose that $v \in (\pi_{V^*}K) \cap (W \cap V^*)$. Then $v = \pi_{V^*}(l - \pi_{W\cap V^*}\pi_{V^*}(l))$ for some l. But then, $v \in W \cap V^*$, hence $\pi_{W\cap V^*}v = v$. Therefore, we have

$$v = \pi_{W\cap V^*}(v) = \pi_{W\cap V^*}\pi_{V^*}(l) - \pi_{W\cap V^*}\pi_{V^*}(l) = 0,$$

as desired.

Step 2 : We wish to show that $(W \cap V^*)^\perp = \pi_V(W)$. Firstly, suppose $v \in \pi_V(W)$. Then $v = \pi_V(u)$ for some $u \in W$. Write $u = v + v'$, where $v \in V, v' \in V^*$. Now suppose that $w \in W \cap V^*$. Then because W is Lagrangian, we must have that $\langle w, u\rangle = 0$. Therefore,

$$0 = \langle w, u\rangle = \langle w, v + v'\rangle = \langle w, v\rangle$$

hence $v = \pi_V(u) \in (W \cap V^*)^\perp$.

Secondly, suppose that $v \in (W \cap V^*)^\perp$. Choose a basis $k_1, ..., k_s$ of K (recall that K has finite dimension). For each i, write $k_i = v_i + v_i^*$, for $v_i \in V, v_i^* \in V^*$. Now, choose $w_1^*, ..., w_s^* \in V^*$ such that $(w_i^*, v_j) = \delta_{i,j}$ as in the proof of Lemma 3.2.1. We claim that $v + \sum_{i=1}^{s}\langle v_i^*, v\rangle w_i^*$ is in W. Indeed, if $w \in W \cap V^*$, we have

$$\langle v + \sum_{i=1}^{s}(v_i^*, v)w_i^*, w\rangle = 0$$

because $v \in (W \cap V^*)^\perp$ and $w_i^* \in V^*$. If $w \in K$, write $w = \sum_{j=1}^{s} a_j k_j$. Then

$$\begin{aligned}
\langle v + \sum_{i=1}^{s}(v_i^*, v)w_i^*, \sum_{j=1}^{s} a_j k_j\rangle &= \langle v, \sum_{j=1}^{s} a_j k_j\rangle - \langle \sum_{j=1}^{s} a_j k_j, \sum_{i=1}^{s}(v_i^*, v)w_i^*\rangle \\
&= (v, \sum_{j=1}^{s} a_j v_j^*) - (\sum_{j=1}^{s} a_j v_j, \sum_{i=1}^{s}(v_i^*, v)w_i^*) \\
&= \sum_{j=1}^{s} a_j(v_j^*, v) - \sum_{i,j=1}^{s}(v_i^*, v)(w_i^*, a_j v_j) \\
&= \sum_{j=1}^{s} a_j(v_j^*, v) - \sum_{j=1}^{s} a_j(v_j^*, v) = 0.
\end{aligned}$$

Therefore, $\langle v + \sum_i (v_i^*, v) w_i^*, w \rangle = 0$ for all $w \in W$, hence, $v + \sum (v_i^*, v) w_i^* \in W$ (because W is Lagrangian) and consequently $v = \pi_V(v + \sum (v_i^*, v) w_i^*) \in \pi_V(W)$, as claimed.

Step 3 : We extend $\pi_{V^*}(K)$ to a finite-dimensional subspace M of V^* so that $V^* = W \cap V^* \oplus M$. We can do this because $W \cap V^*$ has finite co-dimension in V^* and because of Eq. (9). We claim that $V = (W \cap V^*)^\perp \oplus M^\perp$.

Firstly, suppose that $v \in (W \cap V^*)^\perp \cap M^\perp$. Then $\langle v, u \rangle = 0$ for all $u \in W \cap V^* + M$. But then, $(u, v) = 0$ for all $u \in V^*$, hence $v = 0$.

Secondly, we show that $V = (W \cap V^*)^\perp + M^\perp$. We do so by dimension counting. Supppose $M = \text{span}\{v_1^*, ..., v_k^*\}$. Consider the map $\phi : V \to \mathbb{F}^k$ defined by

$$\phi(v) = (v_1^*(v), ..., v_k^*(v)).$$

The kernel of this map is precisely $M^\perp$ by definition. Furthermore, ϕ is surjective because of Lemma 3.2.12. Therefore, $\dim V/M^\perp = k = \dim M$ and we conclude that $M^\perp$ has co-dimension k. We wish to show that $\dim(W \cap V^*)^\perp = k$. The dimension of $(W \cap V^*)^\perp$ is at most k because otherwise the sum $(W \cap V^*)^\perp \cap M^\perp \neq \{0\}$. Now, suppose $\dim(W \cap V^*)^\perp < k$ and let $v_1, ..., v_s$ be a basis for $(W \cap V^*)^\perp$. Consider the map, $\psi : V^* \to \mathbb{F}^s$, defined by $\psi(f) = (f(v_1), ..., f(v_s))$. ψ is surjective because $v_1, ..., v_s$ are linearly independent. Furthermore, it is easy to see that $W \cap V^* \subseteq \ker \psi$. We claim that we actually have equality.

Suppose $h \in \ker g$. Then $\langle h, v \rangle = 0$ for all $v \in (W \cap V^*)^\perp = \pi_V(W)$ by Step 2. Now suppose $v \in W$. Then write $v = u + u^*$, $u \in \pi_V(W), u^* \in \pi_{V^*}(W)$. Then $\langle h, v + v^* \rangle = \langle h, v \rangle = 0$ by assumption. Therefore, $h \in W$ because W is Lagrangian, and then $h \in W \cap V^*$, as desired. Therefore, $\ker \psi = W \cap V^*$ and this shows that $s = \dim V^* / \ker \psi = \dim V^*/(W \cap V^*) = k$, hence $k = s$.

Combining the two steps, we conclude that $V = (W \cap V^*)^\perp \oplus M^\perp$.

Step 4 : The last three steps prove that

$$X = (W \cap V^* \oplus M^\perp) \oplus (M \oplus \pi_V(W))$$

We claim that $Sp_{\text{fin}}(X)$ acts on $Gr(X, V^*)$ transitively.

It is easy to see that $M \oplus \pi_V(W)$ is a finite dimensional symplectic space with the restriction of the bilinear form $\langle ., . \rangle$. (The restriction is clearly antisymmetric and non-degeneracy follows from knowing that $M^\perp \cap \pi_V(W) = \{0\}$ and $W \cap V^* \cap M = \{0\}$). In particular, both M and $\pi_V(W)$ are isotropic subspaces of dimension k, hence are Lagrangian subspaces of $M \oplus \pi_V(W)$. Also, $K \subseteq M \oplus \pi_V(W)$ because $\pi_{V^*}(K) \subseteq M$, and $\pi_V(K) = \pi_V(W)$. Since $K \subseteq W$, we have that K is isotropic. In fact, we claim that K is Lagrangian in $M \oplus \pi_V(W)$.

Indeed, suppose that $a \in M + \pi_V(W)$ such that $\langle a, K \rangle = 0$. Since both $M \subseteq V^*$ and $W \cap V^*$, we see that $\langle M, W \cap V^* \rangle = 0$. Furthermore,

$$\langle \pi_V(W), W \cap V^* \rangle = \langle (W \cap V^*)^\perp, W \cap V^* \rangle = 0$$

Then we have

$$\langle a, K + W \cap V^* \rangle = \langle a, W \cap V^* \rangle \in \langle M + \pi_V(W), W \cap V^* \rangle = 0$$

and then $\langle a, W \rangle = 0$. Therefore, $a \in W$ because W is Lagrangian. Then write $a = u + v$ where $u \in W \cap V^*$ and $v \in K$. Then $u = a - v \in W$ and $a \in M \oplus \pi_V(W)$ by assumption. But we know that $W \cap (M \oplus \pi_V(W)) = \{0\}$, hence $u = 0$ and $a = v \in K$. Therefore, K is a Lagrangian subspace.

Now, from the theory of finite dimensional symplectic spaces [7, Lemma 2.1], there exists $g'' \in Sp(M + \pi_V W)$ such that $Mg'' = K$. Extend g'' to a map g' on X linearly such that it restricts to the identity on $W \cap V^* \oplus M^\perp$. It is easy to check that $g' \in Sp(X)$. In particular, $g' \in Sp_{\text{fin}}(X)$. Then we have $V^* g' = W$ and the action of $Sp_{\text{fin}}(X)$ on $Gr(X, V^*)$ is transitive.

Step 5 : We complete the proof of the the lemma. Suppose $g \in Sp(X, V^*)$. Then we claim that $V^* g = V^*(\gamma + \delta) \in Gr(X, V^*)$. Indeed, it is clear that $V^* g$ is isotropic. To see why it is Lagrangian, suppose $v \in X$ such that $\langle v, V^* g \rangle = 0$. Then $\langle v g^{-1}, V^* \rangle = 0$ implies that $v g^{-1} \in V^*$ or equivalently that $v \in V^* g$. It remains to prove that $V^* g \cap V^*$ has finite co-dimension in both V^* and $V^* g$. It is enough to prove that $[V^* g \cap V^* : V^* g] < \infty$ because $[V^* g \cap V^* : V^*] = [V^* \cap V^* g^{-1} : V^* g^{-1}]$. Suppose that $x = v + v^* = yg \in V^* g$. Choose elements $f_1^*, ..., f_n^*$ such that $f_1^* \gamma, ..., f_n^* \gamma$ is a basis for $\operatorname{Im} \gamma$. Then, $v = (a_1 f_1^* + ... + a_n f_n^*)\gamma$ for some $a_1, ..., a_n$. In particular,

$(y - a_1 f_1^* - ... - a_n f_n^*)g \in V^* \cap V^* g$. Therefore, $V^* \cap V^* g + \text{span}\{f_1^* g, ..., f_n^* g\} = V^* g$, hence $[V^* \cap V^* g : V^* g] < \infty$.

Now, by step 4, there exists $g' \in Sp_{\text{fin}}(X)$ such that $V^* g = V^* g'$ by transitivity. Therefore, $V^* g g'^{-1} = V^*$ or equivalently $g g'^{-1} \in P$. We conclude that $g = pg'$ for some $p \in P, g' \in Sp_{\text{fin}}(X)$. This completes the proof of this lemma. □

The next lemma is the last one we will need before we can prove the main result of this chapter.

Lemma 3.2.15. *If g_1, g_2 are in $Sp_{\text{fin}}(X)$, then there exists V_1, V_2 such that $V = V_1 \oplus V_2^\perp$, $V^* = V_2 \oplus V_1^\perp$, $V_1^\perp \oplus V_2^\perp$ is fixed pointwise by g_1, g_2 and $V_1 \oplus V_2$ is finite-dimensional and g_1, g_2 invariant.*

Furthermore, if $J \subseteq V$ is an arbitrary finite-dimensional subspace, then we can choose V_1 and V_2 so that $J \subseteq V_1$.

Proof. Write $g_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$ and $g_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}$. Similarly to Proposition 3.2.11, it can be proven that $\text{rk}(\beta_1)$, $\text{rk}(\beta_2)$, $\text{rk}(\alpha_1 - I_V)$, $\text{rk}(\alpha_2 - I_V)$, $\text{rk}(\gamma_1)$, $\text{rk}(\gamma_2)$, $\text{rk}(\delta_1 - I_{V^*})$, $\text{rk}(\delta_2 - I_{V^*})$ are all finite. We claim that we can choose $M, N \subseteq V$ such that $V = M \oplus N$,

$$M_1 := \text{Im}(\alpha_1 - I_V) + \text{Im}(\alpha_2 - I_V) + \text{Im}(\gamma_1) + \text{Im}(\gamma_2) \subseteq M,$$

and

$$M_2 := \text{Im}(\beta_1) + \text{Im}(\beta_2) + \text{Im}(\delta_1 - I_{V^*}) + \text{Im}(\delta_2 - I_{V^*}) \subseteq N^\perp.$$

In fact, we prove the following more general statement : we can choose M, N such that $V = M \oplus N$ and for any finite-dimensional subspaces $M_1 \subseteq V$ and $M_2 \subseteq V^*$, we have that $M_1 \subseteq M$ and $M_2 \subseteq N^\perp$.

Choose a basis $\{w_1^*, ..., w_n^*\}$ for M_2. For $n = 0$, there is nothing to show (one simply takes $A = M_1$). Choose a complementary subspace N for M_1. If $M_2 \subseteq N^\perp$, there is nothing to show, hence assume this is not the case. From Lemma 3.2.12 there exists linearly independent vectors $v_1, ..., v_n \in V$ such that $\langle w_i^*, v_j \rangle = \delta_{i,j}$. Suppose that $\{n_i\}_{i \in I}$ is a basis for N. Then for each $i \in I$, let

$$n_i' = n_i - \sum_{j=1}^{n} \langle w_j^*, n_i \rangle v_j$$

and set $M' = C_1 \oplus \mathbb{F}v_1 \oplus ... \oplus \mathbb{F}v_n$ as well as $N' = \text{span}\{b_i'\}_{i\in I}$. M' is clearly still finite-dimensional, $M_1 \subseteq M'$, and we still have $M' + N' = V$. Furthermore, we now have $w_i \in N'^{\perp}$ by construction. To obtain a direct sum $M' \oplus N'$, it is enough to extend a basis of the intersection of $M' \cap N'$ to a basis of M', and then remove the elements from the intersection of $M' \cap N'$. What we have left is a basis for a subspace N'' which still satisfies $M_2 \subseteq N''^{\perp}$ (because $N'' \subseteq N'$ implies that $N'^{\perp} \subseteq N''^{\perp}$), but also satisfies $M' \oplus N'' = V$.

Now, one can write g_1 in the following way

$$T = \begin{bmatrix} A & 0 & C & 0 \\ A' & \mathbf{1}_N & C' & 0 \\ B & 0 & D & 0 \\ B' & 0 & D' & \mathbf{1}_{M^\perp} \end{bmatrix}$$

where $A : M \to N$, $A' : N \to M$, $C : M \to N^\perp$, $C' : N \to N^\perp$, $B : N^\perp \to M$, $B' : N^\perp \to M$, $D : N^\perp \to N^\perp$ and $D' : M^\perp \to N^\perp$. We obtain zeroes in the other entries because M and $N^\perp$ contain the images of the maps $\beta_1, \gamma_1, \alpha_1 - I_V$ and $\delta_1 - I_{V^*}$. If $v \in X$, then $v = v_1 + v_2 + v_1^* + v_2^*$ for $v_1 \in M, v_2 \in N, v_1^* \in M^\perp, v_2^* \in N^\perp$ and

$$vg_1 = \begin{bmatrix} v_1 & v_2 & v_2^* & v_1^* \end{bmatrix} T.$$

Clearly, A, B, C, D are maps between finite dimensional spaces, hence their transposes exist and are unique. In fact, the transpose of all the other maps also exist : for example, $D'^* = (\delta^* - D^*)\pi_N : M \to N$. Indeed, if $v \in M^\perp$ and $w \in M$, we have

$$\langle vD', w\rangle = \langle v(D'+D) - vD, w\rangle = \langle v\delta - vD, w\rangle = \langle v, w(\delta^* - D^*)\rangle = \langle v, w(\delta^* - D^*)\pi_N\rangle.$$

Therefore, D'^* exist. A similar argument holds for all the other transposes.

Now, we can also write g_1^{-1} using Corollary 3.1.4 and uniqueness of the transpose to obtain

$$\begin{bmatrix} D^* & D'^* & -C^* & -C'^* \\ 0 & \mathbf{1}_N & 0 & 0 \\ -B^* & -B'^* & A^* & A'^* \\ 0 & 0 & 0 & \mathbf{1}_{M^\perp} \end{bmatrix}.$$

Taking the product $g_1^{-1}g_1 = I$, we obtain $A' = 0$, $C' = 0$, $B' = 0$, $D' = 0$.

Therefore, g_1 is actually given by the matrix

$$T = \begin{bmatrix} A & 0 & C & 0 \\ 0 & \mathbf{1}_N & 0 & 0 \\ B & 0 & D & 0 \\ 0 & 0 & 0 & \mathbf{1}_{M^\perp} \end{bmatrix}$$

and $V_1 \oplus V_2$ is then g_1-invariant. A similar proof holds for g_2.

The second statement of the lemma follows, but with

$$M_1 := \mathrm{Im}(\alpha_1 - I_V) + \mathrm{Im}(\alpha_2 - I_V) + \mathrm{Im}(\gamma_1) + \mathrm{Im}(\gamma_2) + J \subseteq M$$

instead. □

Theorem 3.2.16. *The map $g \mapsto T_g$ is a projective representation of $Sp(X, V^*)$ on $S(V)$.*

Proof. First, we claim that $T_g f \in S(V)$ for any $g \in Sp(X, V^*)$ and $f \in S(V)$. By Lemma 3.2.14, we have that $g^{-1} = p^{-1}g'^{-1}$ for $p \in P$ and $g' \in Sp_{\mathrm{fin}}(X)$ and then $g = g'p$. Therefore, by Corollary 3.2.9, $T_g f \sim T_{g'} T_p f$. Write $g' = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$ and $p = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & \delta_2 \end{bmatrix}$. Then

$$(T_p f)(x) = S_p(x) f(x\alpha_2) = \psi(\frac{1}{2}\langle x\alpha_2, x\beta_2\rangle) f(x\alpha_2) = ((x\beta_2, 0) \cdot f)(x\alpha_2).$$

However, α_2 is invertible and linear by Lemma 3.2.7 and $(x\beta_2, 0) \cdot f \in S(V)$, hence $T_p f \in S(V)$. Next, suppose $W \subseteq V$ is finite-dimensional and $h \in S(V)$. By Lemma 3.2.15, choose V_1 and V_2 such that $V = V_1 \oplus V_2^\perp$, $V^* = V_2 \oplus V_1^\perp$, $V_1^\perp \oplus V_2^\perp$ is fixed pointwise by g' and $V_1 \oplus V_2$ is finite-dimensional, g'-invariant and $W \subseteq V_1$ (take $g_1 = g'$ and g_2 to be the identity). Define $h' = h|_{V_1}$. Clearly, $h' \in S(V_1)$ by definition. Then $(T_{g'} h)|_W = T'_{g'}(h')|_W$ where T' is the finite-dimensional Weil representation on the finite dimensional symplectic space $V_1 \oplus V_2$. However, by [7, Lemma 3.2 (3)], $T'_{g'} h' \in S(V_1)$, hence $T_{g'} h|_W \in S(W)$. Therefore, $T_{g'} h \in S(V)$. We conclude that $T_g f \in S(V)$.

Next, we claim that T_g is a projective representation. Suppose $g_1, g_2 \in Sp(X, V^*)$. From Lemma 3.2.14, we have $g'_1 \in Sp_{\mathrm{fin}}(X)$ such that $g_1 = p_1 g'_1$ for some $p_1 \in P$.

From the same lemma, we have $(g_2')^{-1} \in Sp_{\text{fin}}(X)$ such that $g_2^{-1} = p_2^{-1}(g_2')^{-1}$ for some $p_2 \in P$. Equivalently, $g_2 = g_2'p_2$. Now, from Corollary 3.2.9, we have $T_{g_1}T_{g_2} \sim T_{p_1}T_{g_1'}T_{g_2'}T_{p_2}$. But now, from Lemma 3.2.15, there exists V_1, V_2 such that $V = V_1 \oplus V_2^{\perp}$, $V^* = V_2 \oplus V_1^{\perp}$, $V_1^{\perp} \oplus V_2^{\perp}$ is fixed pointwise by g_1 and g_2 and $V_1 \oplus V_2$ is finite-dimensional and g_1,g_2 invariant.

Suppose that $f \in S(V)$. For each $v_2 \in V_2^{\perp}$, define $f_{v_2} : V_1 \to \mathbb{C}$ by $f_{v_2}(v_1) = f(v_1 + v_2)$. It is not hard to see that $f_{v_2} \in S(V_1)$. Then $T_{g_1'}f(v_1 + v_2) = T'_{g_1'}f_{v_2}(v_1)$ where $T'_{g_1'}$ is the Weil representation on the finite dimensional symplectic space $V_1 \oplus V_2$. Now, we know from the finite-dimensional case ([7], Theorem 4.1 (5)) that $T'_{g_1'}T'_{g_2'} = c(g_1', g_2')T'_{g_1'g_2'}$ for some number c. Therefore, for every $f \in S(V)$, $v_1 \in V_1$ and $v_2 \in V_2^{\perp}$, we have

$$T_{g_1'}T_{g_2'}f(v_1 + v_2) = T'_{g_1'}T'_{g_2'}f_{v_2}(v_1) = c(g_1', g_2')T'_{g_1'g_2'}f_{v_2}(v_1) = c(g_1', g_2')T_{g_1'g_2'}f(v_1 + v_2).$$

In particular, $T_{g_1'}T_{g_2'} \sim T_{g_1'g_2'}$. We conclude that

$$T_{g_1}T_{g_2} \sim T_{p_1}T_{g_1'}T_{g_2'}T_{p_2} \sim T_{p_1}T_{g_1'g_2'}T_{p_2} \sim T_{p_1g_1'g_2'p_2} = T_{g_1g_2}$$

where the last $\sim$ follows from Corollary 3.2.9 applied twice. Therefore, $g \mapsto T_g$ is a projective representation. □

The group $Sp(X, V^*)$ is difficult to work with. In most cases, we prefer to restrict to a specific subgroup. Consider $V = (t^{-1}\mathbb{F}[t^{-1}])^{2n}$, the direct sum of $2n$ copies of the vector space of polynomials in t^{-1} with 0 constant term. For an element $p(t) \in \mathbb{R}((t))$, denote by $\text{Res}(p)$ the coefficient of t^{-1}. Then we have the following :

Lemma 3.2.17. *Suppose $V = (t^{-1}\mathbb{F}[t^{-1}])^{2n}$. Consider $W = \mathbb{F}[[t]]^{2n}$, $2n$ copies of the vector space of formal power series. For $v = \sum_{i=1}^{2n} t^{-1}p_i(t^{-1})e_i$ where p_i are polynomials and also for $w = \sum_{i=1}^{2n} q_i(t)e_i$ where q_i are formal power series, define a pairing*

$$(v, w)_W = \sum_{i=1}^{n} \text{Res}(p_iq_{i+n} - p_{i+n}q_i)$$

Then, the pairing results in an isomorphism $W \cong V^*$.

Proof. Clearly, we have a map $\psi : W \to V^*$, defined by $(\psi(w), v) = (w, v)_W$. Linearity of ψ is easy. We show that ψ is injective and surjective.

Suppose $\psi(w) = 0$. Then $(w, v) = 0$ for all $v \in V$. Write $w = \sum_{i=1}^{2n} \sum_{j\geq 0} a_{i,j} t^j e_i$. Then

$$0 = (w, t^{-j} e_i) = \pm a_{i,j-1}.$$

Since we can do this for all $i \in \mathbb{Z}, j \geq 1$, we obtain that $w = 0$.

Now suppose $f \in V^*$. Then set $a_{i+n,j} = -(f, t^{-j-1} e_i)$ for $i \leq n$, $b_{i-n,j} = (f, t^{-j-1} e_i)$ for $i \geq n+1$ and $w = \sum_{i=1}^{n} \sum_{j\geq 0} b_{i,j} t^j e_i + \sum_{i=n+1}^{2n} \sum_{j\geq 0} a_{i,j} t^j e_i$. It is enough to check that $(\psi(w), t^{-j-1} e_i) = (f, t^{-j-1} e_i)$, because the set $\{t^{-j-1} e_i\}_{j\geq 0}$ forms a basis for V.

For $i \leq n$, we have

$$(\psi(w), t^{-j-1} e_i) = \mathrm{Res}(-\sum_{k\geq 0} a_{i+n,k} t^k t^{-j-1}) = -a_{i+n,j} = (f, t^{-j-1} e_i).$$

For $i \geq n+1$, we have

$$(\psi(w), t^{-j} e_i) = \mathrm{Res}(\sum_{k\geq 0} b_{i-n,k} t^k t^{-j-1}) = b_{i-n,j} = (f, t^{-j-1} e_i)$$

as desired. We conclude that ψ is an isomorphism. □

Now, recall that $X = V \oplus V^*$, hence $X = \mathbb{R}((t))^{2n}$, a direct sum of $2n$ copies of the field discussed in Section 2.1. It will be useful to have an explicit formula for the symplectic form on X. Recall that for a pairing $(.,.)$ between a vector space V and its dual V^*,

$$\langle v + v^*, w + w^* \rangle = (v, w^*) - (v^*, w).$$

Corollary 3.2.18. *Suppose* $v = \sum_{i=1}^{2n} p_i(t) e_i$ *and* $w = \sum_{i=1}^{2n} q_i(t) e_i$ *where* $p_i, q_i \in \mathbb{R}((t))$. *Then*

$$\langle v, w \rangle = \sum_{i=1}^{n} \mathrm{Res}(p_i q_{i+n} - p_{i+n} q_i).$$

Proof. Write $p_i(t) = a_i(t) + b_i(t)$ and $q_i(t) = c_i(t) + d_i(t)$ where $a_i, c_i \in t^{-1}\mathbb{R}[t^{-1}]$ and

$b_i, d_i \in \mathbb{R}[[t]]$. Then

$$\begin{aligned}
\langle v, w\rangle &= \left(\sum_{i=1}^{2n} a_i e_i, \sum_{j=1}^{2n} d_i e_i\right)_W - \left(\sum_{i=1}^{2n} b_i e_i, \sum_{j=1}^{2n} c_i e_i\right)_W \\
&= \mathrm{Res}\left(\sum_{i=1}^{n} a_i d_{i+n} - a_{i+n} d_i\right) - \left(\sum_{i=1}^{n} c_i b_{i+n} - c_{i+n} b_i\right) \\
&= \sum_{i=1}^{n} \mathrm{Res}\left(a_i d_{i+n} - a_{i+n} d_i - c_i b_{i+n} + c_{i+n} b_i\right) \\
&= \sum_{i=1}^{n} \mathrm{Res}\left(a_i c_{i+n} + a_i d_{i+n} - a_{i+n} c_i - a_{i+n} d_i - d_i b_{i+n} - c_i b_{i+n} + d_{i+n} b_i + c_{i+n} b_i\right) \\
&= \sum_{i=1}^{n} \mathrm{Res}\left(a_i q_{i+n} - a_{i+n} q_i - q_i b_{i+n} + b_i q_{i+n}\right) \\
&= \sum_{i=1}^{n} \mathrm{Res}\left(p_i q_{i+n} - p_{i+n} q_i\right).
\end{aligned}$$

□

Consider the $\mathbb{R}((t))$-valued form on X

$$\left\langle \sum_{i=1}^{2n} p_i e_i, \sum_{j=1}^{2n} q_j e_j \right\rangle_{\mathbb{R}((t))} = \sum_{i=1}^{n} p_i q_{i+n} - p_{i+n} q_i$$

Then, it is clear by Corollary 3.2.18 that any isometry of $\langle ., . \rangle_{\mathbb{R}((t))}$ is also an isometry of $\langle ., . \rangle$, hence $Sp_{2n}(\mathbb{R}((t))) \subseteq Sp(X)$. However, we actually have the following stronger statement :

Lemma 3.2.19. *$Sp_{2n}(\mathbb{F}((t)))$ is a subgroup of $Sp(X, V^*)$*

Proof. It is well known that $Sp_{2n}(\mathbb{F}((t)))$ is a group. We wish to show that $Sp_{2n}(\mathbb{F}((t)))$ is a subset of $Sp(X, V^*)$. Suppose that $A \in Sp_{2n}(\mathbb{F}((t)))$. Then we can write

$$A = \sum_{i=1}^{2n} \sum_{j=1}^{2n} p_{i,j} e_{i,j}$$

where $p_{i,j} \in \mathbb{F}((t))$.

The product of A with a vector $v^* = \sum_{k=1}^{2n} q_k e_k$ in V^* is $\sum_{i=1}^{2n} (\sum_{j=1}^{2n} p_{i,j} q_j) e_i$. However, v^* is in V^*, hence $v(q_j) \geq 0$ for all j. Then $v(p_{i,j} q_j) \geq v(p_{i,j})$. Let $m = \min\{v(p_{i,j})\}$.

Therefore, we have $v^*A \in (\mathbb{F}t^{-m} \oplus ... \oplus \mathbb{F}t^{-1} \oplus \mathbb{F}[[t]])^{2n}$ for any $v^* \in V^*$. But this means that

$$\operatorname{Im}\gamma \subseteq (\mathbb{F}t^{-m} \oplus ... \oplus \mathbb{F}t^{-1})^{2n}$$

hence $\dim \operatorname{Im}\gamma \leq 2nm < \infty$, as desired.

We can define the Weil representation on $Sp_{2n}(\mathbb{F}((t)))$ to be the restriction of $g \mapsto T_g$ on $Sp_{2n}(\mathbb{F}((t)))$. By Theorem 3.2.16, we know that T_g is a projective representation. □

We will use the following fact about symplectic matrices, a proof of which can be found in [13, p .167] using a polynomial known as the Pfaffian.

Lemma 3.2.20. *Suppose* $g \in Sp_{2n}(\mathbb{F}((t)))$. *Then* $\det(g) = 1$.

Chapter 4

Continuity of the representations

In this chapter, we will establish some continuity results of the Heisenberg representation and the projective representation defined in the previous chapter. We will restrict ourselves to the case $\mathbb{F} = \mathbb{R}$. Let V be an arbitrary vector space. We start by dealing with $\dim V < \infty$ and then we will switch to $V = t^{-1}\mathbb{R}[t^{-1}]$. Henceforth, ψ is a fixed continuous unitary character as in Chapter 3.

For $k \in Z(H) \cong \mathbb{R}$, $v \in V$, $v^* \in V^*$ and $f \in S(V)$, recall the H-action

1. $(k \cdot f)(x) = \psi(k)f(x)$,
2. $(v \cdot f)(x) = f(x + v)$,
3. $(v^* \cdot f)(x) = \psi(\langle x, v^* \rangle)f(x)$.

If $h = (v + v^*, k) \in H$, recall that h can be written as

$$(0, k - \frac{1}{2}\langle v, v^* \rangle)(v, 0)(v^*, 0)$$

in a unique way, as proven in Proposition 3.1.7. Therefore, we can think of the action of an element of H as three actions from three subgroups done in succession. This will be the key idea used in Section 3.1 to prove that the action of H on $S(V)$ is continuous.

4.1 Finite dimensional case

Assume that $\dim V < \infty$, so that $V \cong \mathbb{R}^n$. We shall equip $H = V \oplus V^* \oplus \mathbb{R}$ with the standard Euclidean topology and $S(V)$ with the Fréchet topology on $S(V)$ defined in

Section 2.2. We consider the l^∞ norm on H defined by

$$\left\| \left(\sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i f_i^*, k \right) \right\| = \max\{a_i, b_i, k : 1 \leq i \leq n\}$$

for a basis $e_1, ..., e_n$ of V, $f_1^*, ..., f_n^*$ of V^* and $(0, 1)$ for $Z(H)$.

Lemma 4.1.1. *Suppose* $\|(v_1 + v_1^*, k_1) - (v_2 + v_2^*, k_2)\| < \delta$ *and* $A > 0 \in \mathbb{R}$. *Suppose further that* $\max\{\|(v_1 + v_1^*, k_1)\|, \|(v_2 + v_2^*, k_2)\|\} < A$. *Then we have* $\|v_1 - v_2\| < \delta$, $\|v_1^* - v_2^*\| < \delta$ *and* $\left|(k_1 - \frac{1}{2}\langle v_1, v_1^*\rangle) - (k_2 - \frac{1}{2}\langle v_2, v_2^*\rangle\right| < \delta(An + 1)$.

Proof. Write $v_j = \sum_{i=1}^{n} a_j^i e_i$, $v_j^* = \sum_{i=1}^{n} b_j^i f_i^*$, then we also have $|a_1^i - a_2^i| < \delta$, and $|b_1^i - b_2^i| < \delta$ and $|k_1 - k_2| < \delta$. The first two claims follow. For the third claim,

$$\begin{aligned}
&\left|(k_1 - \frac{1}{2}v_1^*(v_1)) - (k_2 - \frac{1}{2}v_2^*(v_2))\right| = \left|(k_1 - k_2) + \frac{1}{2}\sum_{i=1}^{n}(b_1^i a_1^i - b_2^i a_2^i)\right| \\
&\leq |k_1 - k_2| + \frac{1}{2}\sum_{i=1}^{n}\left|b_1^i a_1^i - b_2^i a_2^i\right| \leq |k_1 - k_2| + \frac{1}{2}\sum_{i=1}^{n}\left|b_1^i a_1^i - b_1^i a_2^i + b_1^i a_2^i - b_2^i a_2^i\right| \\
&\leq |k_1 - k_2| + \frac{1}{2}\sum_{i=1}^{n}\left|b_1^i a_1^i - b_1^i a_2^i\right| + \left|b_1^i a_2^i - b_2^i a_2^i\right| \\
&\leq |k_1 - k_2| + \frac{A}{2}\sum_{i=1}^{n}\left|a_1^i - a_2^i\right| + \left|b_1^i - b_2^i\right| < \delta + An\delta = \delta(An + 1).
\end{aligned}$$

□

Lemma 4.1.2. *Suppose* $f \in S(V)$. *Then for all multi-indices* $\alpha = (\alpha_1, ..., \alpha_n)$ *and* $\beta = (\beta_1, ...\beta_n)$, *we have*

$$x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} ... \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f \in S(V).$$

Proof. This is an easy exercise. □

Lemma 4.1.3. *Suppose* $v \in V$, $A > 0 \in \mathbb{R}$ *and consider the map* $\phi_v : S(V) \to S(V)$ *defined by* $\phi_v(f)(x) = f(x + v)$. *Then there exists a Fréchet open set* $U \subseteq S(V)$ *around the origin, such that, for all* v *with* $\|v\| < A$ *and for all* $\epsilon > 0$, *if* $f - g \in U$, *then* $p_{\alpha,\beta}(\phi_v(f) - \phi_v(g)) < \epsilon$.

Proof. Fix α, β and suppose $f, g \in S(V)$. Then from Lemma 4.1.2, the functions $\tilde{f} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} ... \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f$ and $\tilde{g} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} ... \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} g$ are in $S(V)$. Define an ordering

$$\alpha' \leq \alpha \Leftrightarrow \alpha'_1 \leq \alpha_1, ..., \alpha'_n \leq \alpha_n$$

of multi-indices. Write $v = (v_1, ..., v_n)$. Then we have

$$\begin{aligned} x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} &= (x_1 + v_1 - v_1)^{\alpha_1} (x_2 + v_2 - v_2)^{\alpha_2} ... (x_n + v_n - v_n)^{\alpha_n} \\ &= \sum_{\alpha' \leq \alpha} A_{\alpha'} (x_1 + v_1)^{\alpha'_1} ... (x_n + v_n)^{\alpha'_n} \end{aligned}$$

for some $A_{\alpha'}$. Note that $\|v\| < A$ implies that $|A_{\alpha'}| < A^{|\alpha|}$ (where $|\alpha| = \sum_{i=1}^{n} \alpha_i$). Let $M = |\{\alpha' : \alpha' \leq \alpha\}|$ (note that $M < \infty$). Let $B = A^{|\alpha|}$. Finally, let $N = BM$ and suppose that

$$\left\| \tilde{f} - \tilde{g} \right\|_{\alpha',0} = \left\| x_1^{\alpha'_1} x_2^{\alpha'_2} ... x_n^{\alpha'_n} (\tilde{f} - \tilde{g}) \right\|_{\infty} < \frac{\epsilon}{N} \tag{10}$$

for all $\alpha' \leq \alpha$. Therefore, we obtain

$$\begin{aligned} p_{\alpha,\beta}(\phi_v(f) - \phi_v(g)) &= \sup_{x \in V} \left| x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} (\tilde{f}(x+v) - \tilde{g}(x+v)) \right| \\ &= \sup_{x \in V} \left| \sum_{\alpha' \leq \alpha} A_{\alpha'} (x_1 + v_1)^{\alpha'_1} ... (x_n + v_n)^{\alpha'_n} (\tilde{f}(x+v) - \tilde{g}(x+v)) \right| \\ &\leq \sum_{\alpha' \leq \alpha} |A_{\alpha'}| \sup_{x \in V} \left| (x_1 + v_1)^{\alpha'_1} ... (x_n + v_n)^{\alpha'_n} (\tilde{f}(x+v) - \tilde{g}(x+v)) \right| \\ &= \sum_{\alpha' \leq \alpha} |A_{\alpha'}| \sup_{x \in V} \left| (x_1)^{\alpha'_1} ... (x_n)^{\alpha'_n} (\tilde{f}(x) - \tilde{g}(x)) \right| \\ &< \sum_{\alpha' \leq \alpha} |A_{\alpha'}| \frac{\epsilon}{BM} \leq \sum_{\alpha' \leq \alpha} \frac{\epsilon}{M} < \epsilon. \end{aligned}$$

Thus, if U is chosen to be $\{x \in V : p_{\alpha',\beta}(x) < \frac{\epsilon}{N} : \alpha' \leq \alpha\}$, then f and g satisfy Eq. (10). This proves the lemma. □

The proof of the following lemma can be found in [3, p. 89]

Lemma 4.1.4. *Let $\psi : \mathbb{R} \to \mathbb{C}^{\times}$ be a continuous additive unitary character. Then we have $\psi(k) = e^{i\alpha k}$ for some α.*

Lemma 4.1.5. *Suppose $k \in \mathbb{R}$. Then the map $\phi_k : S(V) \to S(V)$ defined by $\phi_k(f) = \psi(k) f$ is an isometry for any norm $p_{\alpha,\beta}$.*

Proof. This is clear from the fact that ψ is a unitary character. □

Lemma 4.1.6. *Suppose $f \in S(V)$. Then the map $\psi_f : \mathbb{R} \to S(V)$ defined by $\psi_f(k) = \psi(k)f$ is continuous with respect to all norms $p_{\alpha,\beta}$, hence continuous.*

Proof. Suppose $k \in \mathbb{R}$ and fix $p_{\alpha,\beta}$. Then from Lemma 4.1.2, we have that $g = x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}...\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}f \in S(V)$. Let $A > 0$ such that $\|g\|_\infty \leq A$. Then we have

$$\begin{aligned} p_{\alpha,\beta}(\psi(k)f - \psi(k')f) &= \|(\psi(k) - \psi(k'))g\|_\infty \\ &= |\psi(k) - \psi(k')| \, \|g\|_\infty \leq \left|e^{i\alpha k} - e^{i\alpha k'}\right| A. \end{aligned}$$

However, $e^{i\alpha k}$ is a periodic function of k with period $\frac{2\pi}{\alpha}$, hence it is uniformly continuous. Therefore, we can choose δ such that if $|k - k'| < \delta$, then $\left|e^{i\alpha k} - e^{i\alpha k'}\right| < \frac{\epsilon}{A}$, which completes the proof. □

Lemma 4.1.7. *Suppose $f \in S(V)$. Then the map $\psi_f : V \to S(V)$ defined by $\psi_f(v)(x) = f(v + x)$ is continuous with respect to all norms $p_{\alpha,\beta}$, hence continuous.*

Proof. Suppose $v \in V$ and fix $p_{\alpha,\beta}$. From Lemma 4.1.2, we have that $g = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}...\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}f \in S(V)$. Choose R_1 such that if $\|x\| \geq R_1$, then $\|x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}g(x)\| < \frac{\epsilon}{2}$ and $R_2 > R_1$ such that if $\|x\| \geq R_2$, then $\|x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}g(x)\| < \frac{\epsilon}{4}$.

Next, set $A = R_2^{|\alpha|}$ (where $|\alpha| = \sum \alpha_i$). On the disk $D = \{x : \|x\| \leq R_2\}$, g is uniformly continuous (because D is compact), so we can choose δ' such that if $x, y \in D$ and $\|x - y\| < \delta'$, then $|g(x) - g(y)| < \frac{\epsilon}{A}$. Now, let $\delta = \min\{\delta', R_2 - R_1\}$.

Suppose $v' \in V$ such that $\|v - v'\| < \delta$, and then $\|(x + v) - (x + v')\| < \delta$. We wish to compute $|x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}(g(x + v) - g(x + v'))|$. There are three cases.

1. Case 1: If $x + v$, $x + v' \in D$, we have

$$\begin{aligned} |x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}(g(x + v) - g(x + v'))| &= |x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}| \, |(g(x + v) - g(x + v')| \\ &< A\frac{\epsilon}{A} = \epsilon. \end{aligned}$$

2. Case 2 : If, $x + v$, $x + v' \notin D$, we have

$$|x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}g(x + v) - x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}g(x + v')| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon.$$

3. Case 3 : If $x + v \in D$ but $x + v' \notin D$ (or vice-versa), we must have that $\|x + v\| \geq R_1$ by the choice of δ. Therefore,

$$|x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} g(x + v) - x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} g(x + v')| < \frac{\epsilon}{2} + \frac{\epsilon}{4} = \frac{3\epsilon}{4} < \epsilon.$$

Note that δ does not depend on v, hence ψ_f is in fact uniformly continuous. □

Lemma 4.1.8. *The map $\tilde{\psi_f} : V^* \to S(V)$ defined by $\tilde{\psi_f}(v^*) = \psi(\langle x, v^* \rangle) f$ is continuous with respect to all norms $p_{\alpha,\beta}$, hence continuous.*

Proof. The Fourier transform $f \to \mathcal{F}(f)$ is a continuous map on $S(V)$ ([4, p. 250]). Furthermore, if $v^* = \sum_{i=1}^{n} a_i v_i^*$, we can compute using Fubini's theorem and the appropriate normalization of Fourier tranform that

$$\tilde{\psi_f}(v^*) = \mathcal{F}^4(\tilde{\psi_f}(v^*)) = \mathcal{F}^4(\psi(\langle x, v^* \rangle) f) = \mathcal{F}^4(e^{i\alpha(x \cdot (\sum_{i=1}^{n} a_i f_i))} f)) = \mathcal{F}^3(\mathcal{F}(f)(x - \sum_{i=1}^{n} a_i e_i)).$$

Therefore, $\tilde{\psi_f}$ is the composition of continuous maps $\mathcal{F}^3 \circ \psi_{\mathcal{F}(f)}(-\sum a_i e_i))$, where $\psi_{\mathcal{F}(f)}$ is as in Lemma 4.1.7. Then $\tilde{\psi_f}$ is continuous. □

Proposition 4.1.9. *Suppose $f \in S(V)$. Then the map $\phi : H \to S(V)$ defined by $\phi(h) = h \cdot f$ is continuous.*

Proof. Fix a norm $p_{\alpha,\beta}$. Suppose $h_1 \in H$. We will show that the map is continuous around h_1. Suppose $h_2 \in H$ also. Write $h_1 = (v_1 + v_1^*, k_1)$ and $h_2 = (v_2 + v_2^*, k_2)$. Furthermore, write $k_1' = k_1 - \frac{1}{2} v_1^*(v_1)$ and $k_2' = k_2 - \frac{1}{2} v_2^*(v_2)$. First, we note that

$$\begin{aligned}
&\|(k_1' v_1 v_1^* \cdot f - k_2' v_2 v_2^* \cdot f)\|_{\alpha,\beta} \\
&= \|(k_1' v_1 v_1^* \cdot f - k_2' v_1 v_1^* \cdot f + k_2' v_1 v_1^* \cdot f - k_2' v_2 v_1^* \cdot f + k_2' v_2 v_1^* \cdot f - k_2' v_2 v_2^* \cdot f)\|_{\alpha,\beta} \\
&\leq \|(k_1' v_1 v_1^* \cdot f - k_2' v_1 v_1^* \cdot f)\|_{\alpha,\beta} + \|(k_2' v_1 v_1^* \cdot f - k_2' v_2 v_1^* \cdot f)\|_{\alpha,\beta} \\
&+ \|(k_2' v_2 v_1^* \cdot f - k_2' v_2 v_2^* \cdot f)\|_{\alpha,\beta}.
\end{aligned}$$

We want to make each of these three terms less than $\frac{\epsilon}{3}$.

1. For the first term, we know from Lemma 4.1.6 that there exists $\delta_1 > 0$ such that if $|k_1' - k_2'| < \delta_1$, then $p_{\alpha,\beta}((k_1' \cdot v_1 \cdot v_1^* - k_2' \cdot v_1 \cdot v_1^*) < \frac{\epsilon}{3}$.

2. For the second term, by Lemma 4.1.5, $p_{\alpha,\beta}(g-h) = p_{\alpha,\beta}(k_2' \cdot g - k_2' \cdot h)$. Furthermore, from Lemma 4.1.7, there exists δ_2 such that if $\|v_1 - v_2\| < \delta_2$, then $p_{\alpha,\beta}(v_1 \cdot (v_1^* \cdot f) - v_2 \cdot (v_1^* \cdot f)) < \frac{\epsilon}{3}$. The entire term will be less than $\frac{\epsilon}{3}$ with $g = v_1 \cdot v_1^* \cdot f$ and $h = v_2 \cdot v_1^* \cdot f$.

3. For the last term, by Lemma 4.1.5, $p_{\alpha,\beta}(g-h) = p_{\alpha,\beta}(k_2' \cdot g - k_2' \cdot h)$. But now, from Lemma 4.1.3, we also have that whenever $\|v_2\| < A$ for some A, there exists a Fréchet open set U such that $g' - h' \in U$ implies that $p_{\alpha,\beta}(v_2 \cdot g' - v_2 \cdot h') < \frac{\epsilon}{3}$. Finally, from Lemma 4.1.8, we have that there exists δ_3 s.t. if $\|v_1^* - v_2^*\| < \delta_3$, then $v_1^* \cdot f - v_2^* \cdot f \in U$. Now take $g = v_2 \cdot v_1^* \cdot f$ and $h = v_2 \cdot v_2^* \cdot f$, $g' = v_1^* \cdot f$ and $h' = v_2^* \cdot f$ and then $\|(k_2' v_2 v_1^* \cdot f - k_2' v_2 v_2^* \cdot f)\|_{\alpha,\beta}$ will be less than $\frac{\epsilon}{3}$.

Fix the Fréchet open set U corresponding to $A = 2\|v_1\|+1$ as described by Lemma 4.1.3. Choose $B > \|h_1\| + \|v_1\| + 1$ and $\delta = \min\{\|v_1\|+1, \frac{\delta_1}{Bn+1}, \delta_2, \delta_3\}$. For all v_2 such that $\|v_1 - v_2\| < \delta \leq \|v_1\| + 1$, we then have $\|v_2\| < 2\|v_1\| + 1 = A$ by the reverse triangle inequality. Furthermore, with this choice of δ, $\|h_2\| \leq \|v_1\| + 1 < B$. Then by Lemma 4.1.1, $\|v_1 - v_2\| < \delta_2$, $\|v_1^* - v_2^*\| < \delta_3$ and $|k_1' - k_2'| < \frac{\delta_1}{Bn+1}(Bn+1) = \delta_1$, which is enough for $\|(k_1' v_1 v_1^* \cdot f - k_2' v_1 v_1^* \cdot f)\|_{\alpha,\beta}$, $\|(k_2' v_1 v_1^* \cdot f - k_2' v_2 v_1^* \cdot f)\|_{\alpha,\beta}$ and $\|(k_2' v_2 v_1^* \cdot f - k_2' v_2 v_2^* \cdot f)\|_{\alpha,\beta}$ to all be less than $\frac{\epsilon}{3}$, completing the proof. □

4.2 Infinite-dimensional case

We now consider the case $V = (t^{-1}\mathbb{R}[t^{-1}])^{2n}$ so that $V^* = \mathbb{R}[[t]]^{2n}$, $X = \mathbb{R}((t))^{2n}$ and $H = X \times \mathbb{R}$. Recall that by Proposition 2.1.2, a basis for the topology on $\mathbb{R}((t))$ consists of the translations of the sets $U_n = \{x \in \mathbb{R}((t)) : v(x) \geq n\}$, where $v(x)$ is the valuation map. The topology on H is then simply the product topology of $2n$ copies of $\mathbb{R}((t))$ with the Euclidean topology on $\mathbb{R}$. For convenience, we will write the $2n$ components of X as $\mathbb{R}((t_i))$ for $1 \leq i \leq 2n$ to differentiate them from each other.

Lemma 4.2.1. *H is a topological group.*

Proof. We first prove that the group multiplication is continuous. Suppose $U \subseteq H$ is open and $(h_1, h_2) \in H \times H$ such that $h_1 h_2 \in U$. Write $h_1 = (v_1 + v_1^*, k_1)$, $h_2 = (v_2 + v_2^*, k_2)$. Similarly, suppose $h_1' = (w_1 + w_1^*, k_1')$ and $h_2' = (w_2 + w_2^*, k_2')$ are other

elements of H. We know that $v_1 = \sum_{1\leq j\leq 2n, i\in\mathbb{N}^+} a_{i,j}t_j^{-i}$ and $v_2 = \sum_{1\leq j\leq 2n, i\in\mathbb{N}^+} b_{i,j}t_j^{-i}$ for finitely many non-zero $a_{i,j} \in \mathbb{R}$ and $b_{i,j} \in \mathbb{R}$. Because $h_1h_2 \in U$, there exists $M, \epsilon > 0$ such that $(h_1h_2 + \{x \in X : v(x_i) \geq M : 1 \leq i \leq 2n\}) \times (-\epsilon, \epsilon) \subseteq U$.

Now, let $N' = \max(\{M\} \cup \{i : a_{i,j} \neq 0 \text{ or } b_{i,j} \neq 0 \text{ for some } j\})$, $N = N' + 1$ and $U' = \{x \in X : v(x_i) \geq N \text{ for all } 1 \leq i \leq 2n\}$. Suppose further that $(v_1 + v_1^*) - (w_1 + w_1^*) \in U'$ and $(v_2 + v_2^*) - (w_2 + w_2^*) \in U'$. Then $w_1^* = v_1^* + u_1$ where $u_1 \in U'$ and $v_2 = w_2$, hence $\langle w_2, w_1^*\rangle = \langle v_2, v_1^*\rangle$. Similarly, $\langle w_1, w_2^*\rangle = \langle v_1, v_2^*\rangle$. Then $h_1'h_2' - h_1h_2$ is equal to

$$(\tilde{w} - \tilde{v}, \tilde{k} + \frac{1}{2}(\langle v_2, v_1^*\rangle + \langle v_1, v_2^*\rangle + \langle w_2, w_1^*\rangle - \langle w_1, w_2^*\rangle)),$$

where $\tilde{w} = w_1 + w_2 + w_1^* + w_2^*$, $\tilde{v} = v_1 - v_2 - v_1^* - v_2^*$ and $\tilde{k} = -k_1 - k_2 + k_1' - k_2'$. After some simplification, we obtain that

$$h_1'h_2' - h_1h_2 = (u_1 + u_2, -k_1 - k_2 + k_1' + k_2')$$

for some u_2 satisfying $w_2^* = v_2^* + u_2$ and $u_2 \in U'$. Next, write $U_\epsilon' = U' \times (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and consider the set

$$(h_1 + U_\epsilon') \times (h_2 + U_\epsilon') \subseteq H \times H.$$

This set contains (h_1, h_2), is open, and it follows from the above computation that the group operation sends it into U, as desired. Therefore, the group operation is a continuous map from $H \times H$ into H.

To show that the map $h \mapsto h^{-1}$ is continuous, suppose $h_1 = (v_1 + v_1^*, k_1) \in H$ and remember that from Proposition 3.1.5

$$(h_1)^{-1} = (-v_1 - v_1^*, -k_1)$$

Set $U = U_1 \times ... \times U_{2n} \times V$ where $U_1, ..., U_{2n}, V$ are open. In particular, it is easy to prove that $-U_1, ..., -U_{2n}, -V$ are all also open. But now, the inverse map sends $-U_1 \times -U_2 \times ... - U_{2n} \times -V$ into U (from the above formula). Therefore, it is continuous. □

Proposition 4.2.2. *Suppose $f \in S(V)$. Then the map $\phi : H \to S(V)$ defined by $\phi(h) = h \cdot f$ is continuous.*

Proof. As usual, it is enough to check continuity with respect to every seminorm. Fix a seminorm $p_{\alpha,\beta,m}$ and $(v_1+v_1^*, k_1) \in H$. Then $v_1 = \sum_{1\leq j\leq 2n, i\in\mathbb{N}^+} a_{i,j}t_j^{-i}$ for finitely many non-zero $a_{i,j} \in \mathbb{R}$. Let $N' = \max\{i : a_{i,j} \neq 0 \text{ for some } j\}$. Then let $N = \max\{m, N'\}$. From Lemma 4.1.6 applied to $V = V_m$, there exists δ such that if $|k_1' - k_2'| < \delta$, then $p_{\alpha,\beta,m}(k_1' \cdot g - k_2' \cdot g) < \epsilon$. Write

$$U' = \{x \in X : v(x_i) \geq N \text{ for all } 1 \leq i \leq 2n\}.$$

Consider the open set

$$U = (v_1 + v_1^* + U') \times (k_1 - \delta, k_1 + \delta).$$

Now suppose that $(v_2 + v_2^*, k_2) \in U$. We make two observations.

$|k_1' - k_2'| < \delta$: Recall that $k_1' = k_1 - \frac{1}{2}\langle v_1, v_1^*\rangle$, $k_2' = k_2 - \frac{1}{2}\langle v_2, v_2^*\rangle$. Note that if $v_2 + v_2^* \in (v_1 + v_1^* + U')$, then $v_2 + v_2^* = v_1 + v_1^* + v'$ where $v' \in U'$. However, from $N \geq N'$ we have that $\langle v', v_1\rangle = 0$, hence $\langle v_2, v_2^*\rangle = \langle v_1, v_1^*\rangle$ and then $k_1' - k_2' = k_1 - k_2$. Therefore, $|k_1 - k_2| < \delta$ implies that $|k_1' - k_2'| < \delta$. From this proof, we also see that $v_1 = v_2$.

$v_1^* \cdot f(x) = v_2^* \cdot f(x)$ for all $x \in V_m$: Note that by the choice of N, we must have $\langle x, U'\rangle = 0$ for all $x \in V_m$, hence $\langle x, v_2^*\rangle = \langle x, v_1^* + v'\rangle = \langle x, v_1^*\rangle$. It then follows that $v_1^* \cdot f(x) = v_2^* \cdot f(x)$ for all $x \in V_m$.

From these observations we obtain that

$$\begin{aligned} p_{\alpha,\beta,m}(k_1' \cdot v_1 \cdot v_1^* \cdot f - k_2' \cdot v_2 \cdot v_2^* \cdot f) &= p_{\alpha,\beta,m}(k_1' \cdot (v_1 \cdot v_1^* \cdot f) - k_2' \cdot (v_2 \cdot v_1^* \cdot f)) \\ &= p_{\alpha,\beta,m}(k_1' \cdot (v_1 \cdot v_1^* \cdot f) - k_2' \cdot (v_1 \cdot v_1^* \cdot f)) < \epsilon. \end{aligned}$$

Since this is true for any seminorm, we conclude that ϕ is continuous. □

We now wish to show that the map $g \to T_g$ as defined in 3.2.4 is continuous when restricted to $Sp_{2n}(\mathbb{R}((t)))$ with respect to the product topology of $4n^2$ copies of $\mathbb{R}((t))$, obtained from the subspace topology as a subset of $M_{2n,2n}(\mathbb{R}((t)))$. However, for this to make sense, first we should prove the following proposition.

Proposition 4.2.3. *The group $Sp_{2n}(\mathbb{R}((t)))$, with the product topology of $4n^2$ copies of X, is a topological group.*

Proof. Denote the standard basis for $M_{2n,2n}$ by $e_{i,j}$ for $1 \leq i, j \leq 2n$. Let us look at the formula for product of two matrices :

$$\left(\sum_{1\leq i,j\leq 2n} a_{i,j}e_{i,j}\right)\left(\sum_{1\leq k,l\leq 2n} b_{k,l}e_{k,l}\right) = \sum_{1\leq i,l\leq 2n}\left(\sum_{1\leq k\leq 2n} a_{i,k}b_{k,l}\right)e_{i,l}.$$

As usual, it is enough to check that the map is continuous in every component. However, the map sending two matrices $(a_{i,j})$ and $(b_{i,j})$ to $\sum_{1\leq k\leq 2n} a_{i,k}b_{k,l}$ is a composition of projections, additions and multiplications, all of which are continuous. The result is then continuous as well.

We now wish to prove that the inverse map $g \mapsto g^{-1}$ is also continuous. Because $\det(g) = 1$ by Lemma 3.2.20, we know that this map is simply $g \mapsto C(g)^T$, where $C(g)$ is the cofactor matrix of g. However all the entries in the cofactor matrix of g are polynomials in the entries of g, which can be thought of as compositions of projections, additions and multiplications, all continuous maps.

This completes the proof. □

Theorem 4.2.4. *Suppose that Haar measures have been fixed for every possible finite-dimensional subspace of V and $f \in S(V)$. Then the map $\phi_f : Sp_{2n}(\mathbb{R}((t))) \mapsto S(V)$ defined by $\phi_f(g) = T_g f$ is continuous.*

Remark 4.2.5. If the topology on $S(V)$ was induced by a finite family of seminorms $p_{\alpha,\beta,m}$, our proof would show that ϕ_f is locally constant, which corresponds to the idea of smooth representation in p-adic representation theory.

Proof. For any seminorm $p_{\alpha,\beta,m}$, and any element $g \in Sp_{2n}(\mathbb{R}((t)))$, our goal will be to find an open set $U \subseteq Sp_{2n}(\mathbb{R}((t)))$ containing g such that $p_{\alpha,\beta,m}(T_g(f) - T_{g'}(f)) = 0$ (which is clearly less than ϵ). We will prove that on this open set U, $T_g f(x) = T_{g'} f(x)$ for all $x \in V_m$.

Recall the definitions of T_g and S_g from Proposition 3.2.4 :

$$(T_g f)(x) = \int_{V_g} S_g(x + x^*) f(x\alpha + x^*\gamma) d(x^*\gamma),$$

$$S_g(x + x^*) = \psi(\frac{1}{2}\langle x\alpha, x\beta\rangle + \frac{1}{2}\langle x^*\gamma, x^*\delta\rangle + \langle x^*\gamma, x\beta\rangle).$$

Step 1 : Suppose $g \in Sp_{2n}(\mathbb{R}((t)))$ and $V_r \subseteq V$ for some r. Write

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

as usual. Then we claim that there exists a neighbourhood W of g such that for all

$$g' = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix} \in W$$

we have $\gamma' = \gamma$ and $\alpha'|_{V_r} = \alpha|_{V_r}$.

Proof of step 1 : We will first give a more explicit formula for α and γ. Recall that $X = V \oplus V^* = \mathbb{R}((t))^{2n}$. Denote by $e_{i,j}$ the standard basis for $M_{2n,2n}(\mathbb{R}((t)))$, where $1 \leq i, j \leq 2n$. Furthemore, this time we will differentiate the variables t_i by writing them as te_i to be compatible with matrix multiplication. Assume $g = \sum_{1 \leq i,j \leq 2n} a_{i,j} e_{i,j}$, $a_{i,j} \in \mathbb{R}((t))$. Then $a_{i,j} = \sum_k b_{i,j}^k t^k$ where $1 \leq i, j \leq 2n$, $k \in \mathbb{Z}$, $b_{i,j}^k \in \mathbb{R}$ and with the property that only finitely of the $b_{i,j}^k$ such that $k \leq -1$ are non-zero. Assume $\sum_{l \in \mathbb{Z}} a_l t^l \in \mathbb{R}((t))$. Now, we can compute by matrix multiplication :

$$\left(\sum_{l \in \mathbb{Z}} a_l t^l e_m \right) g = \sum_{1 \leq i \leq 2n} \sum_{k,l \in \mathbb{Z}} b_{m,i}^k a_l t^{k+l} e_i.$$

Therefore, we obtain

$$\left(\sum_{l \leq -1} a_l t^l e_m \right) \alpha = \sum_{1 \leq i \leq 2n} \sum_{\substack{k+l \leq -1 \\ l \leq -1}} a_l b_{m,i}^k t^{k+l} e_i,$$

and

$$\left(\sum_{l \geq 0} a_l t^l e_m \right) \gamma = \sum_{1 \leq i \leq 2n} \sum_{\substack{k+l \leq -1 \\ l \geq 0}} a_l b_{m,i}^k t^{k+l} e_i.$$

Note that in both cases the sum over $k+l = p$ is a finite sum for each $p \leq -1$ because only finitely many of the $b_{i,j}^k$ such that $k \leq -1$ are non-zero. From Lemma 3.2.19, we have $\dim V_g < \infty$, hence $V_g \subseteq V_{r'}$ for some r'. Then $(V_r + V_g) \subseteq V_s$ for $s = \max\{r, r'\}$. Let $U = \{x \in \mathbb{R}((t)) : v(x) \geq s\}$ and

$$W = (a_{1,1} + U) \times (a_{1,2} + U) \times ... \times (a_{2n,2n} + U).$$

Clearly, W is open. But then if $g' \in V$, we must have $b'^k_{i,j} = b^k_{i,j}$ for all $k \leq s-1$.

Now we can prove the claims of Step 1. We start with $\gamma' = \gamma$. When $l \geq 0$, $(k+l) \leq -1$ implies that $k \leq -1 \leq s-1$. Then

$$\left(\sum_{l\geq 0} a_l t^l e_m\right)\gamma = \sum_{1\leq i\leq 2n}\sum_{\substack{k+l\leq -1\\ l\geq 0}} a_l b^k_{i,m} t^{k+l} e_i = \sum_{1\leq i\leq 2n}\sum_{\substack{k+l\leq -1\\ l\geq 0}} a_l b'^k_{i,m} t^{k+l} e_i = (t^l e_m)\gamma'$$

as desired (it remains to expand γ by linearity over all $2n$ basis elements e_m). This proves that $\gamma = \gamma'$.

For the second claim, note that whenever $l \geq -s$, we have that $l \leq -1-k$ implies that $k \leq s$, and then

$$\begin{aligned}\left(\sum_{-s\leq l\leq -1} a_l t^l e_m\right)\alpha = \sum_{1\leq i\leq 2n}\sum_{\substack{-s\leq l\leq -1-k\\ l\leq -1}} a_l b^k_{i,m} t^{k+l} e_i &= \sum_{1\leq i\leq 2n}\sum_{\substack{-s\leq l\leq -1-k\\ l\leq -1}} a_l b'^k_{i,m} t^{k+l} e_i \\ &= (t^l e_m)\gamma'\end{aligned}$$

as desired (once again expanding over all e_m). This proves the second claim because $V_r \subseteq V_s$.

Step 2 : Suppose $V_r \subseteq V$. Then there exists a neighbourhood $U \subseteq W$ of g such that $S_g|_{x+V^*} = S_{g'}|_{x+V^*}$ for all $x \in V_r$ and $g' \in U$.

Proof of step 2 : Fix V_r. From the previous step, we have a neighbourhood W of g such that for all $g' \in W$, $\gamma' = \gamma$ and $\alpha'|_{V_r} = \alpha|_{V_r}$. Clearly, there exists $s \geq 0$ such that $\{x, \alpha(x) : x \in V_r\} \subseteq V_s$. In a similar way to the last step, we can compute

$$\left(\sum_{l\leq -1} a_l t^l e_m\right)\beta = \sum_{1\leq i\leq 2n}\sum_{\substack{k+l\geq 0\\ l\leq -1}} a_l b^k_{m,i} t^{k+l} e_i$$

for $l < 0$. For $l \geq 0$, we have

$$\left(\sum_{l\geq 0} a_l t^l e_m\right)\delta = \sum_{1\leq i\leq 2n}\sum_{\substack{k+l\geq 0\\ l\geq 0}} a_l b^k_{m,i} t^{k+l} e_i.$$

Once again, note that in both cases the sum $\sum_{k+l=p}$ is a finite sum for each $p \geq 0$. Let $W' = \{x \in \mathbb{R}((t)) : v(x) \geq 2s\}$ and $\widetilde{W} = (a_{1,1}+W') \times (a_{1,2}+W') \times ... \times (a_{2n,2n}+W')$.

If $g' \in U = W \cap \widetilde{W}$, we must have $b'^k_{i,j} = b^k_{i,j}$ for all $k \leq 2s-1$. Now, note that for any $l \geq -s$ we have that $(k+l) \leq s-1$ imply $k \leq s-l-1 \leq 2s-1$. Therefore, if $y \in V_s$, we have

$$\begin{aligned}\langle y, \left(\sum_{l\leq -1} a_l t^l e_m\right)\beta\rangle &= \langle y, \sum_{1\leq i\leq 2n}\sum_{\substack{0\leq (k+l)\\ l\leq -1}} a_l b^k_{m,i} t^{k+l} e_i\rangle = \langle y, \sum_{1\leq i\leq 2n}\sum_{\substack{0\leq (k+l)\leq s-1\\ l\leq -1}} a_l b^k_{m,i} t^{k+l} e_i\rangle \\ &= \langle y, \sum_{1\leq i\leq 2n}\sum_{\substack{0\leq (k+l)\leq s-1\\ l\leq -1}} a_l b'^k_{m,i} t^{k+l} e_i\rangle = \langle y, \sum_{1\leq i\leq 2n}\sum_{\substack{0\leq (k+l)\\ l\leq -1}} a_l b'^k_{m,i} t^{k+l} e_i\rangle \\ &= \langle y, \left(\sum_{l\leq -1} a_l t^l e_m\right)\beta'\rangle.\end{aligned}$$

The second and fourth equality follows from $\langle y, t^{k+l} e_i\rangle = 0$ for all $k+l \geq s$, $y \in V_s$. Replacing y by $x\alpha = x\alpha'$ (because $U \subseteq W$) and extending by linearity to all basis elements e_m, we obtain $\langle x\alpha, x\beta\rangle = \langle x\alpha', x\beta'\rangle$. Similarly, replacing y by $x^*\gamma = x^*\gamma'$ (because $\gamma = \gamma'$) and extending by linearity again, we obtain $\langle x^*\gamma, x\beta\rangle = \langle x^*\gamma', x\beta'\rangle$. The proof that $\langle x^*\gamma, x^*\delta\rangle = \langle x^*\gamma', x^*\delta'\rangle$ is almost exactly the same computation as for β (the only difference being $l \geq 0$ instead of $l \geq -1$), using $y = x^*\gamma = x^*\gamma'$. By adding those three terms (with a factor $\frac{1}{2}$ in front of the first two), we see that with the open set U chosen, $S_g|_{x+V^*} = S_{g'}|_{x+V^*}$ for all $x \in V_r$ and $g' \in U$.

Step 3 : We prove the statement of the proposition.

Let $g \in Sp_{2n}(\mathbb{R}((t))$ and $p_{\alpha,\beta,m}$ be a semi-norm and fix $f \in S(V)$. We want to show that the map $g \mapsto T_g f$ is continuous. From the previous two steps with $V_r = V_m$, there exists a neighbourhood U of g such that for all $g' \in U$ and $x \in V_m$, $S_{g'}|_{x+V^*} = S_g|_{x+V^*}$, $\gamma = \gamma'$, $V_g = \operatorname{Im}\gamma = \operatorname{Im}\gamma' = V_{g'}$ and $\alpha|_{V_m} = \alpha'|_{V_m}$. In particular, for any function $f \in S(V)$, we have $f(x\alpha + x^*\gamma) = f(x\alpha' + x^*\gamma')$ for all $x \in V_m$. Therefore, for this open set U, and for all $x \in V_m$,

$$\begin{aligned}(T_g f)(x) = \int_{V_g} S_g(x+x^*) f(x\alpha + x^*\gamma) d(x^*\gamma) &= \int_{V_{g'}} S_{g'}(x+x^*) f(x\alpha' + x^*\gamma') d(x^*\gamma) \\ &= (T_{g'} f)(x).\end{aligned}$$

Therefore, $p_{\alpha,\beta,m}(T_g f - T_{g'} f) = 0$ for all $g' \in U$. In particular, $g \mapsto T_g$ is continuous

with respect to the seminorm $p_{\alpha,\beta,m}$. Since this is true for any seminorm $p_{\alpha,\beta,m}$, we conclude that the map $g \mapsto T_g f$ is continuous. $\square$

Chapter 5

A specific central extension

Let $SL_2(\mathbb{R}((t)))$ be the group of 2×2 matrices with coefficients in $\mathbb{R}((t))$ of determinant 1.

Lemma 4.0.1. $Sp_2(\mathbb{R}((t))) = SL_2(\mathbb{R}((t)))$

Proof. Write $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$. Then

$$\langle v, w \rangle_{\mathbb{R}((t))} = v_1 w_2 - v_2 w_1$$

and

$$\begin{aligned} \langle vg, wg \rangle_{\mathbb{R}((t))} &= \langle (v_1 a + v_2 c, v_1 b + v_2 d), (w_1 a + w_2 c, w_1 b + w_2 d) \rangle_{\mathbb{R}((t))} \\ &= v_1 w_2 (ad - bc) - v_2 w_1 (ad - bc) \end{aligned}$$

Therefore, $\langle v, w \rangle_{\mathbb{R}((t))} = \langle vg, wg \rangle_{\mathbb{R}((t))}$ if and only if $ad - bc = 1$. □

In this chapter, we will then denote $Sp_2(\mathbb{R}((t)))$ by $SL_2(\mathbb{R}((t)))$ instead.

5.1 Kubota's theorem

Define the tame symbol

$$\mathbf{t}(t^m \sigma_1, t^n \sigma_2) = (-1)^{mn} c_1^n / c_2^m$$

where $\sigma_1, \sigma_2 \in \mathbb{R}[[t]]^*$ and c_1, c_2 are the constant terms of σ_1, σ_2. Note that $\mathbf{t}(t^m\sigma_1, t^n\sigma_2)$ is never equal to 0. The following proposition gives the most important properties of the tame symbol.

Proposition 5.1.1. *Suppose* $a, b, b_1, b_2 \in \mathbb{R}((t)) \setminus \{0\}$. *Then*

(i) $\mathbf{t}(a, b_1)\mathbf{t}(a, b_2) = \mathbf{t}(a, b_1b_2)$

(ii) $\mathbf{t}(b_1, a)\mathbf{t}(b_2, a) = \mathbf{t}(b_1b_2, a)$

(iii) $\mathbf{t}(a, -a) = 1$

(iv) $\mathbf{t}(a, b) = 1$ *if* $a, b \in \mathbb{R}[[t]]^*$.

(v) $\mathbf{t}(a, 1-a) = 1$ *for* $a \neq 1$

(vi) $\mathbf{t}(a, b)\mathbf{t}(b, a) = 1$

(vii) $\mathbf{t}(a, b)\mathbf{t}(-\frac{b}{a}, a+b) = 1$ *for* $a \neq -b$

(viii) $\mathbf{t}(1, a) = \mathbf{t}(a, 1) = 1$

Proof. Suppose $\sigma_1, \sigma_2, \sigma_3$ are elements of $\mathbb{R}[[t]]^*$ with constant terms c_1, c_2, c_3. For property (i), we have

$$\begin{aligned}\mathbf{t}(t^m\sigma_1, t^n\sigma_2)\mathbf{t}(t^m\sigma_1, t^k\sigma_3) &= (-1)^{mn}c_1^n/c_2^m(-1)^{mk}c_1^k/c_3^m \\ &= (-1)^{m(n+k)}c_1^{n+k}/(c_2c_3)^m = \mathbf{t}(t^m\sigma_1, t^{n+k}\sigma_2\sigma_3).\end{aligned}$$

Property (ii) is similar. For (iii), we have

$$\mathbf{t}(t^n\sigma_1, -t^n\sigma_1) = (-1)^{n^2}c_1^n/(-c_1)^n = (-1)^{n^2-n} = 1.$$

Property (iv) follows directly from the definition. Property (viii) follows from (i) because

$$\mathbf{t}(1, a) = \mathbf{t}(1, a)\mathbf{t}(1, a)$$

and then we obtain $\mathbf{t}(1, a) = 1$ by cancellation. Property (v) depends on three cases. If $v(a) > 0$, then the constant term of $1 - a$ is 1, hence $\mathbf{t}(a, 1-a) = \mathbf{t}(a, 1) = 1$ by property (viii). If $v(a) = 0$, then $v(1-a) = 0$ and then $\mathbf{t}(a, 1-a) = 1$ from property (iv). Finally, if $v(a) < 0$, then $\mathbf{t}(a, 1-a) = \mathbf{t}(a, -a) = 1$. Property

(vi) follows immediately from the definition. For property (vii), $\mathbf{t}(b, \frac{1}{a})\mathbf{t}(b, a) = 1$ by property (viii), hence $\mathbf{t}(b, \frac{1}{a}) = \mathbf{t}(a, b)$ from property (vi). But now, we obtain $\mathbf{t}(-\frac{b}{a}, \frac{1}{a}) = \mathbf{t}(b, \frac{1}{a})\mathbf{t}(-\frac{1}{a}, \frac{1}{a}) = \mathbf{t}(b, \frac{1}{a}) = \mathbf{t}(a, b)$ from property (iii). Then we have

$$\mathbf{t}(a, b)\mathbf{t}(-\frac{b}{a}, a + b) = \mathbf{t}(-\frac{b}{a}, a + b)\mathbf{t}(-\frac{b}{a}, \frac{1}{a}) = \mathbf{t}(-\frac{b}{a}, \frac{a + b}{a}) = \mathbf{t}(-\frac{b}{a}, 1 + \frac{b}{a}) = 1$$

where the last equality follows from property (iv). □

We will frequently use these properties and will not always refer to them. Define

$$\mathbf{x}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} c & : \text{if } c \neq 0 \\ d & : \text{if } c = 0 \end{cases}$$

and for $g_1, g_2 \in SL_2(\mathbb{R}((t)))$,

$$\alpha(g_1, g_2) = \mathbf{t}(\mathbf{x}(g_1 g_2)/\mathbf{x}(g_1), \mathbf{x}(g_1 g_2)/\mathbf{x}(g_2)) \tag{11}$$

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}((t)))$, if $c = 0$, then we must have $d \neq 0$ (because $ad - bc = 1$), hence $\alpha(g_1, g_2)$ is well-defined and never vanishes.

We now wish to verify the cocycle conditions for α as defined in Definition 2.3.5. The first cocycle condition is easy. Indeed,

$$\alpha(g, e) = \mathbf{t}(\mathbf{x}(g_1)/\mathbf{x}(g_1), \mathbf{x}(g_1)) = \mathbf{t}(1, \mathbf{x}(g_1)) = 1$$

by Proposition 5.1.1(viii), and $\alpha(e, g) = 1$ similarly.

The challenge is to verify the second cocycle condition, which will be done in Theorem 5.1.11. The main reference for this section is [5]. Since our definition of α is in appearance different from [5], we should first verify that they are in fact the same.

Lemma 5.1.2.

$$\alpha(g_1, g_2) = \mathbf{t}\left(\mathbf{x}(g_1), \mathbf{x}(g_2)\right) \mathbf{t}\left(-\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}, \mathbf{x}(g_1 g_2)\right) \tag{12}$$

Proof. Indeed, by applying Proposition 5.1.1 (i) and (ii) repeatedly to Eq.(11), we obtain

$$\alpha(g_1, g_2) = \mathbf{t}\left(\mathbf{x}(g_1 g_2), \mathbf{x}(g_1 g_2)\right) \mathbf{t}\left(\mathbf{x}(g_1 g_2), \frac{1}{\mathbf{x}(g_2)}\right) \mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \mathbf{x}(g_1 g_2)\right) \mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \frac{1}{\mathbf{x}(g_2)}\right).$$

However, $\mathbf{t}(\mathbf{x}(g_1g_2), \mathbf{x}(g_1g_2)) = \mathbf{t}(-1, \mathbf{x}(g_1g_2))\mathbf{t}(-\mathbf{x}(g_1g_2), \mathbf{x}(g_1g_2)) = \mathbf{t}(-1, \mathbf{x}(g_1g_2))$ by Proposition 5.1.1(ii) and Proposition 5.1.1(iii). Furthermore,

$$\mathbf{t}\left(\mathbf{x}(g_1g_2), \frac{1}{\mathbf{x}(g_2)}\right)\mathbf{t}\left(\mathbf{x}(g_1g_2), \mathbf{x}(g_2)\right) = \mathbf{t}(\mathbf{x}(g_1g_2), 1) = 1$$

by Proposition 5.1.1(i) and (viii), hence $\mathbf{t}\left(\mathbf{x}(g_1g_2), \frac{1}{\mathbf{x}(g_2)}\right) = \mathbf{t}(\mathbf{x}(g_1g_2), \mathbf{x}(g_2))^{-1} = \mathbf{t}(\mathbf{x}(g_2), \mathbf{x}(g_1g_2))$ by Proposition 5.1.1(vi). Therefore, we obtain

$$\alpha(g_1, g_2) = \mathbf{t}(-1, \mathbf{x}(g_1g_2))\mathbf{t}(\mathbf{x}(g_2), \mathbf{x}(g_1g_2))\mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \mathbf{x}(g_1g_2)\right)\mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \frac{1}{\mathbf{x}(g_2)}\right).$$

But now, by combining the first three terms in this last expression with property (ii) and noting that

$$\mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \frac{1}{\mathbf{x}(g_2)}\right) = \mathbf{t}\left(\mathbf{x}(g_1), \frac{1}{\mathbf{x}(g_2)}\right)^{-1} = \mathbf{t}(\mathbf{x}(g_1), \mathbf{x}(g_2))$$

from Proposition 5.1.1(i), (ii) and (viii), we obtain that Eq.11 is equivalent to Eq.12. □

For the rest of this section, write $g_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in SL_2(\mathbb{R}((t)))$.

Lemma 5.1.3. *If $g_1, g_2 \in SL(2, \mathbb{R}((t)))$ such that $c_1 \neq 0$ but $c_2 = 0$, then*

$$\alpha(g_1, g_2) = \alpha(g_2, g_1) = \mathbf{t}(\mathbf{x}(g_1), \mathbf{x}(g_2)).$$

If both c_1 and c_2 are zero, then

$$\alpha(g_1, g_2) = \mathbf{t}(\mathbf{x}(g_1), \mathbf{x}(g_2))^{-1}.$$

Proof. In the first case, we have

$$g_1g_2 = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ c_1a_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

and

$$g_2g_1 = \begin{bmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ d_2c_1 & d_1d_2 \end{bmatrix}.$$

But note that $\det(g_2) = a_2d_2 = 1$, hence $a_2 = d_2^{-1}$. Then we obtain

$$\mathbf{x}(g_1g_2) = c_1a_2 = \frac{\mathbf{x}(g_1)}{\mathbf{x}(g_2)} \tag{13}$$

and

$$\mathbf{x}(g_2g_1) = d_2c_1 = \mathbf{x}(g_1)\mathbf{x}(g_2). \tag{14}$$

Therefore,

$$\begin{aligned}\alpha(g_1,g_2) &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2))\mathbf{t}\left(-\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)},\mathbf{x}(g_1g_2)\right)\\ &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2))\mathbf{t}\left(-\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)},\frac{\mathbf{x}(g_1)}{\mathbf{x}(g_2)}\right)\\ &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2))\mathbf{t}\left(-\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)},\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)^{-1} = \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2)),\end{aligned}$$

and

$$\begin{aligned}\alpha(g_2,g_1) &= \mathbf{t}(\mathbf{x}(g_1g_2)/\mathbf{x}(g_2),\mathbf{x}(g_1g_2)/\mathbf{x}(g_1))\\ &= \mathbf{t}(\mathbf{x}(g_1)/\mathbf{x}(g_2)^2,\mathbf{x}(g_2))\\ &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2))\mathbf{t}(-1/\mathbf{x}(g_2),\mathbf{x}(g_2))\mathbf{t}(-1/\mathbf{x}(g_2),\mathbf{x}(g_2))\\ &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2)).\end{aligned}$$

This proves the first claim. For the second claim, we have $\mathbf{x}(g_1g_2) = \mathbf{x}(g_1)\mathbf{x}(g_2)$ because $c_1 = c_2 = 0$ and then

$$\begin{aligned}\alpha(g_1,g_2) &= \mathbf{t}(\mathbf{x}(g_1g_2)/\mathbf{x}(g_1),\mathbf{x}(g_1g_2)/\mathbf{x}(g_2))\\ &= \mathbf{t}(\mathbf{x}(g_2),\mathbf{x}(g_1))\\ &= \mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_2))^{-1}\end{aligned}$$

as claimed. □

Lemma 5.1.4. *Suppose* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}((t)))$ *and* $c \neq 0$. *Then there exists a neighbourhood* $U \subseteq \mathbb{R}((t))$ *of* 0 *such that for all* $y \in \mathbb{R}((t) \setminus \{0\}$ *and*

$$h = h(\mu) = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix},$$

where $\mu \in U$, *we have*

$$\mathbf{t}(\mathbf{x}(hg),y) = \mathbf{t}(\mathbf{x}(g),y),$$

$$\mathbf{t}(\mathbf{x}(gh),y) = \mathbf{t}(\mathbf{x}(g),y).$$

In particular, we also have

$$\mathbf{t}(y, \mathbf{x}(hg)) = \mathbf{t}(y, \mathbf{x}(g)),$$

$$\mathbf{t}(y, \mathbf{x}(gh)) = \mathbf{t}(y, \mathbf{x}(g)).$$

Proof. Note that

$$hg = \begin{bmatrix} a & b \\ \mu a + c & \mu b + d \end{bmatrix}$$

and

$$gh = \begin{bmatrix} a + b\mu & b \\ c + d\mu & d \end{bmatrix}$$

Let $M = \max\{v(c) - v(a) + 1, v(c) - v(d) + 1\}$ and

$$U = U_M = \{x \in \mathbb{R}((t)) : v(x) \geq M\}$$

which is clearly non-empty. Then $v(\mu a) = v(\mu) + v(a) > v(c)$ by the choice of M. Similarly, $v(d\mu) > v(c)$. Then $v(\mu a + c) = v(c + d\mu) = v(c)$ and the constant terms of $\mu a + c$, $c + d\mu$ and c are clearly all the same, hence we obtain the statement of the lemma. □

Lemma 5.1.5. *Suppose* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R}((t)))$. *Then there exists a neighbourhood* $U \subseteq \mathbb{R}((t))$ *of* 0 *such that for all*

$$h = h(\mu) = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix},$$

where $\mu \in U$, *we have*

$$\alpha(h, g) = \alpha(g, h) = \begin{cases} 1 & \text{if } c \neq 0, \\ \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g)) & \text{if } c = 0. \end{cases}$$

Proof. If $c = 0$, but $\mu \neq 0$ is arbitrary, we are done by Lemma 5.1.3. If both $c = 0$ and $\mu = 0$, then again by Lemma 5.1.3,

$$\alpha(h, g) = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g))^{-1} = \mathbf{t}(\mathbf{x}(g), 1) = 1 = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g)).$$

But we also have $\alpha(g, h) = \mathbf{t}(\mathbf{x}(g), \mathbf{x}(h))^{-1} = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g))$. This completes the case where $c = 0$ (any non-empty open set will do).

Next we assume $c \neq 0$. Choose U such that Lemma 5.1.4 holds. We obtain

$$\alpha(h,g) = \mathbf{t}(\mathbf{x}(h),\mathbf{x}(g))\mathbf{t}(-\frac{\mathbf{x}(g)}{\mathbf{x}(h)},\mathbf{x}(hg)) = \mathbf{t}(\mathbf{x}(h),\mathbf{x}(g))\mathbf{t}\left(-\frac{\mathbf{x}(g)}{\mathbf{x}(h)},\mathbf{x}(g)\right)$$
$$= \mathbf{t}(\mathbf{x}(h),\mathbf{x}(g))\mathbf{t}(-\mathbf{x}(g),\mathbf{x}(g))\mathbf{t}\left(\frac{1}{\mathbf{x}(h)},\mathbf{x}(g)\right) = \mathbf{t}(1,\mathbf{x}(g)) = 1.$$

The proof that $\alpha(g,h) = 1$ is done similarly. □

Lemma 5.1.6. *Suppose $g_1, g_2 \in SL(2,\mathbb{R}((t)))$. Then there exists a neighbourhood $U \subseteq \mathbb{R}((t))$ of 0 such that for all*

$$h = h(\mu) = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix}$$

where $\mu \in U$, we have

$$\alpha(h,g_1)\alpha(hg_1,g_2) = \alpha(h,g_1g_2)\alpha(g_1,g_2), \tag{15}$$

and

$$\alpha(g_1,g_2)\alpha(g_1g_2,h) = \alpha(g_1,g_2h)\alpha(g_2,h). \tag{16}$$

Proof. First, note that Eq. (15) is equivalent to

$$\alpha(h,g_1)\alpha(hg_1,g_1^{-1}g_2) = \alpha(h,g_2)\alpha(g_1,g_1^{-1}g_2). \tag{17}$$

This can be seen by replacing g_2 with $g_1^{-1}g_2$. Equivalently, we will prove

$$\alpha(hg_1,g_1^{-1}g_2)\alpha(g_1,g_1^{-1}g_2)^{-1} = \alpha(h,g_2)\alpha(h,g_1)^{-1}. \tag{18}$$

We expand the left hand side to obtain

$$\mathbf{t}(\mathbf{x}(hg_1),\mathbf{x}(g_1^{-1}g_2))\mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_1^{-1}g_2))^{-1}\mathbf{t}\left(\frac{-\mathbf{x}(g_1^{-1}g_2)}{\mathbf{x}(hg_1)},\mathbf{x}(hg_2)\right)\mathbf{t}\left(\frac{-\mathbf{x}(g_1^{-1}g_2)}{\mathbf{x}(g_1)},\mathbf{x}(g_2)\right)^{-1} \tag{19}$$

First we suppose that $c_1 \neq 0$ and $c_2 \neq 0$. From Lemma 5.1.5, choose U so that Lemma 5.1.4 and Lemma 5.1.5 hold for g_1 and g_2. Then Expression (19) reduces to

$$\mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_1^{-1}g_2))\mathbf{t}(\mathbf{x}(g_1),\mathbf{x}(g_1^{-1}g_2))^{-1}\mathbf{t}\left(-\frac{\mathbf{x}(g_1^{-1}g_2)}{\mathbf{x}(g_1)},\mathbf{x}(g_2)\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1^{-1}g_2)}{\mathbf{x}(g_1)},\mathbf{x}(g_2)\right)^{-1},$$

which simplifies to 1. However, we also have $\alpha(h, g_2) = \alpha(h, g_1) = 1$. This verifies Eq. (18).

Secondly, suppose $c_1 \neq 0$ but $c_2 = 0$. Then we can choose U so that Lemma 5.1.4 holds for g_1 and Lemma 5.1.5 holds for g_1 and g_2. Then $\alpha(h, g_1) = 1$ and $\alpha(h, g_2) = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g_2))$. Using Eq. (13) on $\mathbf{x}(hg_2)$ and $\mathbf{x}(g_1^{-1}g_2)$, we can simplify Expression (19) to obtain

$$\mathbf{t}\left(-\frac{\mathbf{x}(g_1^{-1})}{\mathbf{x}(g_1)\mathbf{x}(g_2)}, \frac{\mathbf{x}(h)}{\mathbf{x}(g_2)^2}\right).$$

Furthermore, from the formula for the inverse of a 2×2 matrix, we know that $\mathbf{x}(g_1^{-1}) = -\mathbf{x}(g_1)$, so we can simplify even further to obtain

$$\mathbf{t}\left(\frac{1}{\mathbf{x}(g_2)}, \frac{\mathbf{x}(h)}{\mathbf{x}(g_2)^2}\right) = \mathbf{t}\left(\frac{1}{\mathbf{x}(g_2)}, \mathbf{x}(h)\right)\mathbf{t}\left(\frac{1}{\mathbf{x}(g_2)}, \frac{1}{\mathbf{x}(g_2)^2}\right) = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g_2)).$$

This verifies Eq. (18). (Note that $\mathbf{t}(x, x^2) = 1$ for all x because, using Proposition 5.1.1 (i), $\mathbf{t}(x, x^2) = \mathbf{t}(x, -x)\mathbf{t}(x, -x) = 1$).

As third case, suppose $c_1 = 0$ but $c_2 \neq 0$. Then we can choose U such that Lemma 5.1.4 holds for g_2 and Lemma 5.1.5 holds for g_1 and g_2. Then $\alpha(h, g_2) = 1$ and $\alpha(h, g_1) = \mathbf{t}(\mathbf{x}(h), \mathbf{x}(g_1))$. Furthermore, from Eq. (13) and 14, we have that $\mathbf{x}(hg_1) = \frac{\mathbf{x}(h)}{\mathbf{x}(g_1)}$ and $\mathbf{x}(g_1^{-1}g_2) = \mathbf{x}(g_1^{-1})\mathbf{x}(g_2)$. Then we can once again simplify Expression (19) to obtain

$$\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)}, \mathbf{x}(g_1^{-1})\mathbf{x}(g_2)\right)\mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \mathbf{x}(g_1^{-1})\mathbf{x}(g_2)\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1)}{\mathbf{x}(h)}(\mathbf{x}(g_1^{-1})\mathbf{x}(g_2)), \mathbf{x}(g_2)\right)$$
$$\mathbf{t}\left(-\mathbf{x}(g_1)\mathbf{x}(g_2)^{-1}\mathbf{x}(g_1^{-1})^{-1}, \mathbf{x}(g_2)\right)$$

or equivalently, since in this case from the formula of the inverse of a 2×2 matrix we have $\mathbf{x}(g_1^{-1}) = \frac{1}{\mathbf{x}(g_1)}$

$$\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(\frac{\mathbf{x}(g_1)^2}{\mathbf{x}(h)}, \mathbf{x}(g_2)\right) = \mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{1}{\mathbf{x}(g_1)}\right)$$

Now, similarly as in the second case, we obtain

$$\mathbf{t}(\mathbf{x}(h), \mathbf{x}(g_1))^{-1}$$

which verifies Eq. (18).

In the fourth case, where $c_1 = c_2 = 0$, we apply Eq. (13) and (14) multiple times on Expression (19) to obtain

$$\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)}, \mathbf{x}(g_1^{-1}g_2)\right)\mathbf{t}\left(\frac{1}{\mathbf{x}(g_1)}, \mathbf{x}(g_1^{-1}g_2)\right)\mathbf{t}\left(\frac{-\mathbf{x}(g_1)}{\mathbf{x}(h)}\mathbf{x}(g_1^{-1}g_2), \frac{\mathbf{x}(h)}{\mathbf{x}(g_2)}\right)\mathbf{t}\left(\frac{-\mathbf{x}(g_1)}{\mathbf{x}(g_1^{-1}g_2)}, \mathbf{x}(g_2)\right),$$

or equivalently

$$\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \mathbf{x}(g_1^{-1}g_2)\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1)}{\mathbf{x}(h)}\mathbf{x}(g_1^{-1}g_2), \frac{\mathbf{x}(h)}{\mathbf{x}(g_2)}\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1)}{\mathbf{x}(g_1^{-1}g_2)}, \mathbf{x}(g_2)\right).$$

In this case, one can compute from matrix multiplication that $\mathbf{x}(g_1^{-1}g_2) = \mathbf{x}(g_2)\mathbf{x}(g_1)^{-1}$, simplifying this expression to

$$\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_2)}{\mathbf{x}(h)}, \frac{\mathbf{x}(h)}{\mathbf{x}(g_2)}\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1)^2}{\mathbf{x}(g_2)}, \mathbf{x}(g_2)\right)$$

and the middle term equals 1, hence can be removed to yield

$$\begin{aligned}
\mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(-\frac{\mathbf{x}(g_1)^2}{\mathbf{x}(g_2)}, \mathbf{x}(g_2)\right) &= \mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(\mathbf{x}(g_2), -\frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)^2}\right) \\
&= \mathbf{t}\left(\frac{\mathbf{x}(h)}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(\mathbf{x}(g_2), \frac{\mathbf{x}(g_2)^2}{\mathbf{x}(g_1)^2}\right) \\
&= \mathbf{t}\left(\frac{\mathbf{x}(h)\mathbf{x}(g_2)^2}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right) \\
&= \mathbf{t}\left(\frac{\mathbf{x}(g_2)^2}{\mathbf{x}(g_1)^2}, \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right)\mathbf{t}\left(\mathbf{x}(h), \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right) \\
&= \mathbf{t}\left(\mathbf{x}(h), \frac{\mathbf{x}(g_2)}{\mathbf{x}(g_1)}\right),
\end{aligned}$$

which verifies Eq. (18) using Lemma 5.1.5 on the right side.

Note that Eq. (16) is equivalent to

$$\alpha(g_1g_2^{-1}, g_2)\alpha(g_1, h) = \alpha(g_1g_2^{-1}, g_2h)\alpha(g_2, h). \tag{20}$$

This can be seen by replacing g_1 with $g_1g_2^{-1}$. Equivalently, we can prove

$$\alpha(g_1g_2^{-1}, g_2h)\alpha(g_1g_2^{-1}, g_2)^{-1} = \alpha(g_2, h)^{-1}\alpha(g_1, h) \tag{21}$$

The computations are very similar to the ones for Eq. (18). One expands the left side of this equation and then deals with four cases using Lemmas 5.1.3, 5.1.4, 5.1.5. □

Lemma 5.1.7. *Let $g_1, g_2, g_3 \in SL_2(\mathbb{R}((t)))$ and*

$$\beta(g_1, g_2, g_3) = \alpha(g_1, g_2)\alpha(g_1g_2, g_3)\alpha(g_1, g_2g_3)^{-1}\alpha(g_2, g_3)^{-1}$$

Then there exists a neighbourhood $U \subseteq \mathbb{R}((t))$ of 0 such that for all

$$h = h(\mu) = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix}$$

where $\mu \in U$, we have

$$\beta(hg_1, g_2, g_3) = \beta(g_1, g_2, g_3h) = \beta(g_1, g_2, g_3)$$

Proof. Using Lemma 5.1.6, choose U_1 such that the following three equations hold simultaneously.

$$\begin{aligned}
&\alpha(hg_1, g_2) = \alpha(h, g_1)^{-1}\alpha(h, g_1g_2)\alpha(g_1, g_2), \\
&\alpha(hg_1g_2, g_3) = \alpha(h, g_1g_2)^{-1}\alpha(h, g_1g_2g_3)\alpha(g_1g_2, g_3), \\
&\alpha(hg_1, g_2g_3)^{-1} = \alpha(h, g_1)\alpha(h, g_1g_2g_3)^{-1}\alpha(g_1, g_2g_3)^{-1}.
\end{aligned}$$

Then we have

$$\beta(hg_1, g_2, g_3) = \alpha(hg_1, g_2)\alpha(hg_1g_2, g_3)\alpha(hg_1, g_2g_3)^{-1}\alpha(g_2, g_3)^{-1}$$

By replacing $\alpha(hg_1, g_2)$, $\alpha(hg_1g_2, g_3)$ and $\alpha(hg_1, g_2g_3)^{-1}$ with the above three equations and cancelling the terms, we obtain

$$\alpha(g_1, g_2)\alpha(g_1g_2, g_3)\alpha(g_1, g_2g_3)^{-1}\alpha(g_2, g_3)^{-1} = \beta(g_1, g_2, g_3),$$

hence $\beta(hg_1, g_2, g_3) = \beta(g_1, g_2, g_3)$.

The claim that $\beta(g_1, g_2, g_3h) = \beta(g_1, g_2, g_3)$ is similar, but uses Eq.(16) instead and a different open set U_2. Taking $U = U_1 \cap U_2$ as the open set for the statement of the lemma, this completes the proof. □

Let

$$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R}((t)) \right\}$$

Clearly, N is a subgroup of $SL_2(\mathbb{R}((t)))$.

The following lemma is only briefly explained in [5].

Lemma 5.1.8. *Suppose* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}((t)))$. *Then there exist* v, $w \in N$ *such that either*

$$vgw = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \qquad \text{or} \qquad vgw = \begin{bmatrix} 0 & -c^{-1} \\ c & 0 \end{bmatrix}$$

Proof. We deal with two cases.

1. Case 1 : Suppose $c \neq 0$. Then let $v = \begin{bmatrix} 1 & \frac{-a}{c} \\ 0 & 1 \end{bmatrix}$. Now, the product vg is

$$\begin{bmatrix} 0 & b - \frac{ad}{c} \\ c & d \end{bmatrix}$$

   Letting $w = \begin{bmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{bmatrix}$, we obtain the second form after multiplication (note that $-c^{-1} = b - \frac{ad}{c}$).

2. Case 2 : Suppose $c = 0$. Then $d = a^{-1} \neq 0$. Therefore, we can choose $w = I$ and $v = \begin{bmatrix} 1 & -\frac{b}{d} \\ 0 & 1 \end{bmatrix}$ and we obtain the first form, as desired.

□

Notation 5.1.9. *If* $A = (a_{i,j})_{1 \leq i,j \leq m}$ *is a matrix, denote* $\pi_{i,j}(A) = a_{i,j}$.

Lemma 5.1.10. *Suppose* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}((t)))$ *and* $\pi_{2,1}(g) \neq 0$. *Then there exist* $v, v' \in N$ *such that*

$$vg = \begin{bmatrix} 0 & b' \\ c' & d' \end{bmatrix} \qquad \text{and} \qquad gv' = \begin{bmatrix} a'' & b'' \\ c'' & 0 \end{bmatrix}$$

for some $a', b', c', a'', b'', c'' \in \mathbb{R}((t))$.

Proof. The proof is very similar to Lemma 5.1.8 and is an easy exercise. □

We now prove the following theorem, which is the main goal of this section.

Theorem 5.1.11. *([5], Section 2) Let* β *be defined as in Lemma 5.1.7. Then we have* $\beta(g_1, g_2, g_3) = 1$. *In particular,* α *is a cocycle.*

Proof. Let U be as in Lemma 5.1.7 and write

$$h = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} \qquad\qquad h' = \begin{bmatrix} 1 & 0 \\ \mu' & 1 \end{bmatrix}$$

for $\mu, \mu' \in U$. It is easy to see that every non-empty open set of $\mathbb{R}((t))$ is infinite, hence U also is. Consider the products

$$hg_1 = \begin{bmatrix} a & b \\ \mu a_1 + c_1 & \mu b + d \end{bmatrix} \qquad \text{and} \qquad g_1 h = \begin{bmatrix} a_1 + b_1\mu & b_1 \\ c_1 + \mu d_1 & d_1 \end{bmatrix}.$$

We cannot have both $a_1 = 0$ and $c_1 = 0$ (otherwise g_1 would have determinant 0). Therefore, if $c_1 = 0$, then $\pi_{2,1}(hg_1) = \mu a_1 + c_1 = 0$ implies that $\mu = 0$. If $c_1 \neq 0$, then $\mu a_1 + c_1 = 0$ implies that $\mu a_1 = -c_1$ and then $\mu = -c_1 a_1^{-1}$ (note that in this case, $a_1 \neq 0$ as well). In both cases, μ is uniquely determined by a_1 and c_1. Similarly, the condition $\pi_{2,1}(g_1 h) = c_1 + \mu d_1 = 0$ also determines μ uniquely. However, U is infinite so for any finite collection k_i's and l_j's of elements of $SL_2(\mathbb{R}((t)))$, we can choose $\mu \in U$ such that $\pi_{2,1}(hk_i) \neq 0$ and $\pi_{2,1}(l_j h) \neq 0$.

Then choose $\mu \in U$ such that both $\pi_{2,1}(hg_1) \neq 0$ and $\pi_{2,1}(hg_1g_2) \neq 0$ hold. Similarly, one can also choose $\mu' \in U$ such that $\pi_{2,1}(g_3h') \neq 0$, $\pi_{2,1}(g_2g_3h') \neq 0$ and $\pi_{2,1}(hg_1g_2g_3h') \neq 0$. Now, note that from Lemma 5.1.7, we have

$$\beta(hg_1, g_2, g_3h') = \beta(hg_1, g_2, g_3) = \beta(g_1, g_2, g_3).$$

Therefore, we can assume that $\pi_{2,1}(g_1) \neq 0$, $\pi_{2,1}(g_3) \neq 0$, $\pi_{2,1}(g_1g_2) \neq 0$, $\pi_{2,1}(g_2g_3) \neq 0$ and $\pi_{2,1}(g_1g_2g_3) \neq 0$ by replacing g_1 with hg_1 and g_3 with g_3h'.

It is easy to see from a simple computation that for all $v, w \in N$, $\mathbf{x}(vgw) = \mathbf{x}(g)$. Then we have that for any $g, g' \in SL_2(\mathbb{R}((t)))$,

$$\begin{aligned} \alpha(vg, g') &= \mathbf{t}(\mathbf{x}(vg), \mathbf{x}(g'))\mathbf{t}(-\mathbf{x}(vg)^{-1}\mathbf{x}(g'), \mathbf{x}(vgg')) \\ &= \mathbf{t}(\mathbf{x}(g), \mathbf{x}(g'))\mathbf{t}(-\mathbf{x}(g)^{-1}\mathbf{x}(g'), \mathbf{x}(gg')) \\ &= \mathbf{t}(g, g') \end{aligned} \tag{22}$$

and

$$\begin{aligned} \alpha(g, g'v) &= \mathbf{t}(\mathbf{x}(g), \mathbf{x}(g'v))\mathbf{t}(-\mathbf{x}(g)^{-1}\mathbf{x}(g'v), \mathbf{x}(gg'v)) \\ &= \mathbf{t}(\mathbf{x}(g), \mathbf{x}(g'))\mathbf{t}(-\mathbf{x}(g)^{-1}\mathbf{x}(g'), \mathbf{x}(gg')) \\ &= \mathbf{t}(g, g') \end{aligned} \tag{23}$$

and

$$\begin{aligned}\alpha(gv, g') &= \mathbf{t}(\mathbf{x}(gv), \mathbf{x}(g'))\mathbf{t}(-\mathbf{x}(gv)^{-1}\mathbf{x}(g'), \mathbf{x}(gvg')) \\ &= \mathbf{t}(\mathbf{x}(g), \mathbf{x}(vg'))\mathbf{t}(-\mathbf{x}(v)^{-1}\mathbf{x}(vg'), \mathbf{x}(gvg')) \\ &= \alpha(g, vg')\end{aligned} \tag{24}$$

Then we obtain for all $v_1, v_2, v_3 \in N$, $g_1, g_2, g_3 \in SL_2(\mathbb{R}((t)))$ that

$$\begin{aligned}&\beta(v_1 g_1 v_2, v_2^{-1} g_2 v_3, v_3^{-1} g_3 v_4) \\ &= \alpha(v_1 g_1 v_2, v_2^{-1} g_2 v_3)\alpha(v_1 g_1 g_2 v_3, v_3^{-1} g_3 v_4)\alpha(v_1 g_1 v_2, v_2^{-1} g_2 g_3 v_4)^{-1}\alpha(v_2^{-1} g_2 v_3, v_3^{-1} g_3 v_4)^{-1} \\ &= \alpha(g_1 v_2, v_2^{-1} g_2)\alpha(g_1 g_2 v_3, v_3^{-1} g_3)\alpha(g_1 v_2, v_2^{-1} g_2 g_3)^{-1}\alpha(g_2 v_3, v_3^{-1} g_3)^{-1} \\ &= \alpha(g_1, g_2)\alpha(g_1 g_2, g_3)\alpha(g_1, g_2 g_3)^{-1}\alpha(g_2, g_3)^{-1} \\ &= \beta(g_1, g_2, g_3)\end{aligned}$$

Therefore, we can choose v_2^{-1} and v_3 such that $v_2^{-1} g_2 v_3$ is in one of the two forms of Lemma 5.1.8. Then we also have $\pi_{2,1}(g_1 v_2) \neq 0$ and $\pi_{2,1}(v_3^{-1} g_3) \neq 0$, so we can choose v_1 and v_4 so that $v_1 g_1 v_2$ and $v_3^{-1} g_3 v_4$ are in the forms of Lemma 5.1.10. Therefore, it is enough to verify the theorem in the following two cases :

1. $\pi_{2,1}(g_2) \neq 0$:

$$g_1 = \begin{bmatrix} 0 & -a^{-1} \\ a & d \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 & -b^{-1} \\ b & 0 \end{bmatrix}, \qquad g_3 = \begin{bmatrix} e & -c^{-1} \\ c & 0 \end{bmatrix}$$

We obtain from our assumptions that $\mathbf{x}(g_1 g_2) = bd$, $\mathbf{x}(g_2 g_3) = be$, and $\mathbf{x}(g_1 g_2 g_3) = bde - ab^{-1}c$. Therefore, $\beta(g_1, g_2, g_3)$ equals

$$\begin{aligned}&\alpha(g_1, g_2)\alpha(g_1 g_2, g_3)\alpha(g_1, g_2 g_3)^{-1}\alpha(g_2, g_3)^{-1} \\ &= \mathbf{t}(a, b)\mathbf{t}(-a^{-1}b, bd)\mathbf{t}(bd, c)\mathbf{t}(-b^{-1}d^{-1}c, bde - ab^{-1}c) \\ &\mathbf{t}(a, be)^{-1}\mathbf{t}(-a^{-1}be, bde - ab^{-1}c)^{-1}\mathbf{t}(b, c)^{-1}\mathbf{t}(-b^{-1}c, be)^{-1} \\ &= \mathbf{t}(a, e^{-1})\mathbf{t}(-a^{-1}b, bd)\mathbf{t}(-ab^{-1}c(bde)^{-1}, bde - ab^{-1}c)\mathbf{t}(d, c)\mathbf{t}(-b^{-1}c, be)^{-1} \\ &= \mathbf{t}(a^{-1}, e)\mathbf{t}(-a^{-1}b, bd)\mathbf{t}(d, c)\mathbf{t}(-b^{-1}c, be)^{-1}\mathbf{t}(-ab^{-1}c, bde) \\ &= \mathbf{t}(a^{-1}, e)\mathbf{t}(c, bd)\mathbf{t}(d, c)\mathbf{t}(-bc^{-1}, be)\mathbf{t}(-ab^{-1}c, e) \\ &= \mathbf{t}(c, b)\mathbf{t}(c, d)\mathbf{t}(d, c)\mathbf{t}(-bc^{-1}, be)\mathbf{t}(-b^{-1}c, e) \\ &= \mathbf{t}(c, b)\mathbf{t}(-bc^{-1}, b) = \mathbf{t}(-b, b) = 1\end{aligned}$$

as desired.

2. $\pi_{2,1}(g_2) = 0$:

$$g_1 = \begin{bmatrix} 0 & -a^{-1} \\ a & d \end{bmatrix}, \qquad g_2 = \begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix}, \qquad g_3 = \begin{bmatrix} e & -c^{-1} \\ c & 0 \end{bmatrix}$$

In this case, $\mathbf{x}(g_1g_2) = ab^{-1}$, $\mathbf{x}(g_2g_3) = bc$, $\mathbf{x}(g_1g_2g_3) = ab^{-1}e + bcd$. Then

$$\beta(g_1, g_2, g_3) = \alpha(g_1, g_2)\alpha(g_1g_2, g_3)\alpha(g_1, g_2g_3)^{-1}\alpha(g_2, g_3)^{-1}$$

From Lemma 5.1.3 on the first and last terms, we have

$$\begin{aligned} \beta(g_1, g_2, g_3) &= \mathbf{t}(a, b)\mathbf{t}(ab^{-1}, c)\mathbf{t}(-a^{-1}bc, ab^{-1}e, ab^{-1}e + bcd)\mathbf{t}(a, bc)^{-1} \\ &\quad \mathbf{t}(-a^{-1}bc, ab^{-1}e + bcd)^{-1}\mathbf{t}(c, b)^{-1} \\ &= \mathbf{t}(a, b)\mathbf{t}(b, c)\mathbf{t}(ab^{-1}, c)\mathbf{t}(bc, a) \\ &= \mathbf{t}(a, b)\mathbf{t}(a, c)\mathbf{t}(bc, a) \\ &= \mathbf{t}(a, bc)\mathbf{t}(bc, a) = 1 \end{aligned}$$

which completes the proof of the theorem. □

5.2 Splitting of the cocycle over a subgroup

The goal of this section is to show that the cocycle α splits over $SL_2(\mathbb{R}[[t]])$. We do so by proving that it is equal to a coboundary. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define

$$s(g) = \begin{cases} \mathbf{t}(c, d) & \text{if } v(c) > 0 \text{ and } c \neq 0 \\ 1 & \text{if } v(c) = 0, +\infty \end{cases}$$

and

$$\gamma(g_1, g_2) = s(g_1)^{-1}s(g_2)^{-1}s(g_1g_2).$$

Clearly, $\gamma(.,.)$ is a coboundary. Write $N' = N \cap SL_2(\mathbb{R}[[t]])$. It is easy to see that N' is a subgroup of $SL_2(\mathbb{R}[[t]])$.

Lemma 5.2.1. *For all $u, v, w \in N'$ and $g_1, g_2 \in SL_2(\mathbb{R}[[t]])$, we have*

$$\alpha(ug_1v, v^{-1}g_2w) = \alpha(g_1, g_2),$$

and

$$\gamma(ug_1v, v^{-1}g_2w) = \gamma(g_1, g_2).$$

Proof. The first statement is proved more generally in Theorem 5.1.11, as a corollary of Eq. (22), (23) and (24). For the second statement, it is enough to prove that $s(gu) = s(ug) = s(g)$ for all $u \in N'$, $g \in SL_2(\mathbb{R}[[t]])$. Write $u = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$ug = \begin{bmatrix} a + xc & b + xd \\ c & d \end{bmatrix} \text{ and } gu = \begin{bmatrix} a & ax + b \\ c & cx + d \end{bmatrix}$$

Clearly, $s(ug) = s(g)$. If $v(c) = 0$, then we also have $s(gu) = s(g) = 1$. Therefore, suppose $v(c) > 0$. Then $ad - cd = 1$ and $v(cd) > 0$ implies that $v(ad) = 0$, hence $v(d) = 0$. But $v(x) \geq 0$, hence $v(cx) > 0$ and then $v(cx + d) = 0$. Then if $c = \sum_{i \geq 0} a_i t^i$, $d = \sum_{i \geq 0} b_i t^i$ and $cx + d = \sum_{i \geq 0} c_i t^i$, we have $a_0 = 0$ and $b_0 = c_0 \neq 0$. Therefore,

$$s(gu) = \mathbf{t}(c, cx + d) = (-1)^0 a_0^{-v(cx+d)} c_0^{v(c)} = b_0^{v(c)} = \mathbf{t}(c, d) = s(g),$$

as claimed. □

The following is a more specialized variation of Lemma 5.1.8. The main difference is that we have more information on g.

Lemma 5.2.2. *Suppose* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}[[t]])$. *Then there exists* $w \in N'$ *such that*

$$gw = \begin{cases} \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} & \text{if } d \in \mathbb{R}[[t]]^* \\ \begin{bmatrix} a & -c^{-1} \\ c & 0 \end{bmatrix} & \text{if } c \in \mathbb{R}[[t]]^* \end{cases}$$

Similarly, there exists $u \in N'$ *such that*

$$ug = \begin{cases} \begin{bmatrix} d^{-1} & 0 \\ c & d \end{bmatrix} & \text{if } d \in \mathbb{R}[[t]]^* \\ \begin{bmatrix} 0 & -c^{-1} \\ c & d \end{bmatrix} & \text{if } c \in \mathbb{R}[[t]]^* \end{cases}$$

Proof. Since $ad - bc = 1$, we must have either both $a, d \in \mathbb{R}[[t]]^*$ or $b, c \in \mathbb{R}[[t]]^*$. In the first case, we have

$$g \begin{bmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & -\frac{cb}{a} + d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & -\frac{b}{d} \\ 0 & 1 \end{bmatrix} g = \begin{bmatrix} a - \frac{bc}{d} & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} d^{-1} & 0 \\ c & d \end{bmatrix}.$$

In the second case, we have

$$g \begin{bmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -\frac{da}{c} + b \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & -c^{-1} \\ c & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} g = \begin{bmatrix} 0 & -\frac{da}{c} + b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -c^{-1} \\ c & d \end{bmatrix}.$$

□

Theorem 5.2.3. *For all $g_1, g_2 \in SL_2(\mathbb{R}[[t]])$, $\alpha(g_1, g_2) = \gamma(g_1, g_2)$. Consequently, α is a coboundary.*

Remark 5.2.4. The proof is done with a case by case argument. The main reason for this is that the definition of $\mathbf{x}$ depends on whether or not $\pi_{2,1}(g) = 0$ or not, and the definition for $s(g)$ depends on whether or not $v(c) = 0$ or not. Because the definition of α and γ depend on g_1, g_2 and their product $g_1 g_2$, we get a surprisingly large amount of cases on the values they can take. We will use Lemmas 5.2.1 and 5.2.2 to help reduce the number of cases.

Proof. Write

$$g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \qquad g_1 g_2 = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}.$$

From Lemma 5.2.1, we can replace g_1 and g_2 by $u g_1 v$ and $v^{-1} g_2 w$ for $u, v, w \in N'$. We start with two main cases.

Case 1 : Suppose $c_1 \neq 0$ and $c_2 \neq 0$. Then we deal with three more subcases

1. Suppose $v(c_1) = 0$. Then by the proof of Lemma 5.1.8, we can choose $u, v \in N'$ (because here $v(c_1) = 0$) such that $g_1' := ug_1v = \begin{bmatrix} 0 & -c_1^{-1} \\ c_1 & 0 \end{bmatrix}$. By Lemma 5.2.2, there is $w \in N'$ such that

$$g_2' := v^{-1}g_2w = \begin{bmatrix} a_2 & 0 \\ c_2 & a_2^{-1} \end{bmatrix} \qquad \text{or} \qquad g_2' := v^{-1}g_2w = \begin{bmatrix} a_2 & -c_2^{-1} \\ c_2 & 0 \end{bmatrix}$$

Assume the first case. Then $v(a_2) = 0$ and $g_1'g_2' = \begin{bmatrix} -c_1^{-1}a_2 & -c_1^{-1}a_2^{-1} \\ c_1a_2 & 0 \end{bmatrix}$. Then we must have $\mathbf{x}(g_1') = c_1$, $\mathbf{x}(g_2') = c_2$ and $\mathbf{x}(g_1'g_2') = c_1a_2$. Therefore,

$$\alpha(g_1', g_2') = \mathbf{t}(c_1a_2/c_1, c_1a_2/c_2) = \mathbf{t}(a_2, c_1a_2/c_2) = \mathbf{t}(a_2, c_2^{-1}).$$

But also, $s(g_1') = 1$, $s(g_1'g_2') = 1$. If $v(c_2) > 0$, then $s(g_2') = \mathbf{t}(c_2, a_2^{-1}) = \mathbf{t}(a_2, c_2) = \mathbf{t}(a_2, c_2^{-1})^{-1}$. If $v(c_2) = 0$, then both $\mathbf{t}(a_2, c_2^{-1})$ and $s(g_2')$ equal 1. Therefore, $\alpha(g_1', g_2') = \gamma(g_1', g_2')$.

Now suppose the second case. Then $v(c_2) = 0$ and $g_1'g_2' = \begin{bmatrix} -c_1^{-1}c_2 & 0 \\ c_1a_2 & -c_1c_2^{-1} \end{bmatrix}$ This yields three subcases :

(a) $v(a_2) = 0$: Clearly, we have $s(g_1) = s(g_2) = s(g_1g_2) = 1$ and we obtain $\alpha(g_1', g_2') = \mathbf{t}(a_2, -c_1a_2c_2^{-2}) = 1$ because $v(c_1) = v(c_2) = v(a_2) = 0$.

(b) $v(a_2) > 0$: In this case, we still have $s(g_1) = 0$ and $s(g_2) = 0$, but $s(g_1g_2) = \mathbf{t}(c_1a_2, -c_1c_2^{-1})$. Furthermore,

$$\alpha(g_1, g_2) = \mathbf{t}(a_2, c_1a_2/c_2) = \mathbf{t}(a_2, -c_1c_2^{-1})\mathbf{t}(a_2, -a_2) = s(g_1g_2).$$

(c) $a_2 = 0$: Clearly we have $s(g_1) = s(g_2) = s(g_1g_2) = 1$ and we obtain $\alpha(g_1', g_2') = \mathbf{t}(-c_2^{-1}, -c_1c_2^{-2}) = 1$ because $v(c_1) = v(c_2) = 0$.

In each of these subcases, we conclude that $\alpha(g_1', g_2') = \gamma(g_1', g_2')$

2. $v(c_2) = 0$: In this case, choose v^{-1}, w such that $g_2' = v^{-1}g_2w = \begin{bmatrix} 0 & -c_2^{-1} \\ c_2 & 0 \end{bmatrix}$. Then by Lemma 5.2.2, there exists $u \in N'$ such that

$$g_1' = ug_1v = \begin{bmatrix} 0 & -c_1^{-1} \\ c_1 & d_1 \end{bmatrix} \qquad \text{or} \qquad g_1' = ug_2v = \begin{bmatrix} d_1^{-1} & 0 \\ c_1 & d_1 \end{bmatrix}$$

If $v(c_1) = 0$, then we can use the first case, hence we are done by the previous result. Therefore, we can assume $v(c_1) > 0$ and use $g_1' = \begin{bmatrix} d_1^{-1} & 0 \\ c_1 & d_1 \end{bmatrix}$, hence $v(d_1) = 0$ and $g_1'g_2' = \begin{bmatrix} 0 & -(d_1c_2)^{-1} \\ d_1c_2 & -c_1c_2^{-1} \end{bmatrix}$. Then $\mathbf{x}(g_1') = c_1$, $\mathbf{x}(g_2') = c_2$ and $\mathbf{x}(g_1'g_2') = d_1c_2$. Therefore, $\alpha(g_1', g_2') = \mathbf{t}(d_1c_2c_1^{-1}, d_1) = \mathbf{t}(c_1^{-1}, d_1)$. Furthermore, $s(g_1') = \mathbf{t}(c_1, d_1) = \mathbf{t}(c_1^{-1}, d_1)^{-1}$, $s(g_2') = 1$ and $s(g_1'g_2') = 1$. Therefore, $\alpha(g_1'g_2') = \gamma(g_1', g_2')$.

3. $v(c_1) > 0$ and $v(c_2) > 0$. In this case, we must have $v(d) = v(d') = 0$. Therefore, we can choose by Lemma 5.2.2 u, w such that

$$g_1' = ug_1 = \begin{bmatrix} a_1 & 0 \\ c_1 & a_1^{-1} \end{bmatrix} \qquad \text{and} \qquad g_2' = g_2w = \begin{bmatrix} a_2 & 0 \\ c_2 & a_2^{-1} \end{bmatrix}$$

Then $g_1'g_2' = \begin{bmatrix} a_1a_2 & 0 \\ c_1a_2 + a_1^{-1}c_2 & (a_1a_2)^{-1} \end{bmatrix}$. Clearly, we have $v(a_1) = v(a_2) = 0$ and $s(g_1') = \mathbf{t}(a_1, c_1)$, $s(g_2') = \mathbf{t}(a_2, c_2)$. We deal with two cases.

   (a) $c_1a_2 + a_1^{-1}c_2 = 0$: In this case, $c_2 = -(a_1a_2c_1)$, $\mathbf{x}(g_1') = c_1$, $\mathbf{x}(g_2') = c_2$, and $\mathbf{x}(g_1'g_2') = (a_1a_2)^{-1}$, hence

$$\begin{aligned} \alpha(g_1', g_2') &= \mathbf{t}((a_1a_2)^{-1}c_1^{-1}, (a_1a_2)^{-1}c_2^{-1}) = \mathbf{t}(a_1a_2c_1, a_1a_2c_2) \\ &= \mathbf{t}(-c_2, a_1a_2c_2) = \mathbf{t}(-c_2, a_1a_2). \end{aligned}$$

But now, $s(g_1'g_2') = 1$. Therefore,

$$\begin{aligned} s(g_1')^{-1}s(g_2')^{-1} &= \mathbf{t}(c_1, a_1)\mathbf{t}(c_2, a_2) \\ &= \mathbf{t}(-a_1^{-1}a_2^{-1}c_2, a_1)\mathbf{t}(c_2, a_2) \\ &= \mathbf{t}(-a_1^{-1}a_2^{-1}, a_1)\mathbf{t}(c_2, a_1)\mathbf{t}(c_2, a_2) \\ &= \mathbf{t}(a_2^{-1}, a_2)\mathbf{t}(-1, a_1a_2)\mathbf{t}(-1, a_1a_2)\mathbf{t}(c_2, a_1a_2) \\ &= \mathbf{t}(-a_2^{-1}, a_2)\mathbf{t}(-1, a_1)\mathbf{t}(-c_2, a_1a_2) \\ &= \mathbf{t}(-c_2, a_1a_2). \end{aligned}$$

(b) $c_1a_2 + a_1^{-1}c_2 \neq 0$: In this case, $s(g_1'g_2') = \mathbf{t}(c_1a_2 + a_1^{-1}c_2, (a_1a_2)^{-1})$. Indeed, note that the equality holds in both cases where $v(c_1a_2 + a_1^{-1}c_2) = 0$ (because $v(a_1a_2) = 0$) and $v(c_1a_2 + a_1^{-1}c_2) > 0$. Furthermore, we have $\mathbf{x}(g_1'g_2') = c_1a_2 + a_1^{-1}c_2$ and we can compute

$$\begin{aligned} &s(g_1')^{-1}s(g_2')^{-1}s(g_1'g_2') = \mathbf{t}(c_1, a_1)\mathbf{t}(c_2, a_2)\mathbf{t}(c_1a_2 + a_1^{-1}c_2, (a_1a_2)^{-1}) \\ &= \mathbf{t}(c_1, a_1)\mathbf{t}(c_2, a_2)\mathbf{t}(c_1a_2 + a_1^{-1}c_2, -(a_1a_2)^{-1}c_2c_1^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, -c_1c_2^{-1}) \\ &= \mathbf{t}(c_1, a_1)\mathbf{t}(c_2, a_2)\mathbf{t}(c_1a_2, c_2a_1^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1c_2^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, -1) \\ &= \mathbf{t}(c_2, a_2)\mathbf{t}(a_2, c_2a_1^{-1})\mathbf{t}(c_1, c_2)\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1c_2^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1a_2 + a_1^{-1}c_2) \\ &= \mathbf{t}(a_1, a_2)\mathbf{t}(c_1, c_2)\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1c_2^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1a_2 + a_1^{-1}c_2) \\ &= \mathbf{t}(c_1^{-1}, c_2^{-1})\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_2^{-1})\mathbf{t}(c_1^{-1}, c_1a_2 + a_1^{-1}c_2)\mathbf{t}(c_1a_2 + a_1^{-1}c_2, c_1a_2 + a_1^{-1}c_2) \\ &= \mathbf{t}((c_1a_2 + a_1^{-1}c_2)/c_1^{-1}, (c_1a_2 + a_1^{-1}c_2)/c_2^{-1}) = \alpha(g_1', g_2'). \end{aligned}$$

Case 2 : Either $c_1 = 0$ or $c_2 = 0$:

1. Suppose $c_1 = 0$. Then $v(d_1) = 0$.

   (a) If $c_2 = 0$, then by Lemma 5.1.3, we have $\alpha(g_1, g_2) = \mathbf{t}(\mathbf{x}(g_2), \mathbf{x}(g_1)) = \mathbf{t}(d_2, d_1)$. But we must have $v(d_1) = v(d_2) = 0$, hence we also have $\mathbf{t}(d_2, d_1) = 1$ by property (iv). Furthermore, $s(g_1) = s(g_2) = s(g_1g_2) = 1$.

   (b) If $c_2 \neq 0$, then by Lemma 5.1.3, we have $\alpha(g_1, g_2) = \mathbf{t}(\mathbf{x}(g_1), \mathbf{x}(g_2)) = \mathbf{t}(d_1, c_2)$. Now, if $v(c_2) = 0$, $\mathbf{t}(d_1, c_2)$ is just 1 by property (iv), but we also have $s(g_1) = 1$, $s(g_2) = 1$ and $s(g_1g_2) = 1$.

If $v(c_2) > 0$, we have $v(d_1) = v(d_2) = 0$. Furthermore, $s(g_1) = 1$, $s(g_2) = \mathbf{t}(c_2, d_2)$ and $s(g_1g_2) = \mathbf{t}(d_1c_2, d_1d_2)$. Then

$$\begin{aligned}\gamma(g_1, g_2) &= \mathbf{t}(c_2, d_2)^{-1}\mathbf{t}(d_1c_2, d_1d_2)\\ &= \mathbf{t}(d_2, c_2)\mathbf{t}(d_1, d_1)\mathbf{t}(d_1, d_2)\mathbf{t}(c_2, d_1)\mathbf{t}(c_2, d_2)\\ &= \mathbf{t}(d_1, c_2) = \alpha(g_1, g_2).\end{aligned}$$

as claimed.

2. Suppose $c_2 = 0$. Then $v(a_2) = v(d_2) = 0$. If $c_1 = 0$, then we have already dealt with this case. Therefore, assume $c_1 \neq 0$. Then by Lemma 5.1.3,

$$\alpha(g_1, g_2) = \mathbf{t}(\mathbf{x}(g_1), \mathbf{x}(g_2)) = \mathbf{t}(c_1, d_2).$$

If $v(c_1) = 0$, then $\mathbf{t}(c_1, d_2)$ is 1 by Proposition 5.1.1(iv), but we also have $s(g_1) = s(g_2) = s(g_1g_2) = 1$.

If $v(c_1) > 0$, we have $v(d_1) = 0$. Furthermore, $s(g_1) = \mathbf{t}(c_1, d_1)$. Set $u = \begin{bmatrix} d_2 & 0 \\ 0 & a_2 \end{bmatrix}$. Then

$$ug_2 = \begin{bmatrix} 1 & d_2b_2 \\ 0 & 1 \end{bmatrix} \in N'.$$

But then

$$s(g_1g_2) = s(g_1u^{-1}ug_2) = s(g_1u^{-1}) = s\left(\begin{bmatrix} a_1a_2 & b_1d_2 \\ c_1a_2 & d_1d_2 \end{bmatrix}\right) = \mathbf{t}(c_1a_2, d_1d_2).$$

Therefore,

$$\begin{aligned}\gamma(g_1, g_2) &= \mathbf{t}(c_1, d_1)^{-1}\mathbf{t}(c_1a_2, d_1d_2)\\ &= \mathbf{t}(d_1, c_1)\mathbf{t}(c_1, d_1)\mathbf{t}(c_1, d_2)\mathbf{t}(a_2, d_1)\mathbf{t}(a_2, d_2)\\ &= \mathbf{t}(c_1, d_2) = \alpha(g_1, g_2),\end{aligned}$$

as desired. This completes every possible cases.

□

Bibliography

[1] D. Dummit, R. Foote, *Abstract Algebra* John Wiley and Sons, 2004

[2] Y. Eidelman, V. Milman, A. Tsolomitis *Functional Analysis, An Introduction* American Mathemical Society, 2004.

[3] G. Folland, *A course in Abstract Harmonic Analysis*. CRC Press, 1994

[4] G. Folland, *Real analysis, modern techniques and their applications*. CRC Press, 1999

[5] T. Kubota *Topological covering of SL(2) over a local field* J. Math. Soc. Japan Volume 19, Number 1 (1967), 114-121.

[6] C. Moeglin, M.-F.Vigneras, J.L.Waldspurger *Correspondence de Howe sur un corps p-adique*, Springer (1987).

[7] R. Rao *On some explicit formulas in the theory of Weil representation* Pacific Journal of Mathematics Volume 157, No. 2 (1993), 335-371.

[8] A. Weil *Sur certaines groupes d'opérateurs unitaires*, Acta Math., 11 1964, 143-211.

[9] I. E. Segal *Foundations of the theory of dynamical systems of infinitely many degrees of freedom (I)* Mat.-Fys. Medd. Danske Vid. Selsk., 31, no 12 (1959), 1-39.

[10] I. E. Segal *Transforms for operators and symplectic automorphisms over a locally compact abelian group* Math. Scand., 12-13 (1963), 31-43.

[11] D. Shale *Linear symmetries of free boson fields* Trans. Amer. Math. Soc. 103 (1962), 149-167.

[12] A. Weil *Basic Number Theory* Springer-Verlag New York, 1967.

[13] H. Weyl *The Classical Groups. Their Invariants and Representations* Princeton University Press, 1939.

[14] Y. Zhu, *Theta functions and Weil Representations of Loop Symplectic Groups* Duke Mathematical Journal. 143 (2008), No. 1, 17-39