SUPERCONFORMAL CURRENT ALGEBRAS

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1. Superconformal Current Algebras

1.1. **Witt and Virasoro algebra.** A general example of an infinite-dimensional Lie algebra is Vect(M), the Lie algebra of all vector fields on a manifold with the Lie bracket given by the commutator. For example, in the the case of Vect(S^1)

$$\left[f(\theta)\frac{d}{d\theta}, g(\theta)\frac{d}{d\theta}\right] = \left(f(\theta)g'(\theta) - g(\theta)f'(\theta)\right)\frac{d}{d\theta}. \tag{1}$$

The complex vector fields on S^1 can be given by a basis $L_n = -ie^{-in\theta}\frac{d}{d\theta}$ and the algebra generated by this basis is called the Witt algebra denoted by $\mathcal{W}$ that is the complexification $\mathbb{C} \otimes \text{Vect}(S^1)$. Furthermore,

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{2}$$

It is known that Vect(S^1) is the Lie algebra of Diff$^+$(S^1), the group of orientation preserving (conformal) differomorphisms of S^1. This fact connects the Witt algebra to conformal field theory (CFT), the generators of the Witt algebra naturally arise as the generators of conformal transformations on the 2D Minkowski space $\mathbb{R}^{1,1}$ or its compactified version $S^1 \times S^1$. The Witt algebra is also the Lie algebra of derivations $\text{Der}(\mathbb{C}[t, t^{-1}])$ of the (associative) algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ which can be seen by the substitution $t = e^{-i\theta}$ in the expression $L_n = -ie^{-in\theta}\frac{d}{d\theta}$, thereby obtaining a basis for $\text{Der}(\mathbb{C}[t, t^{-1}])$

The Virasoro algebra $\mathcal{V} = \mathcal{W} \oplus \mathbb{C}c$ is obtained by a 1-dimensional central extension of the Witt algebra i.e. adding an element c that commutes with every element in the Lie algebra.

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n}\frac{m(m^2-1)}{12}c$$
$$[L_n, c] = 0$$

The Virasoro algebra can also be obtained as derivations on the loop algebra. We describe its "super" generalization in the next section.

1.2. **Loop algebras and central extensions.** Let $\mathfrak{g}$ be a Lie algebra of a compact Lie group G so that the killing form is negative definite. The super loop algebra $\tilde{\mathfrak{g}}$ is

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}; \theta] \qquad t \in \mathbb{C}^*, \quad \theta^2 = 0 \tag{3}$$

that is all Laurent polynomials in t and a Grassmann variable θ with coefficients in $\mathfrak{g}$. The Lie bracket is given by

$$[x \otimes P(t, t^{-1}; \theta), y \otimes Q(t, t^{-1}; \theta)] = [x, y] \otimes P(t, t^{-1}; \theta)Q(t, t^{-1}; \theta). \tag{4}$$

It has a graded structure with $\mathbb{Z} \times \mathbb{Z}_2$ as the graded group and degrees:

$$\deg \mathfrak{g} = 0, \quad \deg t = 1, \quad \deg \theta = \kappa \in \mathbb{Z}/2 \tag{5}$$

Definition 1.1. *The central extension of a Lie algebra $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is defined to be the vector space $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ with the lie bracket*

$$[X + u, Y + v] = [X, Y] + \psi(X, Y)$$

where the 2-cocycle $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$ is an anti-symmetric, bilinear map satisfying the identity

$$\psi([X, Y], Z) + \psi([Y, Z], X) + \psi([Z, X], Y) = 0 \tag{6}$$

Remark 1.2. *The above definition is equivalent to constructing a short exact sequence (see diagram below) such that the image of ϕ lies in the centre of $\tilde{\mathfrak{g}}$.*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\phi} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

A morphism of central extensions of $\mathfrak{g}$ is a commutative diagram such that the one below, where η_1 and η_2 are Lie algebra homomorphisms. Two central extensions $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are equivalent if the map η_2 is an isomorphism. It will follow easily that η_1 is also an isomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{\phi} & \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathfrak{n}' & \xrightarrow{\phi'} & \tilde{\mathfrak{g}}' & \xrightarrow{\pi'} & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

Let $C^k(\mathfrak{g}, \mathfrak{n}) = \{\psi : \mathfrak{g}^n \to \mathfrak{n} \,|\psi \text{ is k-linear and anti-symmetric}\}$. The second cohomology group is the quotient $H^2_{al}(\mathfrak{g}, \mathfrak{n}) = ker(\delta_2)/im(\delta_1)$ where δ_k is the map

$$\delta_k : C^k \to C^{k+1}, (\delta_k \psi)(X_1, \ldots, X_{k+1}) = \sum_{1 \le i \le j \le k+1} (-1)^{j+1} \psi([X_i, X_j], X_1 \ldots X_{i-1}, X_{i+1}, \ldots, X_{k+1}).$$

In other words the second cohomology group is the equivalence class of 2-cocyles up to 2-coboundaries which are 2-cocycles that can be expressed in terms of a linear map $\alpha : \mathfrak{g} \to \mathfrak{n}$ by a relation $\psi(X, Y) = \alpha([X, Y])$

Theorem 1.3. *The inequivalent central extension of $\mathfrak{g}$ by $\mathfrak{n}$ are in one to one correspondence with the equivalence classes of $H^2_{al}(\mathfrak{g}, \mathfrak{n})$*

Proof. See [3] for the proof

□

We consider the even central extension of $\tilde{\mathfrak{g}}$ denoted by $\hat{\mathfrak{g}}_\kappa$ by adding the co-cycle

$$\omega(x \otimes P(t, t^{-1}; \theta), y \otimes Q(t, t^{-1}, \theta)) = (x, y) f((dP)Q) \tag{7}$$

where (x, y) is the Killing form and the d operator appearing in the RHS is the exterior derivative, and acts on terms in the Laurent expansion as

$$d(t^k) = kt^{k-1}dt, \qquad \text{and} \qquad d(t^k\theta) = kt^{k-1}\theta dt + t^k d\theta.$$

Thus if $P = P_1(t, t^{-1}) + \theta P_2$, and $Q = Q_1(t, t^{-1}) + \theta Q_2(t, t^{-1})$, then

$$f((dP)Q) = (dP_1 - P_1 d\theta)(Q_2 + \theta Q_2).$$

f can be given an explicit form in terms of a basis for one-forms

$$f(P_0 dt - P_1\theta d\theta + P_2 d\theta - P_3\theta dt) = -\oint_{|t|=1} (\alpha P_0 + \beta t^{2\kappa-1} P_1)\frac{dt}{2\pi i} \tag{8}$$

where $P_{0/1} = P_{0/1}(t, t^{-1})$ are Laurent polynomials and α and β are positive. This is equivalent to making the choices:

$$f(t^{-1}dt) = -\alpha \quad f(t^{-2\kappa}\theta d\theta) = -\beta \quad f(t^n dt) = 0\,(n \neq -1) \quad f(t^n\theta d\theta) = 0\,(n \neq -2k), \tag{9a}$$

$$f(t^n d\theta) = 0 \quad f(t^n\theta dt) = 0 \quad \forall\, n \in \mathbb{Z}, \tag{9b}$$

Equations (9a) and (9b) imply that f vanishes on all odd and exact forms, so that the central extension is even. To be more precise

$$\mathrm{Ker}(f) = \mathrm{span}\{t^n dt,\, t^n\theta d\theta,\, t^n d\theta\,, t^n\theta dt\ : n \in \mathbb{Z}\}$$

1.3. **Derivations of the central extension.** Given a derivation of $\mathfrak{g}$, we want to extend it to a derivation of $\tilde{\mathfrak{g}}_\kappa$ such that it acts trivially on the center. Using the graded Leibniz rule (superscript $\epsilon = 0, 1$ stands for even and odd derivations respectively)

$$\begin{aligned} &D^\epsilon[x \otimes P, y \otimes Q] \\ &\quad = [x \otimes D^\epsilon P, y \otimes Q] + [x \otimes (P_0 + (-1)^\epsilon)\theta P_1), y \otimes D^\epsilon Q] \\ &\quad = [x, y]D^\epsilon(PQ) + (x, y)f(dD^\epsilon(P_0 + \theta P_1)Q) + (x, y)f(d(P_0 + (-1)^\epsilon\theta P_1)D^\epsilon Q) \end{aligned}$$

Hence the lifting condition for derivations of $\tilde{\mathfrak{g}}_\kappa$ is

$$f(dD^\epsilon(P_0 + \theta P_1)Q) + f(d(P_0 + (-1)^\epsilon\theta P_1)D^\epsilon Q) = 0$$

Theorem 1.4. *The most general graded odd and even differentiations satisfying*

$$f(dD^\epsilon(P_0 + \theta P_1)Q) + f(d(P_0 + (-1)^\epsilon\theta P_1)D^\epsilon Q) = 0 \tag{10}$$

are

$$D^1_{n+\kappa} = t^n\Big(\sqrt{\frac{\beta}{\alpha}}t^{2\kappa}\frac{\partial}{\partial\theta} - t\sqrt{\frac{\alpha}{\beta}}\theta\frac{\partial}{\partial t}\Big) \tag{11}$$

$$D^0_n = \frac{1}{2}[D^1_{n-\kappa}, D^1_\kappa]_+ = -t^n\{t\frac{\partial}{\partial t} + (\frac{n}{2} + \kappa)\theta\frac{\partial}{\partial\theta}\} \tag{12}$$

Proof. Assuming an expansion of the derivations in terms of the basis $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \theta}$, one can find the polynomial coefficients in the above formula. For the odd operators the most general operator if of the form $D^1 = \sum_{n\in\mathbb{Z}} D^1_n$ where

$$D^1_n = -\alpha_n t^{n+1}\theta\frac{\partial}{\partial t} + \beta_n t^n \frac{\partial}{\partial \theta} \tag{13}$$

where the powers of t have been selected in such a way as to preserve degree of both the terms upto an arbitrary common factor. We assume $P = P_0 + \theta P_1$ and Q in equation (10) takes the form $P = t^{a_0} + \theta t^{a_1}$ and $Q = t^{b_0} + \theta t^{b_1}$ for $a_a, a_1, b_0, b_1 \in \mathbb{Z}$. Linearity of the derivations then ensure the proof works for all P, Q. Applying equation (9a, 9b) to each of the terms of equation (10) when $\epsilon = 1$,

$$f(d(D^1P)Q) = \alpha_{-2\kappa-a_0-b_1}a_0\beta - \beta_{-a_1-b_0}b_0\alpha$$

$$f(d(P_0 - \theta P_1)D^1Q) = \beta_{-a_0-b_1}a_0\alpha - \alpha_{-2\kappa-a_1-b_0}b_0\beta$$

Imposing condition (10) on the above two equations with $a_1 + b_0 = a_0 + b_1$, and renaming $-2\kappa - a_0 - b_1 = n$, yields

$$\beta_{n+2\kappa}\alpha = -\alpha_n\beta$$

Hence if we chose $\alpha_n = \sqrt{\frac{\alpha}{\beta}}$, then $\beta_{n+2\kappa} = -\sqrt{\frac{\beta}{\alpha}}$. Redefining $D^1_{n+\kappa} = -\alpha_n t^{n+1}\theta\frac{\partial}{\partial t} + \beta_{n+2\kappa}t^{n+2\kappa}\frac{\partial}{\partial\theta}$ in lieu of the above equation, and substituting in values of α_n and $\beta_{n+2\kappa}$ gives equation (11).

We apply a similiar procedure for $D^0 = \sum_{n\in\mathbb{Z}} D^0_n$ with a basis element given by

$$D^0_n = -\alpha_n t^{n+1}\frac{\partial}{\partial t} + \beta_n\theta t^n\frac{\partial}{\partial\theta} \tag{14}$$

Applying equation (9a, 9b) for D^0 and $P = t^{a_0} + \theta t^{a_1}$ and $Q = t^{b_0} + \theta t^{b_1}$, we get

$$f((dD^0P)Q) = (\alpha_{-a_0-b_0}a_0b_0\alpha + \alpha_{-2\kappa-a_1-b_1}a_1\beta - \beta_{-2\kappa-a_1-b_1}\beta)$$

$$f((dP)(D^0Q)) = (-\alpha_{-a_0-b_0}a_0b_0\alpha + \alpha_{-2\kappa-a_1-b_1}b_1\beta - \beta_{\beta-2\kappa-a_1-b_1}\beta)$$

Imposing condition (10) gives the equality for $\beta \neq 0$,

$$\alpha_{-2\kappa-a_1-b_1}(a_1 + b_1) = 2\beta_{-2\kappa-a_1-b_1} \quad \forall a_1, b_1 \in \mathbb{Z}$$

Setting $-2\kappa - a_1 - b_1 = n$ and $\alpha_n = 1$

$$\beta_n = -\frac{2\kappa + n}{2} \quad \forall n \in \mathbb{Z}$$

Hence equation the RHS of (12) is obtained.

The first equality in (12) can be proved by making use of $D^1_{n-k} = D^1_{(n-2\kappa)+\kappa}$ and computing the anti-commutator.

□

A change of variables $\theta \to \sqrt{\frac{\beta}{\alpha}} t^{[\kappa]}\theta$ can be used to get rid of the constant factors and reduce the class of superalgebras to $\kappa = \frac{1}{2}$ and $\kappa = 0$.

The super-Virasoro algebra SV_κ is defined as the universal extension of the algebra of differential operators, with the class of $\kappa = 0$ called the Ramond algebra and $\kappa = \frac{1}{2}$ the Neveu-Schwarz algebra. The superconformal current algebra $S_\kappa(G)$ is defined to be the semi-direct product

$$S_\kappa(G) := \hat{\mathfrak{g}_\kappa} \rtimes SV_k \tag{15}$$

with the Lie bracket given by

$$[x \otimes P + \mathcal{D}_1, y \otimes Q + \mathcal{D}_2] = [x \otimes P, y \otimes Q] + y \otimes \mathcal{D}_1 Q - x \otimes \mathcal{D}_2 P. \tag{16}$$

1.4. **Lie Brackets for the superconformal current algebra.** Defining a graded basis of generators allows us to fully describe the algebra in terms of commutation and anti-commutation relations. For $\mathfrak{g}$ of dimension d_G, we choose x_a satisfying

$$(x_a, x_b) = -C_2\delta_{ab}, \qquad [x_a, x_b] = \sum_{c=1}^{d_G} f_{abc}x_c, \qquad a, b, c = 1 \ldots, d_G \tag{17}$$

where C_2 is the eigenvalue of the Casimir operator for the adjoint representations of G. A graded basis for $\tilde{\mathfrak{g}}_\kappa$ can be defined by

$$Q^a_n = ix_a \otimes t^n \qquad\qquad h^a_{n+\kappa} = ix_a \otimes t^n\theta \tag{18}$$

After a rescaling of θ mentioned above, we obtain the following commutation relations

$$[h^a_{n+\kappa}, h^b_{m-\kappa}]_+ = \frac{\lambda}{2}\delta_{m+n}\delta_{ab} \tag{19}$$

$$[Q^a_m, h^b_{m+\kappa}] = \sum_{c=1}^{d_G} if_{abc}h^c_{n+m+\kappa} \tag{20}$$

$$[Q^a_n, Q^b_m] = \sum_{c=1}^{d_G} if_{abc}Q^c_{n+m} + \frac{\lambda}{2}n\delta_{n+m}\delta_{ab} \tag{21}$$

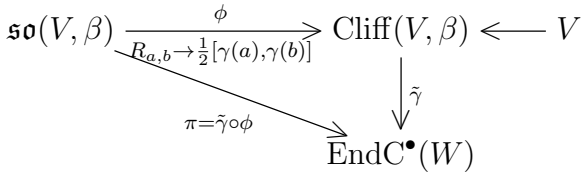
The above equations describe the super loop algebra part of the semi-direct product. The super Virasoro part, and their commutation relation with the superloop algebra generators

are given by

$$[h^a_{m+\kappa}, L_n] = (m + \kappa + \frac{n}{2})h^a_{m+n+\kappa} \tag{22}$$
$$[Q^a_m, L_n] = mQ^a_{m+n} \tag{23}$$
$$[h^a_{m+\kappa}, G_{n-\kappa}]_+ = Q^a_{m+n} \tag{24}$$
$$[Q^a_m, G_{n+\kappa}] = mh^a_{m+n+\kappa} \tag{25}$$
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} \tag{26}$$
$$[G_{m+\kappa}, L_n] = (m + \kappa - \frac{n}{2})G_{m+n+\kappa} \tag{27}$$
$$[G_{m+\kappa}, G_{n-\kappa}]_+ = 2L_{m+n} + \frac{c}{3}\{(\kappa+m)^2 - \frac{1}{4}\delta_{m+n}\} \tag{28}$$
$$m, n = 0, \pm1, \pm2, \ldots; \qquad \kappa = 0 \,\text{or}\, \frac{1}{2} \tag{29}$$

2. Representations of Superconformal Current algebras

2.1. **Spin Representations.** A complete classification of irreducible representations of simple Lie algebras and their lie groups has been done through Dynkin diagrams where each node corresponds to an irreducible representations called the fundamental representations. Other irreducible representations can be obtained by taking tensor product of these representations. In the case of orthogonal groups $SO(2l)$ and $SO(2l+1)$, Clifford algebras can be used to construct representations of the double cover of these groups $Spin(2l)$ and $Spin(2l+1)$ that have extra nodes in their Dynkin diagrams and give projective representations of the orthogonal groups. Two new representations for $\mathfrak{so}(2l)$, each of dimension 2^{l-1} and for $\mathfrak{so}(2l+1)$ a representation of dimension 2^l can be constructed by embedding $\mathfrak{so}(V)$ in $\text{Cliff}(V)$ corresponding to these projective representations. These are called spin representations and will be used in the next section to get a unitary representation of the superconformal algebra.



Consider a vector space V with a non-degenerate symmetric symmetric form β. For $\dim V$ even, we can decompose V as $V = W \oplus W^*$, where W, W^* are maximally isotropic subspaces and W^* is infact the dual space of W. On the other hand if $\dim V$ is odd, then V decomposes into $V = W \oplus \mathbb{C}e_0 \oplus W^*$ (Refer to [4, Chapter 1]). We denote $C^+(W) = \oplus_{p\ \text{even}} C^p(W)$ and $C^-(W) = \oplus_{p\ \text{odd}} C^p(W)$ for the space of antisymmetric multinear functions in even and odd variables respectively. One can identify the space $\wedge^p W^*$ with $C^p(W)$ through

$$\Psi(w_1, \ldots, w_n) = det[< w_i^*, w_j >]$$

This identification can be used to define a contraction and expansion

$$\epsilon(w^*) : C^p(W) \to C^{p+1}(W)$$
$$\epsilon(w^*)\Psi(w_0, \dots, w_p) = \sum (-1)^j < w^*, w_j > \Psi(w_0, \dots w_p)$$

where w_j has been ommited in the RHS of the last equation. Using the correspondence between $\wedge^p W^*$ and $C^p(W)$ a similiar map is defined by evaluation in the first variable

$$i(w) : \wedge^p W^* \to \wedge^{p-1} W^*$$
$$i(w)\Psi(w_2, \dots w_p) = \Psi(w, w_2, \dots w_p)$$

It can be checked that both ϵ, i are anti-commutative. For $\dim V$ even

$$\gamma : V \to \operatorname{End} \mathrm{C}^\bullet(W) \qquad \gamma(x + x^*) = i(x) + \epsilon(x^*) \tag{30}$$

While for odd dimensional V, γ can take two forms

$$\gamma_\pm(w + \lambda e_0 + w^*)u = (e(w) + \epsilon(w^*) + (-1)^p \lambda)u \qquad \text{for } u \in C^p(W) \tag{31}$$

The map γ will be used to define the spin representation, which is a representation on a "spinor space".

Definition 2.1. *If S is a complex vector space and there is a linear map $\alpha : V \to \operatorname{End}(S)$ such that*

$$\{\alpha(x), \alpha(y)\} = \beta(x, y)I$$

for all $x, y, \in V$ and the only subspaces of S invariant under $\alpha(V)$ are 0 and S, then the pair (S, α) is called a spinor space.

As elements of V generate $\operatorname{Cliff}(V, \beta)$, α extends to an irreducible representation

$$\tilde{\alpha} : \operatorname{Cliff}(V, \beta) \to \operatorname{End}(S)$$

This accounts for all the irreducible representations due to the universal property of Clifford algebra.

The goal of this section is to find a representation of $\mathfrak{so}(V, \beta)$ on the space of spinors $\mathrm{EndC}^\bullet(W)$, where $\mathrm{EndC}^\bullet(W)$ is the space of p-multilinear functions on V. First we define an embedding of $\mathfrak{so}(V, \beta)$ in $\operatorname{Cliff}(V)$. The elements of $\mathfrak{so}(V, \beta)$ are spanned by the elements $R_{a,b} \in \operatorname{End}(V)$, where $R_{a,b}v = \beta(b, v)a - \beta(a, v)b$ as a, b vary over V. The embedding is given by $\phi : \mathfrak{so}(V, \beta) \to \operatorname{Cliff}(V, \beta)$

$$\phi(R_{a,b}) = \frac{1}{2}[\zeta(a), \zeta(b)] \tag{32}$$

Here ζ is the defining map from (V, β) for the Clifford algebra i.e a linear map $\zeta : V \to \operatorname{Cliff}(V, \beta)$ and $\zeta(V)$ generates $\operatorname{Cliff}(V, \beta)$ as an associative algebra.

We now define the spin representation of $\mathfrak{so}(V, \beta)$ from the representation of $\operatorname{Cliff}(V, \beta)$ on the space of spinors $\mathrm{C}^\bullet(W)$ when $\dim V$ is even and odd respectively.

When $\dim V$ is even, the map $\gamma(v) : C^\pm(W) \to C^\mp(W)$ defined above interwines the even

and odd sectors and can be extended to $\tilde{\gamma} : \mathrm{Cliff}(V,\beta) \to \mathrm{End}(\mathrm{C}^\bullet(W))$ using the fact that V generates $\mathrm{Cliff}(V,\beta)$. The spin representation is given by the map

$$\Pi(x) : \tilde{\gamma} \circ \phi : \mathfrak{so}(V,\beta) \to \mathrm{End}\mathrm{C}^\bullet(W) \tag{33}$$

As $\phi(x)$ is an even element of the clifford algebra by virtue of definition of the map, it preserves the off and even spaces and one obtains two representations if $\dim V$ is even called the half-spin representations.

$$\Pi^{\pm}(x) = \Pi(x)_{C^{\pm}(W)}$$

For $\dim V = 2l$, these representations have dimension 2^{l-1}.

In the case $\dim V = 2l+1$, the spin representation π of $\mathfrak{so}(V,\beta)$ on $C^\bullet(W)$ is defined using the map γ_+ defined above, and its canonical extension $\tilde{\gamma}_+$ to a representation of $\mathrm{Cliff}(V,\beta)$ on $C^\bullet(W)$ in a similiar manner as the even case: $\pi = \tilde{\gamma}_+ \circ \phi$. The dimension of π is 2^l for $\dim V = 2l+1$.

2.2. **Minimal Unitary Highest Weight Representations (UHWIR).** The anticommutation relation for h^a_n's in equation (22) can be used as a symmetric bilinear form to define an infinite-dimensional clifford algebra. The minimal representation of the superconformal current algebra is then constructed in terms of a Fock space $\mathcal{F}$ acted upon by the Clifford algebra. Set the central charge $\lambda = C_2$. We now construct the generators Q^a_n, G_n, and L_n in terms of the Clifford algebra. The normal ordering operation [5] $: a(t)b(t) :$ is defined as

$$: a(t)b(t) := a(t)_+ b(t) + (-1)^{|a||b|} b(t)a(t)_-$$

where

$$a(t)_+ = \sum_{-(m+\kappa+\frac{1}{2})\geq 0} a_{m+\kappa} t^{-(m+\kappa+\frac{1}{2})} = \sum_{m\leq -1} a_{m+\kappa} t^{-(m+\kappa+\frac{1}{2})}$$

and

$$a(t)_- = \sum_{-(m+\kappa+\frac{1}{2}<0)} a_{m+\kappa} t^{-(m+\kappa+\frac{1}{2})} = \sum_{m>-1} a_{m+\kappa} t^{-(m+\kappa+\frac{1}{2})}$$

are the positive and negative Laurent modes. For $m, n \in \mathbb{Z}$ the normal ordered product can be defined as,

$$: a_{m+\kappa} b_{n+\kappa} ::= \begin{cases} a_{m+\kappa} b_{n+\kappa} & \text{if } m \leq -1 \\ (-1)^{|a||b|} b_{n+\kappa} a_{m+\kappa} & \text{if } m > -1 \end{cases}$$

The explicit form of the normal ordered product in terms of the variable t is related to the normal ordering for the modes given above by,

$$\begin{aligned} : a(t)b(t) : &= \sum_{n\in\mathbb{Z}} : ab :_{(n)} t^{-(n+1)} \\ &= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} : a_{n+\kappa} b_{n-m-\kappa} : t^{-(n+1)}. \end{aligned}$$

The last equality serves as the definition of $: ab :_{(n)}$ and is derived on page no 10.

One defines the superconformal generator and odd generator of the superloop algebra as follows

$$Q_n^a = \frac{i}{C_2} \sum_{a,t} f_{sat} \sum_{m \in \mathbb{Z}} : h^a_{\kappa - m} h^t_{n+m-\kappa} : \tag{34}$$

$$G_{n+\kappa} = \frac{2}{3C_2} \sum_{m \in \mathbb{Z}} : \vec{Q}_{-m} \vec{h}_{m+n+\kappa} : \tag{35}$$

$$L_n = \frac{1}{2}[G_{n-\kappa}, G_\kappa]_+ \tag{36}$$

$$L_0 = \frac{1}{2}[L_1, L_{-1}] \tag{37}$$

where the vector symbol denotes the inner product $\vec{Q}_k \vec{h}_\rho = \sum_{s=1}^{d_G} Q^s_\kappa h^s_\rho$

The motivation for the above definitions comes from physics (c.f. [2, Chapter 5]). One can form fields which have the above operators as Laurent expansion coefficients. We define the conformal stress-energy tensor generated by L_n, the left conserved currents generated by Q^a_n, and the fermi-fields $G(z)$ and $H_a(z)$.

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \tag{38}$$

$$J_a(z) = \sum_{n \in \mathbb{Z}} \frac{Q^a_n}{z^{n+1}} \tag{39}$$

$$G(z) = \sum_{n \in \mathbb{Z}} \frac{G_{n+\kappa}}{z^{n+\kappa+\frac{3}{2}}} \tag{40}$$

$$H_a(z) = \sum_{n \in \mathbb{Z}} \frac{h^a_{n+\kappa}}{z^{n+\kappa+\frac{1}{2}}} \tag{41}$$

Then equation (37) can be written down from the relationship $J^a(z) = \frac{i}{C_2} \sum_b \sum_c f_{abc} : H^b(z) H^c(z) :$. The normal ordering formula given above then reproduces equation (34) as

shown below.

$$\begin{aligned}
J^a(z) &= \sum_{n\in\mathbb{Z}} \frac{Q^a_n}{z^{n+1}} \\
&= \frac{i}{C_2} \sum_b \sum_c f_{abc} : H^b(z) H^c(z) : \\
&= \frac{i}{C_2} \sum_{b,c} \sum_{r,s} f_{abc} : h^b_{r+\kappa} h^c_{s+\kappa} : z^{-(r+s+2\kappa+1)} \\
&= \frac{i}{C_2} \sum_{b,c} \sum_{r\in\mathbb{Z}, r+s+2\kappa=n} f_{abc} : h^b_{r+\kappa} h^c_{n-r-\kappa} : z^{-(n+1)} \\
&= \frac{i}{C_2} \sum_{b,c} \sum_{r\in\mathbb{Z}} f_{abc} : h^b_{\kappa-r} h^c_{n+r-\kappa} : z^{-(n+1)}
\end{aligned}$$

Comparing last equality with (39), once obtains the equation for Q^a_n in terms of the modes, which can be further simplified as follows:

$$\begin{aligned}
Q^a_n &= \frac{i}{C_2} \sum_{a,t} f_{sat} \sum_{m\in\mathbb{Z}} : h^a_{\kappa+m} h^t_{n-m-\kappa} : . \\
&= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\leq -1} h^a_{\kappa+m} h^t_{n-m-\kappa} - \sum_{m>-1} h^t_{n-m-\kappa} h^a_{\kappa+m} \right) \\
&= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^a_{\kappa-m} h^t_{n+m-\kappa} - \sum_{m<1} h^t_{n+m-\kappa} h^a_{\kappa-m} \right) \\
&= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^a_{\kappa-m} h^t_{n+m-\kappa} - \sum_{m\geq -n} h^t_{\kappa-m} h^a_{n+m-\kappa} \right)
\end{aligned}$$

In the last equality a change of summation index $n+m-\kappa = \kappa - m'$ has been done, and both the $\kappa = 0$ and $\kappa = \frac{1}{2}$ cases give the summation condition $m' \geq -n$. This formula will be used in the proof of Theorem 2.2 below to verify the commutation relations.

Such quadratic normal ordered products are canonical in physics for models such as the supersymmetric free fermion and boson model, with a stress energy tensor

$$T(z) = \frac{1}{2}(J_a(z)J^a(z)) + \frac{1}{2}(H_a(z)\partial H^a(z)) \tag{42}$$

One can then define a fermionic field $G(z) = (J_a(z)H^a(z))$, whose Laurent mode expansion gives us the equation (35). Hence equations (34)-(37) which have been constructed in an ad-hoc way by Kac and Todorov in [1] can be obtained conceptually from the stress-energy tensors of certain CFTs and in turn from the Lagrangians using the Noether's procedure [6, Chapter 10].

Theorem 2.2. *The anti-commutation relation (25) and equations (37)-(40) imply the super-commutation relations (25)-(32) with central charges $\lambda = C_2$, $c = \frac{d_g}{2}$*

Proof. Consider the formula:

$$Q_n^a = \frac{i}{C_2} f_{sat} \sum_{m\in\mathbb{Z}} : h^s_{\kappa-m} h^t_{n+m-\kappa} :$$

From the definition of normal and ordering and using [5, Eqn 6.13, p. 178] the above formula can be written as

$$\begin{aligned} Q_n^a &= \frac{i}{C_2} \sum_{a,t} f_{sat} \sum_{m\in\mathbb{Z}} : h^s_{\kappa-m} h^t_{n+m-\kappa} : \\ &= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^s_{\kappa-m} h^t_{n+m-\kappa} - \sum_{m\geq -n} h^t_{n+m-\kappa} h^s_{\kappa-m} \right). \end{aligned}$$

Verifying equation (22) below,

$$\begin{aligned} [Q_n^a, h^b_{r+\kappa}] &= \frac{i}{C_2} \sum_{a,t} f_{sat} \left([\sum_{m\geq 1} h^s_{\kappa-m} h^t_{n+m-\kappa}, h^b_{r+\kappa}] - [\sum_{m\geq -n} h^t_{\kappa-m} h^s_{n+m-\kappa}, h^b_{r+\kappa}] \right) \\ &= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^s_{\kappa-m} [h^t_{n+m-\kappa}, h^b_{r+\kappa}] + \sum_{m\geq 1} [h^s_{\kappa-m}, h^b_{r+\kappa}] h^t_{n+m-\kappa} - t \longleftrightarrow s \right) \\ &= \frac{i}{C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^s_{\kappa-m} [h^t_{n+m-\kappa}, h^b_{r+\kappa}]_+ - \sum_{m\geq 1} [h^s_{\kappa-m}, h^b_{r+\kappa}]_+ h^t_{n+m-\kappa} - t \longleftrightarrow s \right) \\ &= \frac{i\lambda}{2C_2} \sum_{a,t} f_{sat} \left(\sum_{m\geq 1} h^s_{\kappa-m} \delta_{n+m+r} \delta_{t,b} - \sum_{m\geq 1} h^t_{n+m-\kappa} \delta_{r-m+2\kappa} \delta_{s,b} - t \longleftrightarrow s \right) \\ &= \frac{i\lambda}{2C_2} \sum_{a,t} f_{sat} \left((\sum_{m\geq 1} + \sum_{m\geq -n}) h^s_{\kappa-m} \delta_{n+m+r} \delta_{t,b} - h^t_{n+m-\kappa} \delta_{r-m+2\kappa} \delta_{s,b} \right) \\ &= \frac{i\lambda}{2C_2} \sum_{a,t} \big[f_{sat} \left(\sum_{m\geq 1} h^s_{\kappa-m} \delta_{n+m+r} \delta_{t,b} - h^t_{n+m-\kappa} \delta_{r-m+2\kappa} \delta_{s,b} \right) \\ &+ \frac{i\lambda}{2C_2} \left(\sum_{m\geq 1} h^s_{\kappa-m+n+1} \delta_{m-1+r} \delta_{t,b} - h^t_{m-1-\kappa} \delta_{r-m+n+1+2\kappa} \right) \big] \\ &= i f_{abt} h^t_{n+r+\kappa}. \end{aligned}$$

By checking for each case ($n < 0, r < 0$ or $m > 0, r > 0$ or $m < 0, r > 0$ or $m > 0, r < 0$) only two terms survive and using total anti-symmetry of the structure constants f_{abc} the last equality follows. The pre-factor cancels by using $\lambda = C_2$.

□

We now construct the minimal UHWIR on the Fock space. We assume a unique vacuum state $|0>$ annihilated by half the operators. Hermiticity of the fields imply $L_n^* = L_{-n}, G_\rho^* = G_{-\rho}, \dots$ etc. Hence the spectrum of L_0 should be non-negative and their should exists a highest weight vector.

For $\kappa = \frac{1}{2}$, the highest weight is defined by

$$h_\rho^a|0>=0 \quad \text{for}\, \rho \geq \frac{1}{2} \qquad\qquad L_n|0>=0 \quad \text{for}\, n \geq -1 \qquad \rho \in \frac{1}{2} + \mathbb{Z}, n \in \mathbb{Z} \tag{43}$$

For $\kappa = 0$, the minimal UHWIR is given by the spin representation of $\mathfrak{so}(d_G)$, this is because the zero modes of the fields in the Ramond sector satisfy the Clifford algebra $[h_0^a, h_0^b]_+ = \frac{\lambda}{2}\delta_{a,b}$, and the state-operator correspondence gives fields that transform as spinors. The highest weight state is given by

$$h_n^a|R(G)>=0 \quad \text{n} \geq 1, \qquad \vec{z}\vec{h}_0|R(G)>=0 \quad \text{for}\ z \in Z_- \tag{44}$$

Z_- is a fixed maximal isotropic subspace of C^{d_G} (this is the sapce W in the notation of the previous section) of dimension $[\frac{d_G}{2}]$. The full representation space is spanned by $h_0^{a_1} \dots h_0^{a_n}|R(G) > \ (0 \leq n \leq d_G)$ constrained by the equation (47) and is of dimension $2^{[\frac{d_G}{2}]}$.

2.3. **The Metaplectic factorization.** A result by Karl-Hermann Neeb shows that the unitary highest weight module of a special-type of finite-dimensional algebra can be written as a tensor product of a highest weight module of a reductive algebra and a highest weight module of a generalized Heisenberg algebra. This result can be extended to the case of superconformal algebras, and hence used to classify arbitrary UHWIR.

Definition 2.3. *(Generalized Heisenberg algebra) let V and $\mathfrak{z}$ be complex vector spaces with involutive antilinear automorphisms $v \to v^*$ and $z \to z^*$ respectively and a bilinear skew-symmetric map $\Omega : V \times V \to \mathfrak{z}$ satisfying $\Omega(w,v)^* = \Omega(w^*, v^*)$. The generalized Heisenberg algebra $\mathfrak{u} := \mathfrak{h}(V, \mathfrak{z}, \Omega)$ is the cartesian product $\mathfrak{u} := V \times \mathfrak{z}$ with the Lie Bracket $[(v,x),(v',x')] = (0, \Omega(v,v'))$. If we chose a complex subspace $V = V^+ \oplus V^-$, and $\Omega(V^+, V^+) = 0$, then $\mathfrak{b} = V^+ \times \mathfrak{z}$ satisfies $\mathfrak{u} = \mathfrak{g} + \mathfrak{g}^*$ is called a generalized parabolic subalgebra*

A linear functional $\alpha : \mathfrak{z} \to \mathbb{C}$ with $\alpha^* = \alpha$ defines a hermitian form $h_\alpha(v,w) = \alpha([v, w^*])$. One can define a multiplication operator on $P := \text{Pol}(\text{V}^+)$, $m_v f(z) := h_\alpha(z,v)f(z)$ and a differential operator $(\partial_v f)(z) = \frac{d}{dt}|_{t=0} f(z + tv)$. It is easy to see that the map $\rho : \mathfrak{u} \to \text{End}(\text{Pol}(\text{V}^+))$,

$$(v, u, z) \to \partial_v + m_{u^*} + \alpha(z)$$

is a representation of the Heisenberg algebra $\mathfrak{u}$ on the space of polynomials in V^+, $\text{End}(\text{Pol}(\text{V}^+))$. Also ρ defines a generalized highest weight representation of $\mathfrak{u}$ with respect to the parabolic $\mathfrak{b}$ with highest weight $\lambda : \mathfrak{b} \to \mathbb{C}$ defined as $\lambda(v,z) := \alpha(z)$ and generated by the constant function 1. The representation is unitary if h_α is positive semi-definite.

The metaplectic representation of the Jacobi algebra We now describe how the representation discussed above can be use to construct a highest weight representaton of the Jacobi algebra $\mathfrak{hsp}(V,\Omega) := \mathfrak{h}(V,\Omega) \rtimes \mathfrak{sp}(V,\Omega)$, which is essential for the metaplectic factorization in the general case. Let $\mathfrak{g} := \rho(\mathfrak{u}) + \rho(V)^2$, then it is easy to show that $\mathfrak{g}$ is a subalgebra of $\mathrm{End}(P)$ and $\rho(\mathfrak{u}) \cong \mathfrak{u}$ is a nilpotent ideal of $\mathfrak{g}$. Let $e_1, \ldots, e_n$ be an orthonormal basis for V^+, and $z_j :=< z, e_j >:= [z, e_j^*]$. The operators are abbreviated as $\partial_j := \partial_{e_j}$ and $m_j := m_{e_j}$. In terms of this basis

$$\rho(V)^2 = \mathbb{C}\mathbf{1} + \sum_{j,k=1}^{n} \mathbb{C} m_j \partial_k + \sum_{j,k=1}^{n} \mathbb{C}\partial_j\partial_k + \sum_{j,k=1}^{n} \mathbb{C} m_j m_k.$$

The root decomposition of $\mathfrak{g}$ is given by the following lemma

Lemma 2.4. *The subalgebra* $\mathfrak{h} := \mathrm{span}\{m_1\partial_1, \ldots, m_n\partial_n\}$ *is a Cartan subalgebra of* $\mathfrak{g}$*, and the roots are given by*

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm\epsilon_j, \pm\epsilon_j \pm \epsilon_l : 1 \leq j \neq l \leq n\} \backslash \{0\}$$

, and the corresponding root spaces are

$$\mathfrak{g}^{\epsilon_j} = \mathbb{C}\,\partial_j, \quad g^{-\epsilon_j} = \mathbb{C}\,m_j, \quad \mathfrak{g}^{\epsilon_j+\epsilon_l} = \mathbb{C}\,\partial_j\partial_l, \quad \mathfrak{g}^{-\epsilon_j-\epsilon_l} = \mathbb{C}\,m_j m_l, \quad \mathfrak{g}^{\epsilon_j-\epsilon_l} = \mathbb{C}\,m_l\partial_j.$$

The roots can be classified as

$$\Delta_r = \{\pm\epsilon_j : j = 1, \ldots, n\}, \quad \Delta_s = \{\pm 2\epsilon_j, \pm\epsilon_j \pm \epsilon_l : 1 \leq j \neq l \leq n\},$$

and

$$\Delta_k = \{\epsilon_j - \epsilon_l : 1 \leq j \neq l \leq n\}, \quad \Delta_{p,s} = \{\pm(\epsilon_j + \epsilon_l) : j, l = 1, \ldots, n\}$$

Proof. The roots are defined via the linear functionals $\epsilon_j \in \mathfrak{h}^*$ by $\epsilon_j(1) = 0$, and $\epsilon_j(m_k\partial_k) = -\delta_{jk}$. Here Δ_s stands for the semi-simple root and Δ_r for the solvable roots. The root decomposition can be computed by calculating the Lie Brackets w.r.t $\mathfrak{h}$ where we abbreviate $E_j := m_j\partial_j$, turn out to be

$$[E_k, m_j] = \epsilon_j(E_k)m_j \tag{45}$$

$$[E_k, \partial_j] = \epsilon_j(E_k)\partial_j \tag{46}$$

$$[E_k, \partial_j\partial_k] = (\epsilon_j + \epsilon_l)(E_k)\partial_j\partial_k \quad [E_k, m_j m_l] = -(\epsilon_j + \epsilon_l)(E_k)m_j m_k \tag{47}$$

$$[E_k, m_l\partial_j] = (\epsilon_j - \epsilon_l)(E_k)m_l\partial_j \tag{48}$$

Now clearly $\epsilon_j \in \Delta_r$ as these roots come from the Heisenberg part $\mathfrak{u}$ which is nilpotent. The semi-simplicity of the roots in Δ_s can be checked by $\alpha([Z, Z^*]) \neq 0$, where α is the linear functional for the element Z. Δ_k and $\Delta_{p,s}$ corresponds to when this functional is positive and negative respectively.

□

It can be shown that $\mathfrak{g}$ is isomorphic to the complex Jacobi algebra. The representation of $\mathfrak{u}$ on $\mathrm{Pol}(\mathrm{V}^+)$ discussed above extends to a representation of the Jacobi algebra and is called the metaplectic representation. In addition, The real form $\mathfrak{g}_{\mathbb{R}} = \{X \in \mathfrak{g} : X^* = -X\}$ is isomorphic to the Jacobi algebra $\mathfrak{hsp}(V_{\mathbb{R}}, \Omega_{\mathbb{R}})$, where $\Omega_{\mathbb{R}}$ is the restriction of Ω to $V_{\mathbb{R}}$.

Lemma 2.5. $\mathfrak{g}$ *is isomorphic to the complex Jacobi Algebra* $\mathfrak{hsp}(V,\Omega) = \mathfrak{h}(V,\Omega) \rtimes \mathfrak{sp}(V,\Omega)$

Proof. Let $der_{\mathfrak{z}}(\mathfrak{u}) := \{D \in der(\mathfrak{u}) : D.\mathfrak{z} = 0, D.V \subseteq V\} \cong \mathfrak{sp}(V,\Omega)$. Consider the homormophism

$$\delta : \rho(V)^2 \to der_{\mathfrak{z}}(\mathfrak{u}) \qquad \delta(x).y = \rho^{-1}([x,\rho(y)]).$$

It is easy to check that $\ker\delta = \mathbb{C}\,\mathbf{1}$. In view of the formula for $\rho(V)^2$ above Lemma 2.4,

$$\dim(\mathrm{im}\delta) = n(n+1) + n^2 = 2n^2 + n = \dim \mathfrak{sp}(V,\Omega).$$

Hence $\rho(V)^2$ is a one-dimensional extension of $\mathfrak{sp}(V,\Omega)$ that splits, and hence $\rho(V)^2$ contains a subalgebra (say $\mathfrak{l}$) isomorphic to $\mathfrak{sp}(V,\Omega)$, and

$$\mathfrak{g} = \rho(\mathfrak{u}) \rtimes \mathfrak{l} \cong \mathfrak{u} \rtimes \mathfrak{sp}(V,\Omega) = \mathfrak{hsp}(V,\Omega),$$

where use has been made of the fact $\rho(\mathfrak{u}) \cong \mathfrak{u}$. This isomorphism also describes an irreducible unitary representation of $\mathfrak{hsp}(V,\Omega)$. □

Theorem 2.6. *(1)* $\mathfrak{h}(V_{\mathbb{R}},\Omega_{\mathbb{R}}) \cong \rho(V_{\mathbb{R}},\Omega_{\mathbb{R}})$ *is the nilradical of* $\mathfrak{g}_{\mathbb{R}}$

(2) $\Delta^+ := \{\epsilon_j, \epsilon_j \pm \epsilon_l : 1 \le j \le l \le n\}\backslash\{0\}$ *is an adapted positive system of roots.*

(3) Let $\epsilon_0 \in h^*$ *be defined by* $\epsilon_0(\mathbf{1}) = 1$ *and* $\epsilon_0(\tilde{\alpha}) = 0$ *for all* $\alpha \in \Delta$. *Then* $P := \mathrm{Pol}(\mathrm{V}^+)$ *is an irreducible unitary highest weight module with the highest weight*

$$\lambda = \epsilon_0 - \frac{1}{2}\sum_{j=1}^{n}\epsilon_j = \epsilon_0 - \frac{1}{2}\mathrm{tr} \circ \mathrm{ad}_{\mathrm{V}^+}$$

with respect to Δ^+. *The set* $\mathcal{P}_P$ *of* $\mathfrak{h}$*-weights of* P *is given by* $\lambda - \sum_{j=1}^{n}\mathbb{N}_0\epsilon_j$

Proof. (1) The real form of $\mathfrak{u}$, $\mathfrak{u}_{\mathbb{R}} = V_{\mathbb{R}} \times i\mathbb{R}$ gives the real Heisenberg algebra $\mathfrak{h}(V_{\mathbb{R}},V_{\mathbb{R}})$. Also it can be verified from the definition of ρ and δ, $\delta(x)(u)^* = -\delta(x^*).u^*$ for $x \in \mathfrak{l}$ and $u \in \mathfrak{u}$, so that the for the involution $x^* = -x$, $\delta(x)$ preserves the real form $\mathfrak{u}_{\mathbb{R}}$. Thus analogous to the proof of Lemma 2.5, $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{hsp}(V_{\mathbb{R}},\omega_{\mathbb{R}})$.

(2) $\Delta^+ := \{\epsilon_j, \epsilon_j \pm \epsilon_l : 1 \le j \le l \le n\}\backslash\{0\}$ is an adapted positive system of roots. This is equivalent to and hence can be defined by the condition $\Delta_k \cup \Delta_p^+$ is a parabolic system (a root system with all positive(negative) roots and a single negative (positive) root) and satisfies the inequality $\alpha(\Delta_k \cup \Delta_p^+) \le 0, \alpha \in \Delta$.

(3) The constant function $v_\lambda := \mathbf{1}$ generates P. It is also a $\mathfrak{h}$ eigenvector and anihilated by all positive root spaces. Thus P is a unitary highest weight module of $\mathfrak{g}$ generated by v_λ, where $\lambda(\mathbf{1}) = 1$ and $\lambda(E_j) = 0$ for $j = 1, 2 \ldots, n$. This can be written by the formula $\lambda = \epsilon_0 - \frac{1}{2}\sum_{j=1}^{n}\epsilon_j$. The formula $[\partial_j^2, m_j^2] = 4E_j + 2\cdot\mathbf{1}$ as worked out in lemme 2.4 means the co-root

$$(2\epsilon_j)^\vee = -E_j - \frac{1}{2}\mathbf{1}.$$

Since ϵ_0 vanishes on all co-roots by definition, $\epsilon_0(E_j) = -\frac{1}{2}$ and the expression for λ thus holds.

□

We now define a homomorphism γ_α between an involutive lie algebra that can be written as $\mathfrak{g} = \mathfrak{u} \rtimes l = V \times \mathfrak{z} \times l$ and $\mathfrak{hsp}(V,\Omega)$, where the label α stands for a functional $\alpha \in \mathfrak{z}^*$ with $\alpha^* = \alpha$. Here $\mathfrak{l}$ is an involutive lie algebra that acts on $\mathfrak{u}$ through $\delta : l \to \mathrm{der}(\mathfrak{u})$ such that δ preserves V, $\delta(X).\mathfrak{z} = \{0\}$ and $\delta(X^*) = \delta(X)^*$. This map,

$$\gamma_\alpha : \mathfrak{g} = V \times \mathfrak{z} \times \mathfrak{l} \to \mathfrak{hsp}(V,\Omega), \quad (v,z,X) \to (v,\alpha(z),\delta(X)$$

can be used to define a pullback of representations of $\mathfrak{hsp}(V,\Omega)$ to representations of $\mathfrak{g}$ given by $\nu_\alpha := \rho_\alpha \circ \gamma_\alpha$. This pullback allows us to compute the highest weight of a generalized metaplectic representation as given by the following theorem.

Theorem 2.7. *Let $\mathfrak{g}$ be a finite-dimensional involutive with root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$. Let us assume $\mathfrak{g}$ has the Levi decomposition, $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$, where $\mathfrak{l}$ is an $\mathfrak{h}$-invariant reductive involutive subalgebra and $\mathfrak{u}$ is the maximal nilpotent ideal. Let us assume that $\mathfrak{u}$ is a generalized Heisenberg algebra cand can be written as*

$$\mathfrak{u} = V \times \mathfrak{z} \quad \textit{where} \quad V^+ \cong \mathfrak{p}_r^+ = \sum_{\alpha\in\Delta_r^+}\mathfrak{g}^\alpha \quad \mathfrak{z} = \mathfrak{z}(\mathfrak{g})$$

for an adapted positive system Δ^+. A linear functional $\alpha \in \mathfrak{z}^*$ with $\alpha = \alpha^*$ defines the skew-symmetric map $\Omega_\alpha(v,w) := \alpha([v,w])$ with $rad(\Omega_\alpha)$ as its radical. Denote $V_\alpha := V/\mathrm{rad}(\Omega_\alpha)$. The action of l on V factors an action on V_α giving the homorphism

$$\gamma_\alpha : \mathfrak{g} = V \times \mathfrak{z} \times \mathfrak{l} \to \mathfrak{hsp}(V,\Omega), \quad (v,z,X) \to (v + \mathrm{rad}(\Omega_\alpha), \alpha(z), \mathrm{ad}_{V_\alpha}(X)).$$

If the hermitian form $h_\alpha(v,w) := \alpha([v,w^*])$ is positive semi-definite on V^+, then the generalized metaplectic representation $\nu_{alpha} = \rho_\alpha \circ \gamma_\alpha$ of $\mathfrak{g}$ on $\mathrm{Pol}(V_\alpha^+)$ is a unitary highest weight representation with highest weight

$$\lambda_\alpha = \alpha - \frac{1}{2}\mathrm{ad}_{V_\alpha^+},$$

where α is extended by 0 on $\mathfrak{h}\cap\mathfrak{l}$ to an element of $\mathfrak{h}^*$

We now inch close to the theorem on metaplectic factorization which would allow us to factorize a a unitary highest weight representation as a tensor product of such a pull back to a generalized Heisenberg algebra, and a representation of a reductive quotient algebra. We will make use of the following three results in the final proof.

Lemma 2.8. *If $\mathfrak{h}$ is an abelian involutive Lie algebra and V a unitary $\mathfrak{h}-$module such that all weight spaces V^α, are finite-dimensional, then each $\mathfrak{h}$-invariant subspace $W \subseteq V$ satisfies $V = W \oplus W^\perp$.*

We now state the restriction theorem which can be used to write a unitary highest weight module as a orthogonal direct sum of unitary highest weight modules over a subalgebra.

Theorem 2.9. *Let $\mathfrak{g}$ be an **admissible** involutive Lie algebra, Δ^+ an adapted positive system and $L(\lambda,\Delta^+)$ a unitary highest weight module w.r.t Δ^+. Let $\mathfrak{g}_1 \subseteq \mathfrak{g}$ be an involutive subalgebra containing an element $X_0 = X_0^* \in \mathrm{int}(\Delta_p^+)$. Then,*

1. *$L(\lambda,\Delta^+)$is a semisimple $\mathfrak{g}_1$-module with finite multiplicites. It is an orthogonal direct sum of simple $/g_1-$modules.*
2. *Let $\mathfrak{h} \subseteq \mathfrak{g}_1$ be an involutive Cartan subalgebra of $\mathfrak{g}_1$ containing X_0 and $\Delta_1^+ \subseteq \Delta_1$ a positive system with $X_0 \in (\Delta_1^+)^*$. Then each simple $\mathfrak{g}_1$-submodule of $L(\lambda,\Delta^+)$ is a unitary highest weight module with respect to Δ_1^+*
3. *$L(\lambda,\Delta^+)$ is an orthogonal direct sum of unitary highest weight modules of $\mathfrak{g}_1$ with respect to the positive system Δ_1^+.*

Lemma 2.10. *Let V be a $\mathfrak{g}-$module with $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\boldsymbol{1}$ and W a trivial $\mathfrak{g}-$module. Then $\mathrm{End}_{\mathfrak{g}}(V \otimes W) \cong \boldsymbol{1} \otimes \mathrm{End}(W)$*

We now state and prove the main theorem for the metaplectic factorization of unitary highest weigh modules.

Theorem 2.11. *Let $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$ be an involutive admissible Lie algebra with root decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\alpha}, \mathfrak{z} := \mathfrak{z}(\mathfrak{g}), \Delta^+$ an adapted positive system and $\lambda = \lambda^* \in \mathfrak{h}^*$ such that the highest weight module $L(\lambda, \Delta^+)$ with highest weight λ is unitary. Let $h_{\lambda_Z} : (v, w) \to \lambda_Z([v, w^*])$ be a hermitian form on $\mathfrak{p}_{\mathfrak{r}}{}^+ = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{\alpha}$ and $rad(h_{\lambda_Z})$ its radical. We define*

$$\rho_{\lambda_Z} := \frac{1}{2}\mathrm{tr}\,\mathrm{ad}_{\mathfrak{p}_r^+} - \frac{1}{2}\mathrm{tr}\,\mathrm{ad}_{\mathrm{rad}(h_{\lambda_Z})}$$

and $\lambda_U := \lambda_z - \rho_{\lambda_Z}$. Then the highest weight module factorizes as

$$L(\lambda, \Delta^+) \cong L(\lambda_U, \Delta^+) \otimes L(\lambda - \lambda_u, \Delta^+)$$

. One also has the isomorphisms

(1) $L(\lambda_U, \Delta^+) \cong L(\lambda_Z, \mathfrak{z} + \mathfrak{p}_r^+, \mathfrak{u})$ as $\mathfrak{u}$-modules, and
(2) $L(\lambda - \lambda_U, \Delta^+) \cong L(\lambda - \lambda_U, \Delta_l^+)$ as $\mathfrak{l}$-modules as a trivial module acting on this space

Proof. First we obtain a tensor product using the Restriction Theorem (Theorem 2.9), and then show the isomorphisms using the theorem on metaplectic representation (Theorem 2.7), and a lemma on involutive lie algebras with hermitian forms ([7, Lemma IX.1.3]).

Consider the subalgebra $\mathfrak{g}_1 := \mathfrak{u} + \mathfrak{h}$. Then Theorem 2.9 allows us to write $V := L(\lambda, Delta^+)$ as an orthogonal direct sum of irreducible highest weight modules of $\mathfrak{g}_1$. According to the classification of unitary highest weight modules of generalized oscillator algebras ([7, Theorem IX.1.24]) is equivalent as $\mathfrak{u}-$modules to $V_{\mathfrak{u}} := L(\lambda_z, \mathfrak{z} + \mathfrak{p}_r^+, \mathfrak{u})$. Thus $V \cong V_{\mathfrak{u}} \otimes W$, where W acts as a trivial $\mathfrak{l}-$module.

We now obtain a representation π_λ of $\mathfrak{g}$ on V. Call $\nu = \nu_{\lambda_z}$ the corresponding metaplectic representation of $\mathfrak{g}$ on $V_{\mathfrak{u}}$, and consider the mapping

$$\pi' : \mathfrak{g} \to \mathrm{End}(V), \quad X \to \pi_{\lambda(x)} - \nu(X) \otimes \mathbf{1}$$

. Clearly $\mathfrak{u} \subseteq \ker\pi'$. For any $X \in \mathfrak{g}$ and $Y \in \mathfrak{u}$, it can be show by a simple computation

$$[\pi_\lambda(Y), \pi'(X)] = 0.$$

Lemme 2.10 shows that there exists a linear map

$$\rho_W : \mathfrak{g} \to \mathrm{End}(W)$$

where $\pi'(X) = \mathbf{1} \otimes \rho_W(X)$, and hence $\rho_W(\mathfrak{u}) = \{0\}$. We now prove that ρ_W is a representation of $\mathfrak{g}$.

$$\begin{aligned}
\mathbf{1} \otimes [\rho_w(X), \rho_w(Y)] &= [1 \otimes \rho_w(X), \pi_\lambda(Y) - \nu(Y) \otimes \mathbf{1}] \\
&= [\pi_\lambda(X) - \nu(X) \otimes \mathbf{1}, \pi_\lambda(Y)] \\
&= [\pi_\lambda(X), \pi_\lambda(Y)] - [\nu(X) \otimes \mathbf{1}, \nu(Y) \otimes \mathbf{1}] \\
&= \mathbf{1} \otimes \rho_W([X, Y]).
\end{aligned}$$

Hence $\pi_\lambda = \nu \otimes \rho_W$, where ρ_W is a representation of $\mathfrak{l}$ on W. As V is irreducible, W is an irreducible $\mathfrak{l}-$ module. Theorem 2.7 states that ν turns $V_{\mathfrak{u}}$ into a highest weight module $L(\lambda_U, \Delta^+, \mathfrak{g})$. The other isomorphism $W \cong L(\lambda - \lambda_U, \Delta^+)$ follows from [7, Proposition IX.1.13] and [7, Proposition IX.3.12]. Proposition IX.1.13 implies that the set of $\mathfrak{h}-$weights of W are $\mathcal{P}_W \subseteq \lambda - \lambda_U - \mathbb{N}_0[\Delta^+]$. Define $\mu := \lambda - \lambda_U$ and as $\mathcal{P}_W \subseteq \mu - \mathbb{N}_[\Delta^+]$, Proposition IX.3.12 implies that $W \cong L(\mu, \Delta^+)$.

□

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