1 Notation

We use the following notation

$$W = \mathbb{C}[t, t^{-1}]$$
$$A = W \otimes \Lambda\mathbb{C}$$
$$A_0 = W \otimes \mathbb{C}$$
$$A_1 = W \otimes \mathbb{C}\theta$$

2 Derivations of $W \otimes \Lambda\mathbb{C}$

In this section, we attempt to classify all the derivations of $W \otimes \Lambda\mathbb{C}$. We first prove a simple lemma that will be used twice.

Lemma : If D is an odd or even derivation and $u \in A_0$, then $D(u) = D(t)\frac{du}{dt}$

Let $D : A_0 \to A$ be a derivation of A restricted to A_0. Then, we have

$$D(1) = D(1 * 1) = D(1)(1) + (1)D(1) = 2D(1)$$

which implies $D(1) = 0$

We also have

$$0 = D(1) = D((t)(t^{-1})) = D(t)(t^{-1}) + (t)D(t^{-1})$$

which implies $D(t^{-1}) = (-t^{-2})D(t)$

Let us now define a derivation E on A restricted to A_0 by

$$E = D(t)(\frac{d}{dt})$$

It is clear that $E(t) = D(t)$. We also have

$$E(t^{-1}) = D(t)(-t^{-2}) = D(t^{-1})$$

where the last equality follow from the above formula.

Because a derivation is uniquely determined by its values on the generators of an algebra, it is now clear that $D(u) = D(t)\frac{du}{dt}$.

We now consider $A = W \otimes \Lambda\mathbb{C} = W \otimes \mathbb{C} \oplus W \otimes \mathbb{C}e$. In this case, derivations split into odd and even derivations, that is,

$$der(A) = der^-(A) \oplus der^+(A)$$

in the sense that, if $u \in A$, $v \in A_p$, $D^+ \in der^+(A)$, $D^- \in der^-(A)$, we have

$$D^+ : A_l \to A_l, D^+(vu) = D^+(v)u + vD^+(u)$$

as well as

$$D^- : A_l \to A_{l+1}, D^-(vu) = D^-(v)u + (-1)^p vD^-(u)$$

2.1 Even derivations

Proposition : Any even derivation has to obey the following two relations, and these relations define a derivation

$$D^+(f) = pf'$$
$$D^+(\theta) = q\theta$$

where f, p and q are Laurent polynomials.

Proof By the previous fact, we have that $D^+(f) = D^+(t)\frac{df}{dt}$. By restricting the codomain to A_0, we let $D^+(t) = p$. We also let $D^+(\theta) = qe$ as general elements of A_1, respectively. The two relations follow.

Now we must prove that these two relations, once extended by linearity and by Leibniz rule, define a derivation. For elements of the form fe, we obtain

$$D^+(f\theta) = D^+(f)(\theta) + (f)D^+(\theta) = (pf' + fq)\theta$$

. Linearity follows by extension. The only thing that is left to check is the Leibniz rule. We shall examine all 3 possibilities.

1. $D^+((f)(g)$ where f, g, are Laurent polynomials.

$$D^+((f)(g) = (pf')(g) + (f)(pg')$$

$$= (p)((f'g + g'f))$$
$$= D^+(fg)$$

2. $D^+(f(g\theta))$ where f, g are Laurent polynomials.

$$D^+(f(g\theta)) = D^+(f)(g\theta) + (f)D^+(g\theta)$$

$$= (pf')(g\theta) + (f)((pg' + gq)\theta)$$

$$= (pf'g + fpg' + fgq)\theta$$

$$= (p(f'g + fg') + fgq)\theta$$

$$= (p(fg)' + (fg)q)\theta$$

$$= D^+(fg\theta)$$

3. $D^+((f\theta)(g\theta))$ where f, g are Laurent polynomials. We know that this is zero because $\theta^2 = 0$.

$$D^+((f\theta)(g\theta)) = D^+(f\theta)(g\theta) + (f\theta)D^+(g\theta)$$

$$= ((pf' + fq)\theta)(g\theta) + (f\theta)((pg' + gq)\theta)$$

$$= 0$$

We conclude that these relations define an even derivation because the Leibniz rule is verified on a basis t^m or $t^m\theta$ (t^m is also a polynomial and the Leibniz rule is satisfied for arbitrary polynomials).

2.2 Odd derivations

Proposition : Any odd derivation has to obey the following two relations, and these relations define a derivation

$$D^-(f) = (f' p\theta)$$
$$D^-(\theta) = q$$

where f, p and q are Laurent polynomials.

Proof By the previous fact, $D^-(f) = D^-(t \otimes 1)\frac{df}{dt}$ (because $f \in A_0$). Let us write

$$D^-(t) = p\theta$$

as a general element of A_1. Similarly, we write

$$D^-(\theta) = q$$

as a general element of A_0.

We now prove that this defines an odd derivation, once extended by linearity. Once again, this extension uses the Leibniz rule to define the derivation on elements of the form $f\theta$, that is :

$$D^-(f\theta) = (pf'\theta)(\theta) + (f)(q)$$
$$= fq$$

We need to check the Leibniz rule in 4 cases this time :

1. $D^-((f)(g))$ where f, g are Laurent polynomials.

$$D^-((f)(g)) = D^-(f)(g) + (f)D^-(g)$$

$$= (f' p\theta)(g) + (f)(g' p\theta)$$

$$= ((f'g + fg')p\theta) = (fg)' p\theta = D^-(fg)$$

2. $D^-((f)(g\theta))$ where f, g are Laurent polynomials

$$D^-((f)(g\theta)) = D^-(f)(g\theta) + (f)D^-(g\theta)$$

$$= (f' p\theta)(g\theta) + (f)(gq)$$

$$= fgq = D^-(fg\theta)$$

3. $D^-((f\theta)(g))$ where f, g are Laurent polynomials

$$D^-((f\theta)(g)) = D^-(f\theta)(g) - (f\theta)D^-(g)$$

$$= (fq)(g) - (f\theta)(g'p\theta)$$

$$= fgq = D^-(fg\theta)$$

4. $0 = D^-((f\theta)(g\theta))$ where f, g are Laurent polynomials

$$D^-((f\theta)(g\theta)) = D^-(f\theta)(g\theta) - (f\theta)D^-(g\theta)$$

$$= (fq)(g\theta) - (f\theta)(gq)$$

$$= (fgq\theta) - (fgq\theta) = 0$$

We conclude that these definitions satisfy the conditions for an odd derivation.

3 The SuperWitt algebra

We can now define an explicit basis for $Der(A)$, that we will now refer to as the SuperWitt algebra and will sometimes be written as L. The basis is the following

$$d_n = -t^{n+1}\frac{d}{dt}$$

$$f_n = \theta t^n \frac{d}{d\theta}$$

$$\overline{d_n} = -\theta t^{n+1}\frac{d}{dt}$$

$$\overline{f_n} = t^n \frac{d}{d\theta}$$

It is clear that the overlined basis elements are odd derivations, while the others are even. It is not difficult to show that these elements are linearly independent, and that they span all of $Der(A)$. The first fact follows from the linear independence of the basis elements $\{t^n, t^n\theta \mid n \in \mathbb{Z}\}$ (it suffices to apply any linear combination that sums to 0 of these elements to t or θ to find that the coefficients must be 0), while the second one follows from the above classification.

In particular, these basis elements form a Lie Superalgebra. The 10 resulting brackets can be computed to obtain

$[d_m, d_n] = (m-n)d_{m+n}$
$[d_m, f_n] = -nf_{m+n}$
$[d_m, \overline{d_n}] = (m-n)\overline{d_{m+n}}$
$[d_m, \overline{f_n}] = -n\overline{f_{m+n}}$

$[f_m, f_n] = 0$
$[f_m, \overline{d_n}] = \overline{d_{m+n}}$
$[f_m, \overline{f_n}] = -\overline{f_{m+n}}$
$[\overline{d_m}, \overline{d_n}] = 0$
$[\overline{d_m}, \overline{f_n}] = d_{m+n} - nf_{m+n}$
$[\overline{f_m}, \overline{f_n}] = 0$

The computation of the last two brackets require to act on elements of the form f and $f\theta$ separately, where f is a Laurent polynomial. All the others can be done on an arbitrary twice differentiable function in t and θ, using known properties of differentiation.

4 The one-dimensional central extension of the SuperWitt algebra

Proposition : There exists a unique, up to isomorphism, one-dimensional non-trival even central extension of the SuperWitt algebra. This unique central extension is $L' = L \oplus \mathbb{C}c$, with $c \in Z(L')$ an even element and the brackets are

$[d_m, d_n] = (m-n)d_{m+n}$
$[d_m, f_n] = -nf_{m+n} - \delta_{m+n,0}(m^2)c$
$[f_m, f_n] = -2\delta_{m+n,0}mc$
$[\overline{d_m}, \overline{d_n}] = 0$
$[\overline{d_m}, \overline{f_n}] = d_{m+n} - nf_{m+n} - \delta_{m+n,0}(m^2)c$
$[\overline{f_m}, \overline{f_n}] = 0$
$[d_m, \overline{d_n}] = (m-n)\overline{d_{m+n}}$
$[d_m, \overline{f_n}] = \overline{-nf_{m+n}}$
$[f_m, \overline{d_n}] = \overline{d_{m+n}}$
$[f_m, \overline{f_n}] = -\overline{f_{m+n}}$

Proof : Let us suppose that $L' = L \oplus \mathbb{C}c$ is a central extension of L with c an element of the even center of L'. The last 4 brackets are already known. Suppose further that it is non-trivial. We will construct an explicit isomorphism with the above commutation relations. The proof will be in 7 steps, where in the first 6 steps we will use 7 SuperJacobi identities to find restrictions on the possible central extensions and prove uniqueness, and in the last step we will explicitely check the remaining 3 SuperJacobi identities to prove existence, which will turn out to be quite easy to check. Furthermore, since we will be writing quite a large number of SuperJacobi identities, we will take the liberty to replace all possible appearances of c by "..." if it turns out that this appearance does not affect the final result. This will be, we hope, quite clear from the fact that c is central. Furthermore, the basis elements that result at the end of the identities are assumed to cancel each other, because L is a Lie Superalgebra, so we will not write those either.

Here is a list of the 10 SuperJacobi identities that we are going to use and check. All other possible Jacobi identities that results in even elements can be obtained from these by bilinearity and supercommutativity. The 10 is obtained by a 3-combination from a 4-elements set with repetition allowed, and only taking the ones that result in even elements. There are in fact 10 more SuperJacobi identity to check, but they result in odd elements (by that we mean that they

are made of 2 even and 1 odd elements or 3 odd elements, and so c does not appear at the end of the computations and they are already satisfied). Thus, we do not need to check them.

$$[d_i,[d_m,d_n]]+[d_n,[d_i,d_m]]+[d_m,[d_n,d_i]]=0$$
$$[d_i,[f_m,f_n]]+[f_n,[d_i,f_m]]+[f_m,[f_n,d_i]]=0$$
$$[d_i,[f_m,d_n]]+[d_n,[d_i,f_m]]+[f_m,[d_n,d_i]]=0$$
$$[d_i,[\overline{f_m},\overline{f_n}]]-[\overline{f_n},[d_i,\overline{f_m}]]+[\overline{f_m},[\overline{f_n},d_i]]=0$$
$$[d_i,[\overline{d_m},\overline{d_n}]]-[\overline{d_n},[d_i,\overline{d_m}]]+[\overline{d_m},[\overline{d_n},d_i]]=0$$
$$[d_i,[\overline{d_m},\overline{f_n}]]-[\overline{f_n},[d_i,\overline{d_m}]]+[\overline{d_m},[\overline{f_n},d_i]]=0$$
$$[f_i,[\overline{d_m},\overline{f_n}]]-[\overline{f_n},[f_i,\overline{d_m}]]+[\overline{d_m},[\overline{f_n},f_i]]=0$$
$$[f_i,[f_m,f_n]]+[f_m,[f_n,f_i]]+[f_n,[f_i,f_m]]=0$$
$$[f_i,[\overline{d_m},\overline{d_n}]]-[\overline{d_n},[f_i,\overline{d_m}]]+[\overline{d_m},[\overline{d_n},f_i]]=0$$
$$[f_i,[\overline{f_m},\overline{f_n}]]-[\overline{f_n},[f_i,\overline{f_m}]]+[\overline{f_m},[\overline{f_n},f_i]]=0$$

Step 1 In this step, we suppose that $[d_m,d_n]=(m-n)d_{m+n}+a(m,n)c$ and we find possible restrictions on a(m,n) and use a change of basis to make things appear as close as it is in the claim.

First, because $[d_m,d_n]=-[d_n,d_m]$, we note that $a(m,n)=-a(n,m)$.

We define new elements

$$d'_n=\begin{cases} d_0 & \text{if n = 0} \\ d_n-\frac{1}{n}a(0,n)c & \text{otherwise}\end{cases}$$

These new elements along with the unchanged ones still form a basis for L'. We also define

$$a'(m,n)=\begin{cases} a(m,n) & \text{if m+n = 0} \\ a(m,n)+\frac{m-n}{m+n}a(0,m+n) & \text{otherwise}\end{cases}$$

We now have the new commutation relation

$$[d'_m,d'_n]=(m-n)d_{m+n}+a(m,n)c=(m-n)d'_{m+n}+a'(m,n)c$$

We now consider the following SuperJacobi identity, which must be satisfied for all i,m,n.

$$[d'_i,[d'_m,d'_n]]+[d'_n,[d'_i,d'_m]]+[d'_m,[d'_n,d'_i]]=0 \tag{1}$$
$$(m-n)[d'_i,d'_{m+n}+...]+(i-m)[d'_n,d'_{i+m}+...]+(n-i)[d'_m,d'_{n+i}+...]=0$$

which yields

$$(m-n)a(i,m+n)+(i-m)a(n,i+m)+(n-i)a(m,n+i)=0 \tag{2}$$

In particular, it must be satisfied for $i=0$

$$(m-n)a(0,m+n)-ma(n,m)+na(m,n)=0$$

$$(m-n)a(0,m+n) = -(m+n)a(m,n)$$

If $m+n \neq 0$ then $a(m,n) = -\frac{m-n}{m+n}a(0,m+n)$ and $a'(m,n) = 0$. Therefore,

$$a'(m,n) = \begin{cases} a(m,n) & \text{if m+n = 0} \\ 0 & \text{otherwise} \end{cases}$$

We now define $b(m) = a'(m,-m) = a(m,-m)$ to obtain the commutation relation

$$[d'_m, d'_n] = (m-n)d'_{m+n} + \delta_{m+n,0}b(m)c$$

Note that $b(m) = -b(-m)$ and $b(0) = 0$. Equation (2) must still be satisfied with $i = -m-1$ and $n = 1$

$$(m-1)a(-m-1,m+1) - (2m+1)a(1,-1) + (2+m)a(m,-m) = 0$$

or equivalently

$$(m-1)b(-m-1) - (2m+1)b(1) + (2+m)b(m) = 0$$

or

$$(m-1)b(m+1) = (2+m)b(m) - (2m+1)b(1)$$

Note that whenever $m > 1$, $b(m+1) = \frac{(2+m)b(m)-(2m+1)b(1)}{m-1}$ and so all values for $m > 2$ are uniquely determined by $b(1)$ and $b(2)$. Negative values are determined by $b(m) = -b(-m)$. There are then at most 2 solutions and one can verify that $b(m) = \gamma m$ and $b(m) = \beta m^3$ are solutions. Let us consider the general solution $b(m) = \beta m^3 + \gamma m$. We use another change of basis

$$d''_n = \begin{cases} d'_0 + \frac{\beta+\gamma}{2}c & \text{if n = 0} \\ d'_n & \text{otherwise} \end{cases}$$

In the case where $m+n \neq 0$, we have

$$[d''_m, d''_n] = (m-n)d'_{m+n} = (m-n)d''_{m+n} + \delta_{m+n,0}(m^3-m)\gamma c$$

If $m+n = 0$, we have

$$[d''_m, d''_n] = (m-n)d'_{m+n} + \beta m^3 + \gamma mc = 2md'_{m+n} + 2m\frac{(\beta+\gamma)}{2} + (\beta m^3 - \gamma m)c$$

$$= (m-n)d''_{m+n} + (m^3-m)\gamma c = (m-n)d''_{m+n} + \delta_{m+n,0}(m^3-m)\gamma c$$

In general then,

$$[d''_m, d''_n] = (m-n)d''_{m+n} + \delta_{m+n,0}(m^3-m)\gamma c \tag{3}$$

Conclusion of step 1 : Using a change of basis, we obtain elements d''_n that satisfy equation (3). From now on however, we will drop the primes for increased readability. We will not have to change this basis again. We will later find that γ must be 0. One can also verify that equation (3) satisfies the corresponding equation (1) in general. Hence, we have also cleared one of our 10 SuperJacobi identities for existence.

Step 2 In this step, we suppose that $[f_m, f_n] = a(m,n)c$ and we find a restriction on $a(m,n)$.

First of all, it is clear that $a(m,n) = -a(n,m)$ from antisymmetry.

Now, we consider the following Jacobi identity, which must be true for all i,m,n

$$[d_i, [f_m, f_n]] + [f_m, [f_n, d_i]] + [f_n, [d_i, f_m]] = 0 \tag{4}$$

$$[d_i, ...] + n[f_m, f_{n+i} + ...] - m[f_n, f_{m+i} + ...] = 0$$

$$na(m, n+i) - ma(n, m+i) = 0 \tag{5}$$

In particular, this must be true for $i = 0$

$$na(m,n) - ma(n,m) = 0$$

$$(n+m)a(m,n) = 0$$

which implies that $a(m,n) = 0$ unless $m+n = 0$. We now let $b(m) = a(m,-m)$. Note that $b(m) = -b(-m)$ and that this also means that $b(0) = 0$ and

$$[f_m, f_n] = \delta_{m+n,0} b(m) c$$

. With $i = 1$, $n = -m-1$, equation (5) gives us

$$-(m+1)a(m,-m) - ma(-m-1, m+1) = 0$$

or equivalently

$$-(m+1)b(m) + mb(m+1) = 0$$

For $m > 0$, we have $b(m+1) = \frac{m+1}{m} b(m)$, and so all values of $b(m)$ for $m > 1$ are uniquely determined by $b(1)$ and negative values are given by $b(m) = -b(-m)$. Hence there is at most one solution to this recurrence relation. One can check that $b(m) = \alpha m$ is a solution. Hence, we have

$$[f_m, f_n] = \delta_{m+n,0} \alpha m c$$

Once again, one can easily check that this satisfies (4).

Conclusion of step 2 : We have have used a second Jacobi identity to find a restriction of $[f_m, f_n]$. We are now at 2 out of 10 Jacobi identities.

Step 3 In this step, we suppose that $[f_m, d_n] = m f_{m+n} + a(m,n)c$ and we find a restriction on $a(m,n)$ to make it appear as simple as possible. As in step 1, a change of basis will turn out to be necessary.

We define new elements

$$f'_n = \begin{cases} f_0 & \text{if n = 0} \\ f_n + \frac{1}{n} a(n,0)c & \text{otherwise} \end{cases}$$

These new elements along with the unchanged ones still form a basis for L'. We also define

$$a'_n = \begin{cases} a(m,n) & \text{if m+n = 0} \\ a(m,n) - \frac{m}{m+n} a(m+n,0) & \text{otherwise} \end{cases}$$

We now have the new commutation relation

$$[f'_m, d_n] = mf'_{m+n} + a'(m,n)c = mf_{m+n} + a(m,n)c$$

Now consider the following Jacobi identity, which must be true for all i,m,n.

$$[d_i, [f'_m, d_n]] + [f'_m, [d_n, d_i]] + [d_n, [d_i, f'_m]] = 0 \tag{6}$$

$$[d_i, mf'_{m+n} + ...] + [f'_m, (n-i)d_{i+m} + ...] + [d_m, -mf'_{i+m} + ...] = 0$$

$$\Rightarrow -ma(m+n,i) + (n-i)a(m,i+n) + ma(i+m,n) = 0 \tag{7}$$

Setting $i = 0$, this reduces to

$$-ma(m+n,0) + (n+m)a(m,n) = 0 \tag{8}$$

If $n + m \neq 0$, we have $a(m,n) = \frac{m}{m+n}a(m+n,0)$ and so $a'(m,n) = 0$. We now let $b(m) = a'(m,-m) = a(m,-m)$. We have the new commutation relation

$$[f'_m, d_n] = mf'_{m+n} + \delta_{m+n,0}b(m)c$$

Equation (7) must still be true. We set $i = -m-1$ and $n = 1$ to obtain

$$-ma(m+1,-m-1) + (m+2)a(m,-m) + ma(-1,1) = 0$$

$$-mb(m+1) + (m+2)b(m) + mb(-1) = 0$$

Setting $m = 0$, we find $b(0) = 0$. For the other values, we consider two cases. If $m > 0$, we have $b(m+1) = \frac{m+2}{m}b(m) + b(-1)$, and so $b(m)$ for $m > 0$ is uniquely determined by the values at $b(-1)$ and $b(1)$. For negative values, we have $b(m) = \frac{m}{m+2}(b(m+1) - b(-1))$ and so $b(m)$ for $m < 0$ is uniquely determined by the values at $b(-1)$ and $b(-2)$. It can be checked that in both cases, $b(m) = m$ and $b(m) = m^2$ are solutions. The general solution for all m is then

$$b(m) = \begin{cases} \alpha_1 m + \beta_1 m^2 & m \leq 0 \\ \alpha_2 m + \beta_2 m^2 & m \geq 0 \end{cases}$$

Setting $i = -m-n$ in equation (7), we get

$$-ma(m+n,-m-n) + (m+2n)a(m,-m) + ma(-n,n) = 0$$

$$-mb(m+n) + (m+2n)b(m) + mb(-n) = 0$$

By setting $m = n = 1$ we find after simplification that $\alpha_2 - \beta_2 = \alpha_1 - \beta_1$. Similarly, by setting $m = 1$, $i = 2$, we find after simplifications that $2\alpha_2 - 4\beta_2 = 2\alpha_1 - 4\beta_2$. From these 2 equations it is easy to see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Then, $b(m) = \alpha' m + \beta' m^2$. Things can be made simpler yet again with another change of basis

$$f''_n = \begin{cases} f'_0 + \alpha' c & \text{if n = 0} \\ f'_n & \text{otherwise} \end{cases}$$

If $m + n \neq 0$, we have

$$[f''_m, d_n] = mf'_{m+n} = mf''_{m+n} + \delta_{m+n,0}(m^2)\beta' c$$

If $m + n = 0$, we have

$$[f''_m, d_n] = mf'_{m+n} + (\alpha' m + \beta' m^2)c = m(f'_{m+n} + (\alpha')c) + \beta'(m^2)c$$

$$= mf''_{m+n} + \delta_{m+n,0}\beta'(m^2)c$$

Using a change of basis, we have obtained a much simpler commutation relation. If we drop the primes and use antisymmetry, we obtain

$$[d_m, f_n] = -nf_{m+n} - \delta_{m+n,0}m^2\beta c$$

which is very close to what is claimed. One again, one can check that this satisfies (6).

Step 4 In this step, we write $[\overline{f_m}, \overline{f_n}] = a(m,n)c$ and use a 4th Jacobi identity to find conditions on $a(m,n)$

First note that, unlike step 1 and 2, we have $a(m,n) = a(n,m)$ instead of $a(m,n) = -a(n,m)$.

From Super Jacobi identity, we have, for all i,m,n

$$[d_i, [\overline{f_m}, \overline{f_n}]] - [\overline{f_n}, [d_i, \overline{f_m}]] + [\overline{f_m}, [\overline{f_n}, d_i]] = 0 \tag{9}$$
$$[d_i, ...] + m[\overline{f_n}, \overline{f_{m+i}}] + n[\overline{f_m}, \overline{f_{n+i}}] = 0$$
$$ma(n, m+i) + na(m, n+i) = 0$$

In particular, if $i = 0$, $(m+n)a(m,n) = 0$ which implies that $a(m,n) = 0$ unless $m + n = 0$. Now write $b(m) = a(m,-m)$ with $b(m) = b(-m)$. Let $i = -m - n$. Then, we have

$$ma(n,-n) + na(m,-m) = 0$$

$$mb(n) + nb(m) = 0$$

which must be true for all m,n. If $n = 1$, $m = 0$, we have $b(0) = 0$. If instead $n = m$, we have $2mb(m) = 0$ which implies that $b(m) = 0$ whenever $m \neq 0$. Then, $b(m) = 0$ for all m and we are done. It is also clear that (9) will be satisfied.

Step 5 This is another easy step similar to step 4. This time, we let $[\overline{d_m}, \overline{d_n}] = a(m,n)c$. Once again, we have $a(m,n) = a(n,m)$.

From Super Jacobi identity, we have, for all i,m,n

$$[d_i, [\overline{d_n}, \overline{d_m}]] - [\overline{d_m}, [d_i, \overline{d_n}]] + [\overline{d_n}, [\overline{d_m}, d_i]] = 0 \tag{10}$$
$$[d_i, ...] - [\overline{d_m}, (i-n)\overline{d_{i+n}} + ...] + [\overline{d_n}, (m-i)\overline{d_{m+i}} + ...] = 0$$
$$(n-i)[\overline{d_m}, \overline{d_{i+n}}] + (m-i)[\overline{d_n}, \overline{d_{m+i}}] = 0$$
$$(n-i)a(m, i+n) + (m-i)a(n, m+i) = 0 \tag{11}$$

If $i = 0$, this reduces to $(n+m)a(m,n) = 0$ which implies that $a(m,n) = 0$ unless $m+n = 0$. For the fifth time, we let $b(m) = a(m,-m)$ with commutation relation

$$[\overline{d_m}, \overline{d_n}] = \delta_{m+n,0}b(m)c$$

Setting $i = -m - n$ in equation (11) yields

$$(m+2n)b(m) + (n+2m)b(n) = 0 \tag{12}$$

which must be true for all m,n. Then, setting $m = 0$

$$2nb(0) + nb(n) = 0$$

$$\Rightarrow b(n) = -2b(0)$$

unless $n = 0$. However, if $m = 2$ and $n = -1$ in (12), we have

$$0 + 3b(-1) = 0 \Rightarrow -6b(0) = 0 \Rightarrow b(0) = 0$$

Therefore, $b(m) = 0$ for all m and we are done once again. It is also clear that (10) holds.

Step 6 We are ready to find restrictions on $[\overline{d_m}, \overline{f_n}] = d_{m+n} - nf_{m+n} + a(m,n)c$. By doing so, we will also complete the construction of the isomorphism with the central extension defined in the claim. Unlike all previous steps, we will require not 1, but 2 Jacobi identities to do so. One interesting thing to note here : the change of basis that we have done earlier does appear in this resulting brackets, but we can simply absorb the missing coefficients inside $a(m,n)$.

Remember that $[\overline{d_m}, \overline{f_n}] = [\overline{f_n}, \overline{d_m}]$

The first Super Jacobi identity that we use, which must be true for all i,m,n is

$$[d_i, [\overline{d_m}, \overline{f_n}]] - [\overline{f_n}, [d_i, \overline{d_m}]] + [\overline{d_m}, [\overline{f_n}, d_i]] = 0 \tag{13}$$

$$[d_i, d_{m+n} - nf_{m+n} + ...] - [\overline{f_n}, (i-m)\overline{d_{i+m}}] + [\overline{d_m}, n\overline{f_{n+i}}] = 0$$

$$\delta_{i+m+n,0}(i^3 - i)\gamma - n\delta_{i+m+n,0}i^2\beta + (m-i)a(i+m,n) + na(m,n+i) = 0 \tag{14}$$

Setting $i = 0$ in the last expression, we obtain

$$ma(m,n) + na(m,n) = (m+n)a(m,n) = 0$$

which implies that $a(m,n)$ unless $m + n = 0$. For the sixth and final time, we let $b(m) = a(m,-m)$. Then, with $i = -m - n$ in (14), we have

$$((-m-n)^3 + m + n)\gamma - n(m+n)^2\beta + (2m+n)b(-n) + nb(m) = 0 \tag{15}$$

Setting $n = 0$ in (15) yields

$$(-m^3 + m)\gamma + 2mb(0) = 0 \tag{16}$$

$m = 1$ in (16) implies that $b(0) = 0$. Then, $m = 2$ in (16) implies that $\gamma = 0$.

We now want to find an explicit expression for $b(m)$. We use a second Jacobi identity, which must be true for all i,m,n.

$$[f_i, [\overline{d_m}, \overline{f_n}]] - [\overline{f_n}, [f_i, \overline{d_m}]] + [\overline{d_m}, [\overline{f_n}, f_i]] = 0 \tag{17}$$

$$[f_i, d_{m+n} - nf_{m+n} + ...] - [\overline{f_n}, \overline{d_{m+i}}] + [\overline{d_m}, \overline{f_{n+i}}] = 0$$

We let $i = -m - n$, otherwise this equation is already satisfied. This yields

$$(m+n)^2\beta + n\alpha(m+n) - b(-n) + b(m) = 0 \tag{18}$$

If $n = 0$ in (18), we have

$$b(m) = -m^2\beta \tag{19}$$

If $m = 0$ in (18), we have

$$b(-n) = n^2(\beta + \alpha) \tag{20}$$

or equivalently

$$b(m) = m^2(\beta + \alpha) \tag{21}$$

Substracting (21) from (19), we have

$$0 = (-2\beta - \alpha)m^2 \tag{22}$$

Which implies $\alpha = -2\beta$ (with $m = 1$). It can be checked that (19) satisfies (17). We now replace (19) in (15).

$$n(m+n)^2\beta+(2m+n)(-\beta(n^2))+n(-\beta(m^2)) = \beta(n(m+n)^2-n(m^2+2mn+n^2)) = 0 \tag{23}$$

and so (13) is satisfied.

It can be now checked that $\alpha = -2\beta$, $\gamma = 0$ and $b(m) = -m^2\beta$ satisfy also (17). Furthermore, we have found all our 6 brackets and the commutation relations must be

$[d_m, d_n] = (m - n)d_{m+n}$
$[d_m, f_n] = -nf_{m+n} - \delta_{m+n,0}(m^2)\beta c$
$[f_m, f_n] = -2\delta_{m+n,0}\beta mc$
$[\overline{d_m}, \overline{d_n}] = 0$
$[\overline{d_m}, \overline{f_n}] = d_{m+n} - nf_{m+n} + \delta_{m+n,0} - (m^2)\beta c$
$[\overline{f_m}, \overline{f_n}] = 0$

If β were to be 0, the extension would be trivial which is a contradiction. Hence, β must be non-zero. We can now do one last change of basis

$$c' = \beta c$$

to obtain the required commutation relations. This completes the proof of uniqueness.

Part 7 It remains to check 3 Jacobi identities to prove that this is a central extension. In the previous steps, most of the Super Jacobi identities were left to be checked by the reader. Most of the calculations is already there in its most $a(m, n)$ general form. It is enough to replace the found $a(m, n)$ functions to be convinced of existence.

The first unchecked Jacobi identity is actually quite simple, because it would be satisfied no matter what.

$$\begin{aligned}[f_i, [f_m, f_n]] + [f_n, [f_i, f_m]] + [f_m, [f_n, f_i]] \\ = [f_i, ...] + [f_n, ...] + [f_m, ...] = 0\end{aligned} \tag{24}$$

The second unchecked Jacobi identity is satisfied thanks to step 5.

$$\begin{aligned}[f_i, [\overline{d_m}, \overline{d_n}]] - [\overline{d_n}, [f_i, \overline{d_m}]] + [\overline{d_m}, [\overline{d_n}, f_i]] \\ = [f_i, ...] - [\overline{d_n}, \overline{d_{i+m}}] + [\overline{d_m}, \overline{d_{n+i}}] = 0\end{aligned} \tag{25}$$

Finally, the final unchecked Jacobi identity is satisfied thanks to step 6.

$$\begin{aligned}[f_i, [\overline{f_m}, \overline{f_n}]] - [\overline{f_n}, [f_i, \overline{f_m}]] + [\overline{f_m}, [\overline{f_n}, f_i]] \\ = [f_i, ...] + [\overline{f_n}, \overline{f_{i+m}}] + [\overline{f_m}, \overline{f_{n+i}}] = 0\end{aligned} \tag{26}$$

Hence, all 10 Jacobi identities are satisfied and this indeed defines a central extension.

5 The two-dimensional central even and odd extension of the SuperWitt algebra

The last section classified the case where we had only an even center. In this section, we show that adding an odd center does not change anything. In other words, we still get the following 10 commutation relations.

Claim : There exists a unique, up to isomorphism, two-dimensional non-trival central extension of the SuperWitt algebra where the center has one-dimensional even part and one-dimensional odd part. This unique central extension is $L' = L \oplus \mathbb{C}c \oplus \mathbb{C}\bar{c}$, with $c \in Z(L')$ an even element and $\bar{c} \in Z(L')$ an odd element. The commutation relations are

$[d_m, d_n] = (m-n)d_{m+n}$
$[d_m, f_n] = -nf_{m+n} - \delta_{m+n,0}(m^2)c$
$[f_m, f_n] = -2\delta_{m+n,0}mc$
$[\overline{d_m}, \overline{d_n}] = 0$
$[\overline{d_m}, \overline{f_n}] = d_{m+n} - nf_{m+n} - \delta_{m+n,0}(m^2)c$
$[\overline{f_m}, \overline{f_n}] = 0$
$[d_m, \overline{d_n}] = (m-n)\overline{d_{m+n}}$
$[d_m, \overline{f_n}] = -n\overline{f_{m+n}}$
$[f_m, \overline{d_n}] = \overline{d_{m+n}}$
$[f_m, \overline{f_n}] = -\overline{f_{m+n}}$

Proof : The first 6 brackets are obtained in the same way as in the last section. Similarly, the 10 even Jacobi identities are satisfied. Hence, the only thing we have left to do is to construct an explicit isomorphism that gives us the last 4 brackets to prove uniqueness while preserving the even brackets. Existence of this central extension is clear. The proof will be in 4 steps.

Step 8 In this step, we find restrictions on $[d_m, \overline{d_n}] = (n-m)\overline{d_{m+n}} + a(m,n)\bar{c}$, and by using a change of basis we will obtain something close to what is claimed. Consider the following Jacobi identity.

$$[d_i, [d_m, \overline{d_n}]] + [d_m, [\overline{d_n}, d_i]] + [\overline{d_n}, [d_i, d_m]] = 0 \tag{27}$$

$$[d_i, (m-n)\overline{d_{m+n}} + ...] + [d_m, (n-i)\overline{d_{n+i}} + ...] + [\overline{d_n}, (i-m)d_{i+m} + ...] = 0$$

$$(m-n)[d_i, \overline{d_{m+n}}] + (n-i)[d_m, \overline{d_{n+i}}] + (i-m)[\overline{d_n}, d_{i+m}] = 0$$

$$(m-n)a(i, m+n) + (n-i)a(m, n+i) + (m-i)a(i+m, n) = 0 \tag{28}$$

If $i = 0$,$n = 1$,$m = -1$ in the previous equation, we have $a(0,0) = 0$. If $i = 0$,$m + n \neq 0$, we have $a(m,n) = \frac{n-m}{m+n}a(0, m+n)$

We define new elements

$$\overline{d'_n} = \begin{cases} \overline{d_0} & \text{if n = 0} \\ \overline{d_n} - \frac{1}{n}a(0,n)\bar{c} & \text{otherwise} \end{cases}$$

These new elements along with the unchanged ones still form a basis for L' and preserve the even brackets. We also define

$$a'(m,n) = \begin{cases} a(m,n) & \text{if m+n} = 0 \\ a(m,n) + \frac{m-n}{m+n} a(0,m+n) & \text{otherwise} \end{cases}$$

We now have the new commutation relation

$$[\overline{d'_m}, \overline{d'_n}] = (m-n)\overline{d_{m+n}} + a(m,n)c = (m-n)\overline{d'_{m+n}} + a'(m,n)\overline{c}$$

But we know that $a'(m,n) = 0$ when $m+n \neq 0$. Then, we can define $b(m) = a'(m,-m) = a'(m-m)$ to get the commutation relation

$$[\overline{d'_m}, \overline{d'_n}] = (m-n)\overline{d'_{m+n}} + \delta_{m+n,0} b(m)\overline{c}$$

By setting $i = -m$, $n = 0$ in (28), we have

$$mb(-m) + mb(m) = 0$$

which implies that $b(m) = -b(-m)$ for all m, because $b(0) = 0$. The rest of the proof is the same as Step 1, because (28) is the same as (2) and we have just proven antisymmetry. Then, there is a change of basis that gives us the following relation

$$[\overline{d'_m}, \overline{d'_n}] = (m-n)\overline{d'_{m+n}} + \delta_{m+n,0}(m^3 - m)\overline{\gamma c} \tag{29}$$

We will show later that $\overline{\gamma}$ must be 0.

Step 9 In this step, we find restrictions on $[\overline{f_m}, d_n] = m\overline{f_{m+n}} + a(m,n)\overline{c}$, and by using a change of basis we will obtain something close to what is claimed. Consider the following Jacobi identity.

$$[d_i, [\overline{f_m}, d_n]] + [\overline{f_m}, [d_n, d_i]] + [d_n, [d_i, \overline{f_m}]] = 0 \tag{30}$$

$$[d_i, m\overline{f_{m+n}} + ...] + [\overline{f_m}, (n-i)d_{i+m} + ...] + [d_m, -m\overline{f_{i+m}} + ...] = 0$$

$$\Rightarrow -ma(m+n,i) + (n-i)a(m,i+n) + ma(i+m,n) = 0 \tag{31}$$

Since (30) is the same as (7) and the assumptions are the same, but with the odd element instead of the even one, step 9 can be done in exactly the same way as step 3, to obtain a new basis such that

$$[d_m, \overline{f_n}] = -n\overline{f_{m+n}} + \delta_{m+n,0} m^2 \overline{\beta c} \tag{32}$$

Step 10 In this step, we find restrictions on $[f_m, \overline{f_n}] = -\overline{f_{m+n}} + a(m,n)\overline{c}$. We start by using the following Jacobi identity

$$[d_i, [f_m, \overline{f_n}]] + [f_m, [\overline{f_n}, d_i]] + [\overline{f_n}, [d_i, f_m]] = 0 \tag{33}$$

$$[d_i, -\overline{f_{m+n}} + ...] + [f_m, n\overline{f_{n+i}} + ...] + [\overline{f_n}, -mf_{m+i} + ...] = 0$$

$$-\delta_{m+n+i,0}(m+n)^2\overline{\beta} + na(m,n+i) + ma(m+i,n) = 0 \tag{34}$$

Setting $i = 0$ gives us

$$-\delta_{m+n,0}(m+n)^2\overline{\beta} + na(m,n) + ma(m,n) = 0$$

$$\Rightarrow (m+n)a(m,n) = 0$$

which implies that $a(m,n) = 0$ unless $m+n = 0$. As usual, let $b(m) = a(m,-m)$ to get commutation relation

$$[f_m, \overline{f_n}] = -\overline{f_{m+n}} + \delta_{m+n,0} b(m)\overline{c}$$

(34) must still be satisfied with $i = -m-n$

$$-(m+n)^2\overline{\beta} + nb(m) + mb(-n) = 0 \tag{35}$$

Setting $n = 0$, $m = 1$, we have

$$-\overline{\beta} + b(0) = 0 \tag{36}$$

Setting $n = 0$, $m = 2$, we have

$$-4\overline{\beta} + 2b(0) = 0 \tag{37}$$

From these two equations it is easy to deduce that $\overline{\beta} = 0$ and $b(0) = 0$. Then, (35) reduces to

$$nb(m) + mb(n) = 0 \tag{38}$$

With $n = m$, we have $2mb(m) = 0 \Rightarrow b(m) = 0$ for all m.

Two out of four commutation relations are cleared.

Step 11 In this step, we finally find our last bracket, and also prove that $\overline{\gamma}$ must be 0. Suppose $[\overline{d_m}, f_n] = -\overline{d_{m+n}} + a(m,n)\overline{c}$. We use the following Jacobi identity first

$$[\overline{d_m}, [f_n, f_i]] + [f_n, [f_i, \overline{d_m}]] + [f_i, [\overline{d_m}, f_n]] = 0 \tag{39}$$

$$[\overline{d_m}, ...] + [f_n, \overline{d_{m+i}} + ...] + [f_i, \overline{d_{m+n}}] = 0$$

$$-a(m+i,n) + a(i,m+n) = 0 \tag{40}$$

Setting $i = 0$ gives us

$$a(m,n) = a(0,m+n) \tag{41}$$

It seems our usual trick did not work this time! However, we can still set $b(m) = a(0,m)$ and we obtain the relation

$$[\overline{d_m}, f_n] = -\overline{d_{m+n}} + b(m+n)\overline{c}$$

It turns out that this satisfies (39), so we shall need to use another Jacobi identity.

$$[\overline{d_m}, [d_n, f_i]] + [d_n, [f_i, \overline{d_m}]] + [f_i, [\overline{d_m}, d_n]] = 0 \tag{42}$$

$$[\overline{d_m}, -if_{n+i} + ...] + [d_n, \overline{d_{m+i}} + ...] + [f_i, (m-n)\overline{d_{m+n}}] = 0$$

$$-ia(m,n+i) + \delta_{m+n+i,0}(m^3 - m)\overline{\gamma} + (n-m)a(m+n,i) = 0$$

$$(n-m-i)b(m+n+i) + \delta_{m+n+i,0}(m^3 - m)\overline{\gamma} \tag{43}$$

Let $n = -m$ and $i = 0$ in (43). Then

$$2mb(0) + (m^3 - m)\overline{\gamma} = 0 \tag{44}$$

If m = 1, we have

$$2b(0) = 0 \Rightarrow b(0) = 0$$

Hence

$$(m^3 - m)\overline{\gamma} = 0$$

and clearly this can only be satisfied for all m if $\overline{\gamma} = 0$

Then, (43) becomes

$$(n - m - i)b(m + n + i) = 0 \tag{45}$$

If $m = n = 0$, we have

$$ib(i) = 0 \Rightarrow b(i) = 0$$

for all i, because $b(0) = 0$.

This completes the proof because all odd brackets have been shown to be trivial.

6 An explicit isomorphism to the N = 2 Ramond Algebra

If we define

$$\begin{aligned} L_m &= d_m - \frac{m}{2} f_m - \delta_{m,0} \frac{c}{4} \\ J_m &= f_m \\ G_m^+ &= \overline{d_m} \\ G_m^- &= \overline{f_m} \\ c' &= -6c \end{aligned}$$

we have the following 10 commutation relations

$$\begin{aligned} [L_m, L_n] &= (m-n)d_{m+n} + \frac{n^2}{2} f_{m+n} - \frac{m^2}{2} f_{m+n} + \delta_{m+n,0}(\frac{nm^2}{2} - \frac{mn^2}{2} - \frac{2mn}{4})c \\ &= (m-n)L_{m+n} + \delta_{m+n,0} \frac{-m^3 + m}{2} c \\ &= (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c' \\ [L_m, J_n] &= [d_m, f_n] - \frac{m}{2}[f_m, f_n] \\ &= -n f_{m+n} - \delta_{m+n,0} m^2 c + \frac{m}{2} \delta_{m+n,0} 2mc \\ &= -n f_{m+n} \\ &= -n J_{m+n} \\ [J_m, J_n] &= -2\delta_{m+n,0} mc \\ &= \delta_{m+n,0} \frac{m}{3} c' \end{aligned}$$

$$
\begin{aligned}
[G^+_m, G^-_n] &= d_{m+n} - n f_{m+n} - \delta_{m+n,0} m^2 c \\
&= d_{m+n} - \frac{m+n}{2} f_{m+n} + \frac{m-n}{2} f_{m+n} - \delta_{m+n,0} m^2 c \\
&= L_{m+n} + \frac{m-n}{2} J_{m+n} - \delta_{m+n,0} m^2 c + \delta_{m+n,0} \frac{c}{4} \\
&= L_{m+n} + \frac{m-n}{2} J_{m+n} - \delta_{m+n,0} \frac{c'}{6} (m^2 - \frac{1}{4}) \\
[G^+_m, G^+_n] &= [\overline{d_m}, \overline{d_n}] \\
&= 0 \\
[G^-_m, G^-_n] &= [\overline{f_m}, \overline{f_n}] \\
&= 0 \\
[L_m, G^+_n] &= [d_m, \overline{d_n}] - \frac{m}{2} [f_m, \overline{d_n}] \\
&= (m-n) \overline{d_{m+n}} - \frac{m}{2} \overline{d_{m+n}} \\
&= (\frac{m}{2} - n) G^+_n \\
[L_m, G^-_n] &= [d_m, \overline{f_n}] - \frac{m}{2} [f_m, \overline{f_n}] \\
&= (-n) \overline{f_{m+n}} + \frac{m}{2} \overline{f_{m+n}} \\
&= (\frac{m}{2} - n) G^-_n \\
[J_m, G^+_n] &= \overline{d_{m+n}} = G^+_{m+n} \\
[J_m, G^-_n] &= -\overline{f_{m+n}} = -G^-_{m+n}
\end{aligned}
$$

which is exactly the N = 2 Ramond algebra.