UNDERSTANDING QUANTUM IMMANANTS AND HIGHER CAPELLI IDENTITIES

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Abstract. Andrei Okounkov (Fields Medalist) gave a generalization to the celebrated Capelli identity. The goal of this paper is to develop the necessary tools so that we might be able to understand and use Okounkov's generalization.

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1. Introduction

The Capelli Identity, named after Alfredo Capelli, is celebrated within the realm of representation theory, especially the representations of the Lie algebra of $\mathfrak{gl}(n)$. In 1996 Andrei Okounkov [6] went on give a generalization of this Capelli Identity in the form of the Higher Capelli Identities he also defined the so-called quantum immanants, which are remarkable central elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}(n))$ and described their properties.

To understand Okounkov's paper we will go through important topics like the universal enveloping algebra, representations of the symmetric groups, and symmetric polynomials. A particular focus throughout this paper is to fill in details that are brushed over in [6] and are not so obvious to a non-expert. We wish to supply proofs for these details and give appropriate references to be able to completely understand the concepts leading up to the primary theorem.

This paper is aimed towards undergraduates with a moderate experience in algebra. Particularly, we assume that the reader has some knowledge of group theory, some representation theory, and Lie Theory. There are several concepts that are well above the undergraduate level but the hope is that these are developed in such a way that the undergraduate reader would be able to grasp these concepts and any missing pieces might be found in the references supplied.

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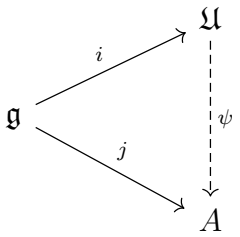
2. Preliminaries

We begin by first defining the primary objects that we will be dealing with throughout this paper. We will go on to develop the initial concepts needed in Okounkov's Higher Capelli Identities.

Definition 2.1 (Universal Enveloping Algebra)**.** Let $\mathfrak{g}$ be a Lie algebra over some field $\mathbb{K}$. Then the *universal enveloping algebra* of $\mathfrak{g}$ is a pair $(\mathfrak{U}, i)$, where $\mathfrak{U}$ is an associative unital algebra over $\mathbb{K}$, $i\colon \mathfrak{g} \to \mathfrak{U}$ is a linear map such that for all $x, y \in L$

$$i([x,y]) = i(x)i(y) - i(y)i(x), \tag{$\star$}$$

and the following universal property holds: for any associative unital $\mathbb{K}$-algebra A, and any linear map, $j\colon \mathfrak{g} \to A$, satisfying $(\star)$, there exists a unique homomorphism of unital associative algebras $\psi\colon \mathfrak{U} \to A$ such that $\psi \circ i = j$. That is, there is a unique ψ such that the following diagram commutes.



Given the universal property in the definiton of $\mathfrak{U}(\mathfrak{g})$, the reader might expect that the universal enveloping algebra is unqiue up to an isomorphism, such an expectation is, indeed, correct. To see the uniqueness, suppose that $(\mathfrak{U}, i)$ and $(\mathfrak{B}, j)$ both satisfy 2.1 for some Lie algebra, $\mathfrak{g}$. Since they both satisfy the universal property, there are unique algebra homomorphisms $\phi\colon \mathfrak{U} \to \mathfrak{B}$ and $\psi\colon \mathfrak{B} \to \mathfrak{U}$ such that

$$\phi \circ i = j \text{ and } \psi \circ j = i.$$

Furthermore, $\phi \circ \psi$ and $\psi \circ \phi$ are unique algebra homomorphisms such that $\phi \circ \psi \circ j = j$ and $\psi \circ \phi \circ i = i$, but clearly the identity map satisfies both of these. Therefore $\phi \circ \psi = 1_{\mathfrak{B}}$ and $\psi \circ \phi = 1_{\mathfrak{U}}$. Thus $(\mathfrak{U}, i) \cong (\mathfrak{B}, j)$.

Another question one should ask is "does such an algebra $\mathfrak{U}$ even exist?". To answer that, we give a general construction of the universal enveloping algebra of $\mathfrak{g}$:

Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$ and let J be the ideal generated by $x \otimes y - y \otimes x - [x,y]$ for all $x, y \in \mathfrak{g}$. Then $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/J$ equipped with the map $i\colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$, which is a restriction of π, the canonical projection map, to $\mathfrak{g}$ does, in fact, satisfy Definition 2.1. For more details and a proof that this construction is valid see [4].

Theorem 2.2 (Poincaré-Birkhoff-Witt)**.** Let $(x_1, x_2, x_3, \dots)$ be an ordered basis of $\mathfrak{g}$. Then the elements $x_{i(1)} \cdots x_{i(m)} = J + x_{i(1)} \otimes \cdots \otimes x_{i(m)}$, with $m \in \mathbb{Z}^+$ and $i(1) \leq i(2) \leq \cdots \leq i(m)$, along with $J + 1$ form a basis of $\mathfrak{U}(\mathfrak{g})$.

The basis that is formed above is referred to as a *PBW basis*. The proof of Theorem 2.2 is technical and is skipped here, see [4] for more information.

Example 2.3. Let $\mathfrak{g} = \mathfrak{gl}(n)$ with the standard basis $(E_{11}, \dots, E_{1n}, E_{21}, \dots, E_{nn})$. By above, the universal enveloping algebra of $\mathfrak{gl}(n)$, $\mathfrak{U}(\mathfrak{gl}(n))$, admits PBW basis

$$\{E_{11}^{a_{11}} \cdots E_{nn}^{a_{nn}} \mid a_{ij} \in \mathbb{Z}_{\geq 0}\}.$$

Consider the representation L of $\mathfrak{U}(\mathfrak{gl}(n))$ on the algebra $\mathbb{K}[M(n)]$, the set of polynomials in $x_{i,j}$ variables for $1 \leq i, j \leq n$, where L is defined on the generators $E_{i,j}$ for all $i, j = 1, \dots, n$ by

$$L(E_{i,j}) = \sum_{k=1}^{n} x_{i,k} \partial_{j,k}, \tag{i}$$

we call the right hand side *polarization operators*. Here we identify $\mathbb{K}[M(n)]$ with $S(\mathbb{C}^n \otimes \mathbb{C}^n)$, the symmetric algebra of $C^n \otimes \mathbb{C}^n$, where $\mathbb{C}^n$ is the standard representation of $\mathfrak{gl}(n)$.

In general, let G be a group, V a vector space, and $\rho_L \colon G \times V \to V$ and $\rho_R \colon V \times G \to V$ be left and right actions respectively defined for any $v \in V$ and $g \in G$ by $\rho_L(g, v) = gv$ and $\rho_R(v, g) = vg^{-1}$. Then for any $g_1, g_2 \in G$

$$\rho_L(g_1, \rho_R(v, g_2)) = g_1(vg_2^{-1}) = \rho_R(\rho_L(g_1, v), g_2).$$

That is, the left and right actions commute. In particular, the action L commutes with the right action of $GL(n)$ on $\mathbb{K}[M(n)]$ since L is the derived representation of the usual left matrix action of $\mathfrak{gl}(n)$. Thus the image of L must rest in the algebra of differential operators with polynomial coefficients on the space $M(n)$ which commute with the right action of $GL(n)$. In fact, L maps $\mathfrak{U}(\mathfrak{gl}(n))$ isomorphically onto the previously mentioned algebra, this is a consequence of the Double Commutant Theorem which is proved in [3, Theorem 4.1.13 on p. 184].

Next consider the following formal matrices, $[L(E)]$, X, and D which each have the (i, j)-th entry as $L(E_{i,j})$, $x_{i,j}$ and $\delta_{i,j}$, respectively. Thus by (i) the following can be written

$$[L(E)] = XD',$$

where D' is the transpose of D. Furthermore, we consider the following element of $\mathfrak{U}(\mathfrak{gl}(n))$

$$C = \sum_{s \in S(n)} \operatorname{sgn}(s) E_{1,s(1)} \left(E_{2,s(2)} + \delta_{2,s(2)}\right) \cdots \left(E_{n,s(n)} + (n-1)\delta_{n,s(n)}\right)$$

and notice that using Capelli identity, [1], the following equality holds

$$L(C) = \det X \det D, \tag{ii}$$

where det is the standard determinant.

Given this expression, we will show that the right hand side of this equation commutes with the action of $GL(n)$ by left multiplication. To see this, consider the following for some $A \in GL(n)$ and some function, ϕ, to be evaluated at Y

$$\begin{aligned}\left(\rho_L(A)\cdot\det(X)\cdot\rho_L(A)^{-1}\cdot\phi\right)(Y) &= \det(A^{-1}Y)\left(\rho_L(A)^{-1}\phi(A^{-1}Y)\right)\\ &= \det(A^{-1})\det(Y)\phi((A^{-1})^{-1}A^{-1}Y)\\ &= \det(A)^{-1}\det(Y)\phi(Y) = \left(\det(A)^{-1}\cdot\det(X)\cdot\phi\right)(Y)\end{aligned}$$

and we also have

$$\begin{aligned}\frac{\partial}{\partial x_{ij}}\left(\rho_L(A)^{-1}\phi\right)(Y) &= \lim_{\epsilon\to 0}\frac{\rho_L(A)^{-1}\phi(Y+\epsilon E_{ij})-\rho_L(A)^{-1}\phi(Y)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{\phi(AY+\epsilon AE_{ij})-\phi(AY)}{\epsilon}\\ &= A_{1i}\frac{\partial\phi}{\partial x_{1j}}(AY)+\cdots+A_{ni}\frac{\partial\phi}{\partial x_{nj}}(AY)\\ &= [A]_i\cdot[D\phi]_j\\ &= (A'\cdot D\phi)_{ij},\end{aligned}$$

where A_{ij} is the $(i,j)^{th}$ entry of A and A_i is the i^{th} row of A. Equiped with these we have the following for any $A \in GL(n)$ and function, ϕ,

$$\begin{aligned}\left(\rho_L(A)\cdot\det X\det D\right)\left(\rho_L(A)^{-1}\phi\right) &= \rho_L(A)\det X\rho_L(A)\rho_L(A)^{-1}\det D\left(\rho_L(A)^{-1}\phi\right)\\ &= (\det A)^{-1}\det X\det(A'D)\left(\phi\right)\\ &= \det X\det D(\phi).\end{aligned}$$

Let $\psi = \rho_L(A)^{-1}\phi$ to get

$$\left(\rho_L(A)\cdot\det X\det D\right)(\psi) = \det X\det D(\rho_L(A)\phi).$$

That is, the right side of (ii) commutes with the left action of $GL(n)$. Furthermore, we know that L commutes with ρ_R and anything that commutes with both ρ_L and ρ_R must come from the image of central elements in $\mathfrak{U}(\mathfrak{gl}(n)$, this is a consequence of the *Double-Commutant Theorem*. Thus, since the map $L\colon \mathfrak{U}(\mathfrak{gl}(n)) \to \mathbb{K}[M(n)]$ is injective then C must be central in $\mathfrak{U}(\mathfrak{gl}(n))$.

The following are some definitions that will be used to later rewrite C and the Capelli Identity more generally. Let

$$E(u) = [E_{i,j} - u\delta_{i,j}]_{i,j=1}^{n} \ \in M_n(\mathfrak{U}(\mathfrak{gl}(n))),$$

where $u \in \mathbb{C}$.

If A is some formal $n\times n$ matrix with entries $a_{i,j}$ from some noncommutative algebra, $\mathbf{A}$, then any such A can be written as

$$A = \sum_{i,j} a_{i,j}\otimes e_{i,j} \in \mathbf{A}\otimes M(n).$$

Furthermore, this can be extended for any tensor product of two such matrices A and B by

$$A \otimes B = \sum_{i,j,k,\ell} a_{i,j} b_{k,\ell} \otimes e_{i,j} \otimes e_{k,\ell} \in \mathbf{A} \otimes M(n)^{\otimes 2}.$$

We define the action of the group $S(k)$ acting on any k-fold tensor, $V^{\otimes k}$, this action is defined for all $\sigma \in S(k)$ and $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ by $\sigma \cdot v_1 \otimes \cdots \otimes v_k = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$. This representation leads to a representation of $\mathbb{K}[S(k)]$ on $M(n)^{\otimes k}$ which will be important later.

That is, given the original representation on $V^{\otimes k}$ we have that $S(k) \subseteq GL(V^{\otimes k})$. We clearly know that $GL(V^{\otimes k}) \subseteq \mathrm{End}(V^{\otimes k})$ and $\mathrm{End}(V^{\otimes k}) \cong \mathrm{End}(V)^{\otimes k} = M(n)^{\otimes k}$ using the standard isomorphism and when $\dim(V) = n$. Thus we can then use standard multiplication in $M(n)^{\otimes k}$ to describe the action of $\mathbb{K}[S(k)]$ on $M(n)^{\otimes k}$.

To explicitly calculate the representation of $\mathbb{K}[S(k)]$ on $M(n)^{\otimes k}$, consider the construction as follows. Let $\mathcal{B} = \{e_i \mid 1 \leq i \leq n\}$ be the standard basis for C^n and

$$B^k = \{b_{i_1} \otimes \cdots \otimes b_{i_k} \mid \ 1 \leq i_1, \ldots, i_k \leq n\}$$

the standard basis for $(\mathbb{C}^n)^{\otimes k}$. Now we can write find $[\sigma]_{B^k} \in M_{n^k}(\mathbb{C})$, the matrix corresponding to $\sigma \in S(k)$ acting on $(\mathbb{C}^n)^{\otimes k}$, and we can write this matrix as a sum on simple k-tensors by identifying $[\sigma]_{B^k}$ as an $n \times n$ matrix whose coefficients are $n^{k-1} \times n^{k-1}$ matrices. Furthermore, these matrix coefficients can be identified the same way and continue this process until scalar coefficients. With these identifications we can then write $[\sigma]_{B^k} = \sum c_{i_1 j_1, \ldots i_k j_k} e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k}$, where $c_{i_1 j_1, \ldots, i_k j_k} \in \mathbb{C}$ and each $e_{i_\ell j_\ell} \in \{e_{ij} \mid 1 \leq i, j \leq n\}$.

With this identification the action of $\mathbb{K}[S(k)]$ on $M(n)^{\otimes k}$ is via pointwise multiplication.

Understandably the identification above can be confusing. To clarify the situation, we consider the following example for $n = k = 2$.

Example 2.4. Take $\mathbb{C}^2$ to have the standard basis and $(\mathbb{C}^2)^{\otimes 2} = \mathrm{span}_{\mathbb{C}}\{e_i \otimes e_j \mid 1 \leq i, j \leq 2\}$. Now $S(2) = \{\mathrm{id}, (1\ 2)\}$ to which we can then write the corresponding matrices and begin the idetification described above:

$$\mathrm{id} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} := e_{11} \otimes e_{11} + e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + e_{22} \otimes e_{22}$$

$$(1\ 2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} := e_{11} \otimes e_{11} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{22} \otimes e_{22}$$

Given the above definitions it can be shown that with $\mathbf{A} = \mathfrak{U}(\mathfrak{gl}(n))$,

$$C = (n!)^{-1} \mathrm{tr}\left(E \otimes E(-1) \otimes \cdots \otimes E(-n+1) \cdot \mathrm{Alt}\right),$$

where Alt is defined as

$$\text{Alt} = \sum_{s \in S(n)} \text{sgn}(s) s \in \mathbb{K}[S(n)],$$

and the trace of an element of $\mathbf{A} \otimes M(n)^{\otimes n}$ is defined by

$$\text{tr}\left(\sum_{i_1, j_1, \ldots, i_n, j_n} a_{i_1, j_1, \ldots, i_n, j_n} \otimes e_{i_1, j_1} \otimes \cdots \otimes e_{i_n, j_n} \right) = \sum_{i_1, \ldots, i_n} a_{i_1, i_1, \ldots, i_n, i_n} \in \mathbf{A}.$$

Thus C can be rewritten again,

$$C = (n!)^{-1} \sum_{i_1, \ldots, i_n} \sum_{s \in S(n)} \text{sgn}(s) E_{i_1, i_{s(1)}} \cdots (E_{i_n, i_{s(n)}} + (n-1)\delta_{i_n, i_{s(n)}}). \tag{iii}$$

Furthermore, we can write the following for the right side of (ii) as

$$\det X \det D = (n!)^{-1} \text{tr}\left(X^{\otimes n} (D')^{\otimes n} \cdot \text{Alt} \right).$$

Thus we can restate the Capelli identity, (ii), as

$$\text{tr}\left(E \otimes E(-1) \otimes \cdots \otimes E(-n+1) \cdot \text{Alt} \right) = \text{tr}\left(X^{\otimes n} (D')^{\otimes n} \cdot \text{Alt} \right), \tag{iv}$$

where we now identify the elements of $\mathfrak{U}(\mathfrak{gl}(n))$ on the LHS with their differential operators under the action of L.

Remark 2.5. Furthermore, we can extend (iv) such that it is also true for the action of $GL(n)$ on rectangular $n \times m$ matrices, where both X and D will also be $n \times m$ matrices. This extension changes the polarization operators from sums up to n to sums up to m.

3. Partitions and Young tableaux

We briefly switch gears to recall definitions and facts concerning representations of $S(k)$. We then use these while working with representations of $\mathbb{K}[S(k)]$.

Definition 3.1 (Partition)**.** Let $k \in \mathbb{Z}^+$ then a *partition*, λ, of k is a non-zero and weakly decreasing sequence, $(a_1, a_2, \ldots, a_\ell) \in \mathbb{Z}^+$, such that $a_1 + a_2 + \cdots + a_\ell = k$.

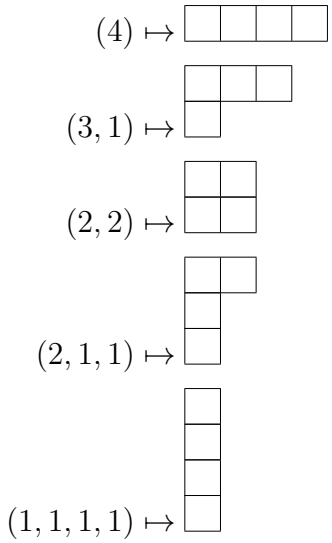
The following are all possible partitions for $k = 4$.

$$(4)$$
$$(3,1)$$
$$(2,2)$$
$$(2,1,1)$$
$$(1,1,1,1)$$

Throughout the following we assume that $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition of k

Definition 3.2 (Young Diagram)**.** The *Young diagram* of λ consists of k boxes placed into ℓ rows such that the i^{th} row from the top contains exactly λ_i boxes.

Following the above example, here are the Young diagrams corresponding to the partitions on $k = 4$:



Definition 3.3 (Young Tableau)**.** A *Young tableau of shape* λ is the Young diagram corresponding to λ such that each box is labelled using $\{1, \ldots, k\}$ and each label is used exactly once. We call this a λ-*tableau*.

Let our partition be $\lambda = (3, 1)$. Some examples of Young tableaux of shape λ are as follows:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Clearly when given a partition, λ, there are several possible Young tableaux of shape λ. We will be concerned with *standard Young tableaux*, which are Young tableaux whose columns and rows are always increase as you move right or down. That is, all of the standard tableaux of shape $(3, 1)$ are:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$$

4. Specht Representations

Of course now that we have these tableaux we can define an action of $S(k)$ on all tableaux containing k squares. Let $\sigma \in S(k)$ then σT is a λ-tableaux such that the labels are permutated according to σ. For example let take

$$T = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Then we have the following examples:

$$(1\ 3)\ T = \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad (1\ 4)(2\ 3)\ T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad (1\ 4\ 3)\ T = \begin{array}{|c|c|c|}\hline 3 & 2 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Definition 4.1 (Column/Row stabilizer)**.** Let T be a λ-tableau. Then the column (respectively, row) stabilizer of T is the subgroup of $S(k)$ whose action preserves the columns (respectively, rows) of T. Denoted C_T (respectively, R_T).

Consider the tableau

1	2	3
4		

which admits the following column and row stabilizers

$$C_T = \{\mathrm{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \text{ and}$$
$$R_T = \{\mathrm{id}, (1\ 4)\}.$$

Define the equivalence relation, $\sim$, on the set of λ-tableaux by $T_1 \sim T_2$ if and only if $R_{T_1} = R_{T_2}$.

Definition 4.2 (Tabloid)**.** An equivalence class of λ-tableaux under $\sim$ is called a λ-*tabloid* or a *tabloid of shape* λ.

We denote the tabloid of T, some λ-tableaux, by $[T]$. The set of all λ-tabloids is denoted T^λ. Lastly, denote T_λ as the tabloid with $1, \ldots, \lambda_1$ in row 1, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in row 2, and in general $\sum_{k=1}^{i-1} \lambda_k + 1, \ldots, \sum_{k=1}^{i-1} \lambda_k + \lambda_i$ in row i.

Observe that given a λ-tableau, T, and a $\sigma \in S(k)$, the action $\sigma[T] = [\sigma T]$ is well defined as a result of the fact

$$C_{\sigma T} = \sigma C_t \sigma^{-1} \text{ and } R_{\sigma T} = \sigma R_T \sigma^{-1}.$$

Define $M^\lambda = \mathbb{C}T^\lambda$. Thus M^λ is a representation of $S(k)$.

Definition 4.3 (Polytabloid)**.** Let T be a Young Tableau and $[T]$ its tabloid. Then the associated *polytabloid*, e_T, is

$$e_T = \sum_{\sigma \in C_T} \mathrm{sgn}(\sigma)\sigma[T] \in M^\lambda.$$

Definition 4.4 (Specht Representation)**.** The subspace $S^\lambda = \mathrm{span}_{\mathbb{C}}\{e_T \mid T \text{ a } \lambda\text{-tableau}\}$ of M^λ is also a representation of $S(k)$, called the *Specht Representation* associated to λ.

Remark 4.5. The representation S^λ will be an irreducible representation whose dimension is equal to the number of standard tableaux of shape λ. We will not include a proof of this remarkable statement although it can be found in [10, Ch 10].

Example 4.6. Consider the partition $\lambda = (2, 1)$ which has the following tabloids:

1	2	
3		

1	3	
2		

2	3	
1		

We calculate the polytabloids,

$$e_1 = e\,\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} + \operatorname{sgn}((1\ 3))(1\ 3)\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$= \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

$$e_2 = e\,\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \operatorname{sgn}((1\ 2))(1\ 2)\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$= \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

$$e_3 = e\,\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} + \operatorname{sgn}((2\ 1))(2\ 1)\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

$$= \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ .$$

Notice that $e_2 = -e_3$ and it is easy to check the following relations:

	e_1	e_2
id	e_1	e_2
$(1\ 2)$	$e_1 - e_2$	$-e_2$
$(1\ 3)$	$-e_1$	$e_2 - e_1$
$(2\ 3)$	e_2	e_1
$(1\ 2\ 3)$	$-e_2$	$e_1 - e_2$
$(1\ 3\ 2)$	$e_2 - e_1$	$-e_1$

Thus it should be clear that S^λ, where $\lambda = (2,1)$, is an irreducible representation of $S(3)$ of dimension 2. Indeed it is $\mathbb{C}^3/\operatorname{span}_{\mathbb{C}}\{(1,1,1)\}$.

Definition 4.7 (Character)**.** Let ϕ be a representation of a group G. Then the *character* of a group element, g, with respect to some representation ϕ is $\chi(g) = \operatorname{tr}(\phi(g))$.

We have classified the irreducible representations of $S(k)$ and now let χ^λ to be the character corresponding to the irreducible representation classified by the partition of k λ.

5. Young's Orthonormal Basis

To continue on towards our main goal, Higher Capelli Identities, let μ be a partition of k containing no more than n parts. Identify the group algebra of $S(k)$ with $\mathbb{K}[S(k)]$ equipped with inner product

$$\langle \phi, \psi \rangle_{S(k)} = \left(\frac{1}{k!}\right) \sum_{s \in S(k)} \phi(s)\psi(s^{-1}).$$

The characters, χ^μ, can be identified with the following element of $\mathbb{K}[S(k)]$

$$\chi^\mu = \sum_{s \in S(k)} \chi^\mu(s) s$$

and furthermore, these characters, χ^μ, are then orthonormal with respect to the inner product above. More can be found on this in [10, Ch 4].

Definition 5.1 (Content)**.** Let λ be a tableau and let $\alpha = (i, j)$ be a cell of λ, where i is the row and j is a column of α. Then $c_\lambda(\alpha) = j - i$ is called the *content* of α.

Upon discovering these irreducible representations Young also showed that the set of tabloids, where each tabloid corresponds to a standard tableaux, form a canonical orthogonal basis. For verification on this and more information see [5, pg. 114]. Furthermore, the following relation will be important for seeing how transpositions act on this standard basis. Let $\sigma_i \in S(k)$ be the transposition $(i\ i+1)$, T be a standard Young tableaux and ξ_T is the young basis element corresponding to standard tableaux T. Then the action of σ_i on each ξ_T is as follows

$$\sigma_i \xi_T = \begin{cases} \xi_T & \text{if } (i\ i+1) \in R_T, \\ -\xi_T & \text{if } (i\ i+1) \in C_T \text{ and} \\ \left(c_T(i+1) - c_T(i)\right)^{-1} \xi_T + \sqrt{1 - \left(c(i+1) - c(i)\right)^{-2}} \xi_{\sigma_i T} & \text{else.} \end{cases}$$

6. Higher Capelli Identities

Take T to be a standard tableau of shape μ and ξ_T to be the corresponding vector of Young orthonormal basis. Consider

$$P_T = \frac{\dim \mu}{k!} \psi_T \in \mathbb{K}[S(k)],$$

where

$$\psi_T = \sum_{s \in S(k)} (s \cdot \xi_T, \xi_T) s \in \mathbb{K}[S(k)].$$

The element P_T will act as an orthogonal projection onto ξ_T in the irreducible $S(k)$-module corresponding to μ and as the zero map in other irreducible $S(k)$-modules.

Proof. To justify that it is a projection, take T, S and S' to be standard tableaux of shape μ and consider the following.

$$\begin{aligned} (P_T \xi_S, \xi_{S'}) &= \left(\sum_{\sigma \in S(k)} (\sigma \xi_T, \xi_T) \sigma \xi_S, \xi_{S'} \right) \\ &= \sum_{\sigma \in S(k)} (\sigma \xi_T, \xi_T)(\sigma \xi_S, \xi_{S'}) \\ &= \sum_{\sigma \in S(k)} \sigma_{TT} \sigma_{S'S} \end{aligned} \tag{v}$$

Here σ_{TT} and $\sigma_{S'S}$ are the TT and $S'S$ respective coefficients of the matrix associated to σ. The final equality is via the fact that all of the $\{\xi_T \mid T \text{ is a standard tableau}\}$ form an orthonormal basis. That is, $(\xi_{S'}, \xi_S) = \delta_{S'S}$.

Next using Corollary 3 from [9, pg. 14] which states that for any representations $\rho_1(g) = (g_{i_1 j_1})$ and $\rho_2(g) = (g_{i_2 j_2})$ of some group G and ρ_1 and ρ_2 are isomorphic then

$$\frac{1}{|G|}\sum_{g\in G}(g^{-1})_{i_1 j_1} g_{i_2 j_2} = \frac{\delta_{i_2 i_1}\delta_{j_2 j_1}}{|G|}.$$

Asserting this for $G = S(k)$ and $i_1, j_1 = T, i_2 = S'$, and $j_2 = S$, standard tableaux, we get

$$\frac{1}{|S(k)|}\sum_{\sigma\in S(k)}(\sigma^{-1})_{TT}\sigma_{S'S} = \frac{\delta_{S'T}\delta_{ST}}{|S(k)|}.$$

Recall that with Young's orthonormal basis, the matrices associated to each σ is orthonormal. That is, $\sigma^{-1} = \sigma^t$ and since we are on the diagonal we have $(\sigma^{-1})_{TT} = \sigma_{TT}$. Thus we have

$$\sum_{\sigma\in S(k)}\sigma_{TT}\sigma_{S'S} = \delta_{S'T}\delta_{ST}.$$

Knowing all this we can conclude that (v) holds the following equality

$$(P_T\xi_S, \xi_{S'}) = \sum_{\sigma\in S(k)}\sigma_{TT}\sigma_{S'S} = \delta_{S'T}\delta_{ST}.$$

Which implies that $P_T\xi_S = \delta_{TS}\xi_T$. Thus we know that P_T acts as an orthogonal projection onto ξ_T in S^μ.

Similarly, we show that P_T is the zero map in other irreducible $S(k)$-modules. To see this, take Corollary 2 from [9, pg. 14] which states that for $\rho_1(g) = (g_{i_1 j_1})$ and $\rho_2(g) = (g_{i_2 j_2})$ of some group G with ρ_1 and ρ_2 non-isomorphic then

$$\frac{1}{|G|}\sum_{g\in G}(g^{-1})_{i_1 j_1} g_{i_2 j_2} = 0.$$

We assert this for ρ_1 and ρ_2 non-isomorphic irreducible representations of $S(k)$ corresponding to partitions λ^1 and λ^2. Thus we have that for any standard tableaux T and T' of shape λ^1 and standard tableaux S and S' of shape λ^2

$$\frac{1}{|S(k)|}\sum_{\sigma\in S(k)}(\sigma^{-1})_{T'T}\sigma_{S'S} = 0.$$

Using this result while applying similar calculations to above to (v) we get the following

$$(P_T\xi_S, \xi_{S'}) = \sum_{\sigma\in S(k)}\sigma_{TT}\sigma_{S'S} = 0$$

where T is a standard tableau of shape λ^1 and S and S' are of shape λ^2. But this norm is non-degenerate and this is true for any such tableaux, thus $P_T\xi_S = 0$ for any T and any S. That is, P_T is the zero map in other non-isomorphic irreducible $S(k)$-modules. □

Example 6.1. Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ then we have the following

$$\begin{aligned}
\psi_T &= \sum_{s\in S(k)} (s\cdot \xi_T, \xi_T)s \\
&= \mathrm{id} + (1\ 2) - \frac{1}{2}(2\ 3) - \frac{1}{2}(1\ 2)(2\ 3) - \frac{1}{2}(2\ 3)(1\ 2) - \frac{1}{2}(1\ 2)(2\ 3)(1\ 2) \\
&= \mathrm{id} + (1\ 2) - \frac{1}{2}(2\ 3) - \frac{1}{2}(1\ 2\ 3) - \frac{1}{2}(1\ 3\ 2) - \frac{1}{2}(1\ 3) \text{ and} \\
P_T &= \frac{2}{3!}\psi_T \\
&= \frac{2}{3!}\left(\mathrm{id} + (1\ 2) - \frac{1}{2}(2\ 3) - \frac{1}{2}(1\ 2\ 3) - \frac{1}{2}(1\ 3\ 2) - \frac{1}{2}(1\ 3)\right).
\end{aligned}$$

Furthermore, in $S^{(2,1)}$ notice that for standard tableaux $\xi_T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\xi_{(2\ 3)T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ we have the following

$$\begin{aligned}
P_T\xi_T &= \frac{2}{3!}\left(\mathrm{id} + (1\ 2) - \frac{1}{2}(2\ 3) - \frac{1}{2}(1\ 2\ 3) - \frac{1}{2}(1\ 3\ 2) - \frac{1}{2}(1\ 3)\right)\xi_T \\
&= \frac{1}{3}\left(\xi_T + \xi_T - \frac{1}{2}\left(\frac{-1}{2}\xi_T + \frac{\sqrt{3}}{2}\xi_{(2\ 3)T}\right) - \frac{1}{2}\left(\frac{-1}{2}\xi_1 - \frac{\sqrt{3}}{2}\xi_{(2\ 3)T}\right)\right. \\
&\qquad \left. - \frac{1}{2}\left(\frac{-1}{2}\xi_t + \frac{\sqrt{3}}{2}\xi_{(2\ 3)T}\right) - \frac{1}{2}\left(\frac{1}{2}\xi_T - \frac{\sqrt{3}}{2}\xi_{(2\ 3)T}\right)\right) \\
&= \frac{1}{3}\left(3\xi_T\right) = \xi_T \text{ and}
\end{aligned}$$

$$\begin{aligned}
P_T\xi_{(2\ 3)T} &= \frac{2}{3!}\left(\mathrm{id} + (1\ 2) - \frac{1}{2}(2\ 3) - \frac{1}{2}(1\ 2\ 3) - \frac{1}{2}(1\ 3\ 2) - \frac{1}{2}(1\ 3)\right)\xi_{(2\ 3)T} \\
&= \frac{1}{3}\left(\xi_{(2\ 3)T} - \xi_{(2\ 3)T} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\xi_T + \frac{1}{2}\xi_{(2\ 3)T}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\xi_T - \frac{1}{2}\xi_{(2\ 3)T}\right)\right. \\
&\qquad \left. - \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\xi_T - \frac{1}{2}\xi_{(2\ 3)T}\right) - \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\xi_T + \frac{1}{2}\xi_{(2\ 3)T}\right)\right) \\
&= 0.
\end{aligned}$$

We can check that P_T acts as a zero map in $S^{(3)} = \mathrm{span}_{\mathbb{C}}\{\ytableaushort{123}\}$ and $S^{(1,1,1)} = \mathrm{span}_{\mathbb{C}}\{\ytableaushort{1,2,3}\}$,

$$P_T\ytableaushort{123} = P_T\xi_1 = \frac{2}{3!}\left(\xi_1 + \xi_1 - \frac{1}{2}\xi_1 - \frac{1}{2}\xi_1 - \frac{1}{2}\xi_1 - \frac{1}{2}\xi_1\right) = 0$$

$$P_T\ytableaushort{1,2,3} = P_T\xi_2 = \frac{2}{3!}\left(\xi_2 - \xi_2 + \frac{1}{2}\xi_2 - \frac{1}{2}\xi_2 - \frac{1}{2}\xi_2 + \frac{1}{2}\xi_2\right) = 0.$$

That is, P_T acts as expected, as a projection onto ξ_T in $S^{(2,1)}$ and a zero map in other irreducible representations of $S(3)$.

Theorem 6.2 (Higher Capelli Identities)**.** For any partitions μ of at most n parts and any standard tableau T of shape μ,

$$\text{(vi)} \qquad \mathrm{tr}\left(E \otimes E(c_T(2)) \otimes \cdots \otimes E(c_T(k)) \cdot P_T\right) = \mathrm{tr}\left(X^{\otimes k}(D')^{\otimes k} \cdot \frac{\chi^\mu}{k!}\right).$$

In the above, it follows that the left side is independent on the choice of T since the right hand side is independent of T.

Example 6.3. Let us construct an example of the above theorem. Let $n = 2$, $\mu = (1,1)$, and $T = \ytableaushort{1,2}$. Then Theorem 6.2 says that

$$\mathrm{tr}\left(E \otimes E(c_T(2)) \cdot P_T\right) = \mathrm{tr}\left(X^{\otimes 2}(D')^{\otimes 2} \cdot \frac{\chi^\mu}{2!}\right),$$

where $\frac{\chi^\mu}{2} = P_T = \frac{1}{2}\left(\mathrm{id} - (1\ 2)\right)$. To verify this, on the LHS we have:

$$\begin{aligned}
\mathrm{tr}\left(E \otimes E(c_T(2)) \cdot P_T\right) &= \frac{1}{2}\left(E_{11}(E_{22}+1) + E_{22}(E_{11}+1) - E_{21}E_{12} - E_{12}E_{21}\right) \\
&= \frac{1}{2}((x_{11}\partial_{11} + x_{12}\partial_{12})(x_{21}\partial_{21} + x_{22}\partial_{22}) + (x_{21}\partial_{21} + x_{22}\partial_{22})(x_{11}\partial_{11} + x_{12}\partial_{12}) \\
&+ (x_{11}\partial_{11} + x_{12}\partial_{12}) + (x_{21}\partial_{21} + x_{22}\partial_{22}) - (x_{21}\partial_{11} + x_{22}\partial_{21})(x_{11}\partial_{21} + x_{12}\partial_{22}) \\
&- (x_{11}\partial_{21} + x_{12}\partial_{22})(x_{21}\partial_{11} + x_{22}\partial_{12})) \\
&= x_{11}x_{22}\partial_{11}\partial_{22} - x_{11}x_{22}\partial_{12}\partial_{21} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22}
\end{aligned}$$

Similarly on the RHS we have:

$$\begin{aligned}
\mathrm{tr}\left(X^{\otimes 2}(D')^{\otimes 2} \cdot \frac{\chi^\mu}{2!}\right) &= \frac{1}{2}(x_{11}x_{22}\partial_{11}\partial_{22} - x_{11}x_{22}\partial_{12}\partial_{21} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22} \\
&\quad + x_{11}x_{22}\partial_{11}\partial_{22} - x_{11}x_{22}\partial_{12}\partial_{21} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22}) \\
&= x_{11}x_{22}\partial_{11}\partial_{22} - x_{11}x_{22}\partial_{12}\partial_{21} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22}
\end{aligned}$$

Thus we can clearly see that both sides are, in fact, equal and the identity holds in this case.

Before moving on notice that with a bit of manipulation we have that

$$\begin{aligned}\operatorname{tr}\left(E\otimes E(c_T(2))\cdot P_T\right) &= x_{11}x_{22}\partial_{11}\partial_{22} - x_{11}x_{22}\partial_{12}\partial_{21} + x_{12}x_{21}\partial_{12}\partial_{21} - x_{12}x_{21}\partial_{11}\partial_{22}\\ &= (x_{11}x_{22}-x_{12}x_{21})(\partial_{11}\partial_{22}-\partial_{12}\partial_{21})\\ &= \det\left(\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\right)\det\left(\begin{bmatrix}\partial_{11} & \partial_{12}\\ \partial_{21} & \partial_{22}\end{bmatrix}\right).\end{aligned}$$

Which we expected since when $\mu = (1,1)$ or any similar partition we have the original Capelli Identity.

Example 6.4. Let $n = 1$ and $\mu = (k)$. Then Theorem 6.2 states

$$x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\cdots\left(x\frac{d}{dx}-k+1\right) = x^k\frac{d^k}{dx^k}.$$

To verify this, first notice that for any k we have

$$\left(\frac{d^k}{dx^k}\right)x\frac{d}{dx} = k\frac{d^k}{dx^k} + x\frac{d^{k+1}}{dx^{k+1}}.$$

By induction, for $k = 1$ this is clear. Assume this is true for some $k-1$, that is

$$\begin{aligned}\left(\frac{d^{k-1}}{dx^{k-1}}\right)x\frac{d}{dx} &= (k-1)\frac{d^{k-1}}{dx^{k-1}} + x\frac{d^k}{dx^k}\\ \left(\frac{d^k}{dx^k}\right)x\frac{d}{dx} &= (k-1)\frac{d^k}{dx^k} + \frac{d^k}{dx^k} + x\frac{d^{k+1}}{dx^{k+1}}\\ &= k\frac{d^k}{dx^k} + x\frac{d^{k+1}}{dx^{k+1}}.\end{aligned}$$

Thus we can use this to show our particular case. For $k = 1$ equality is clear. Assume above for some k and consider the following

$$\begin{aligned}\left(x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\cdots\left(x\frac{d}{dx}-k+1\right)\right)\left(x\frac{d}{dx}-k\right) &= x^k\frac{d^k}{dx^k}\left(x\frac{d}{dx}-k\right)\\ &= x^k\frac{d^k}{dx^k}\left(x\frac{d}{dx}\right) - kx^k\frac{d^k}{dx^k}\\ &= kx^k\frac{d^k}{dx^k} + x^{k+1}\frac{d^{k+1}}{dx^{k+1}} - kx^k\frac{d^k}{dx^k}\\ &= x^{k+1}\frac{d^{k+1}}{dx^{k+1}}.\end{aligned}$$

Therefore (vi) does, in fact, give the appropriate results for $n = 1$ and $\mu = (k)$.

7. Symmetric Polynomials

For our final section we look at symmetric polynomial, which on the surface are not directly related to 6.2, but will be particular important when describing the action of quantum immanants. For the following, let $x_1, \ldots, x_n$ be variables.

Definition 7.1 (Symmetric Function)**.** A polynomial, $p(x_1, \ldots, x_n)$, in $x_1, \ldots, x_n$ variables is said to be a *symmetric function* if it is invariant under the permuation of the variables. That is, for all $\sigma \in S(n)$

$$p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = p(x_1, \ldots, x_n).$$

Example 7.2. Some examples include:

$$\varphi_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$$
$$\phi_k(x_1, \ldots, x_n) = \sum_{a_1+\cdots+a_n=k} x_1^{a_1} \cdots x_n^{a_n}.$$

Definition 7.3 (Elementary Symmetric Functions)**.** Define $e_i = e_i(x_1, \ldots, x_n)$, for $i = 1, \ldots, n$, to be polynomials such that

$$p(t) = \prod_{i=1}^{n}(1 + tx_i) = 1 + \sum_{i=1}^{n} e_i t^i.$$

These polynomials can be seen to be defined as $e_i = \sum_{1 \le a_1 < \cdots < a_i \le n} x_{a_1} x_{a_2} \cdots x_{a_i}$.

Definition 7.4 (Alternating Function)**.** A polynomial, $p(x_1, \ldots, x_n)$, in $x_1, \ldots, x_n$ variables is said to be an *alternating function* if it for all $\sigma \in S(n)$ we have

$$p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \operatorname{sgn}(\sigma) p(x_1, \ldots, x_n).$$

Example 7.5. The following polynomial, called the *Vandermonde Determinant*, is an important example of an alternating function

$$V(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j).$$

An alternate presentation is $V(x_1, \ldots, x_n) = \det(x_i^{n-j})$.

Example 7.6. Another alternating function, which can be attributed to the Jacobi-Trudi identities, is defined for any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ to be

$$T_\lambda(x_1, \ldots, x_n) = \det(x_i^{\lambda_j + n - j}).$$

A simple example when $\lambda = (2, 1, 1)$ we get

$$T_{(2,1,1)}(x_1, x_2, x_3) = \det\left(\begin{bmatrix} x_1^4 & x_1^2 & x_1 \\ x_2^4 & x_2^2 & x_2 \\ x_3^4 & x_3^2 & x_3 \end{bmatrix}\right) = x_1^4(x_2^2 x_3 - x_2 x_3^2) - x_2^4(x_1^2 x_3 - x_1 x_3^2) + x_3^4(x_1^2 x_2 - x_1 x_2^2).$$

Proposition 7.7. A polynomial, $f(x_1, \ldots, x_n)$, is an alternating function if and only if it is of the form

$$f(x_1, \ldots, x_n) = V(x_1, \ldots x_n) g(x_1, \ldots x_n),$$

for some symmetric function $g(x_1, \ldots x_n)$.

The proof of this can be see in [7, pg. 28]

Consequently we can see then that the set of all antisymmetric polynomials, A, is a rank 1 module over the ring of symmetric polynomials, S, and is generated by $V(x_1, \dots x_n)$. That is, $A = V(x_1, \dots x_n)S$ and every alternating function is divisible by the Vandermonde determinant.

Knowing this we can then define *Schur Functions*.

Definition 7.8 (Schur Functions)**.** Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be some partition then the *Schur function* associated to λ is

$$s_\lambda(x_1, \dots, x_n) = \frac{T_\lambda(x_1, \dots, x_n)}{V(x_1, \dots, x_n)} = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})}.$$

These Schur functions have a particularly important role in the characters of highest weight irreducible representations of $\mathfrak{gl}(n)$. More on this can found in [2].

Our interest rests in a variation of the Schur functions called *shifted Schur function*. These polynomial use *factorial powers* which is defined for $n \in \mathbb{Z}^+$ and x a variable by

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1).$$

Definition 7.9 (Shifted Schur Function)**.** Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be some partition then the *shifted Schur fucntion* associated to λ is

$$s^*_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\underline{\lambda_j+n-j}})}{\det(x_i^{\underline{n-j}})}.$$

Let $c_\mu = \operatorname{tr}(E \otimes E(c_T(2)) \otimes \cdots \otimes E(c_T(k)) \cdot P_T)$, the LHS from (vi), and (π, V) be an irreducible representation of $\mathfrak{gl}(n)$. Thus V has highest weight $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$, where $(\lambda_1, \dots, \lambda_n)$ is a partition. That is, for any $v \in V\backslash\{0\}$ we have $\pi(\operatorname{diag}(a_1, \dots, a_n))v = (\lambda_1 a_1 + \cdots + \lambda_n a_n)v$ for all $a_1, \dots, a_n \in \mathbb{C}$ and for any strictly upper triangular matrix, η, we have $\pi(\eta)v = 0$. We know $c_\mu \in \mathcal{Z}(\mathcal{U}(\mathfrak{gl}(n))$ and since (π, V) is irreducible we know by Schur's Lemma that $\pi(c_\mu)$ acts on V by a scalar. In particular, Okounkov proved in [6, pg. 118] that

$$\pi(c_\mu) = s^*_\mu(\lambda).$$

This remarkable proposition forms a direct link between these Schur functions and Theorem 6.2. Furthermore, it describes exactly how these c_μ act on our irreducible representations.

Further directions that might be pursued related to this include working on the Capelli eigenvalue problem for other classical Lie algebras. There is much work already being done related to this problem, for example [8], despite this work the problem still remains open in general.

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