Supergeometry and Lie Supergroups

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1 Introduction

Supergeometry is an extension of classical geometry where one makes a distinction between even and odd coordinates, with the property that the odd coordinates are anticommutative. Usually, the transition from the usual theory to the super theory is done by adding a sign factor whenever the order of two odd elements is changed. Supergeometry is used extensively in theoretical physics especially in supersymmetry and other super theories. Supersymmetry gives rise to a symmetry between bosons and fermions, such that every elementary particle has a partner of opposite spin parity. In traditional physics, the symmetries arise from the tensor representations of the Poincaré group, while the supersymmetries instead use the spinor group.

Lie groups are groups that are also smooth manifolds such that the group structure is compatible with the smooth manifold structure. They are closely related to Lie algebra, whose underlying vector space is the tangent space of the Lie group. In the context of supergeometry, a superstructure A is a $\mathbf{Z}_2$-graded structure, with a direct sum decomposition of the form $A = A_0 \oplus A_1$. We encounter for example, Lie supergroups, supermanifolds, Lie superalgebras and so on.

This work is divided into two major parts. The first part covers Sections 2 through 12 and presents briefly the important definition, theorems and concepts related to supergeometry. The second part is covered in Section 13 and presents an Iwasawa decomposition of Lie Supergroups using isomorphisms between sheaves.

2 Super Vector Spaces

2.1 Gratings

Definition 2.1. An algebraic structure X is said to be graded if it possesses a grading, *i.e.* a decomposition into a direct sum $X = \bigoplus_{i \in I} X_i$ of structures, where I is an indexing ensemble. The elements that are purely in a structure X_i are said to be homogeneous of degree i.

Definition 2.2. A graded vector space is vector space that has a grading, that is, a decomposition of the vector space into a direct sum of vector subspaces.

Example 2.3. Let us consider a $\mathbb{Z}/2\mathbb{Z} = \mathbf{Z_2}$ graded vector space. Now $\mathbf{Z_2}$ has two elements, $\{0, 1\}$. Let V be a vector space such that V can be written in the form $V = V_0 \oplus V_1$ where V_0, V_1 are subspaces of V, then V is a graded vector space.

2.2 Super Vector Spaces

Definition 2.4. A super vector space is a $\mathbb{Z}/2\mathbb{Z}$ graded vector space. A super vector space V can be written as $V = V_0 \oplus V_1$. The elements of V_0 are called even whilst those of V_1 are called odd.

Remark 2.5. We will use the notation SVS to refer to super vector spaces.

Definition 2.6. Let $v \in V$, then the parity of v is denoted $|v|$ and is defined as follows:

$$|v| = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}$$

We note that the parity is only defined for homogeneous elements (*i.e.* elements of either V_0 or V_1).

Definition 2.7. The superdimension of a SVS is the pair (p, q), where $p = \dim(V_0)$ and $q = \dim(V_1)$.

Remark 2.8. When transitioning from the usual theory to the super theory a sign factor is usually introduced whenever the order of two odd elements is changed.

2.3 A Categorical Approach

Definition 2.9. A category $\mathcal{C}$ consist of three things: a collection of objects, a collection morphisms (also called arrows) between each pair of objects, and finally a binary operation called the composition and defined on compatible pairs of morphisms.

Definition 2.10. Functors are structure-preserving mappings between categories.

Definition 2.11. We define the parity reversing functor, Π, in the category of SVS as follows: let V be a SVS, then

$$\Pi(V \to \Pi V)$$

with $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$.

Definition 2.12. The tensor product in the category of SVS is defined as follows:

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$

$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

3 Tensor Product and Tensor Algebra

Here we review basic definitions such as algebras and modules, before looking at the tensor product of vector spaces and the tensor algebra constructed using this product.

3.1 Review of basic concepts

Definition 3.1. An algebra is a vector space A with a bilinear multiplication

$$\tau : A \otimes A \to A.$$

A superalgebra is a SVS with the same multiplication morphism.

Definition 3.2. A left-module M (over a ring R) is an additive abelian group with a product $R \times M \to M$ such that for $r, s \in R$ and $x, y \in M$:

1. $r(sx) = (rs)x$
2. $1x = x$
3. $(r + s)x = rx + sx$
4. $r(x + y) = rx + ry$

A right-module can be defined similarly.

Definition 3.3. A free module (or vector space) is a module (vector space) that has a basis.

3.2 The Tensor Product

Definition 3.4. Let U and V be free vector spaces over a field k with basis $\{e_i\}$ and $\{f_j\}$ respectively. Then $W = U \otimes V$ is the tensor product of U and V and the elements of W are of the form $\sum_{i,j} c_{i,j} e_i \otimes f_j$. Moreover, any element $u \otimes v$ of W for $u \in U$ and $v \in V$ can be obtained by expanding $u \otimes v$ in terms of the original basis of U and V.

Definition 3.5. The tensor algebra $T(V)$ of a vector space V is the algebra formed by all the tensors on V and with the tensor product as the multiplication.

Remark 3.6. Recall that $V^{\otimes n} = V \otimes \cdots \otimes V$ where the tensor product of V is repeated n times.

Example 3.7. Let V be a SVS with vectors denoted v_i. Applying a permutation s from the group of permutations S_n on $V^{\otimes n}$ yields:

$$s \cdot v_1 \otimes \cdots \otimes v_n = (-1)^{|s|} v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)}$$

Definition 3.8. Let $\mathfrak{a}_n$ be the subspace generated by the elements $v_1 \otimes \cdots \otimes v_n - s \cdot v_1 \otimes \cdots \otimes v_n$ for $s \in S_n$ and let V be a SVS, then the symmetric n-power is defined as: $\mathrm{Sym}^n(V) = V^{\otimes n}/\mathfrak{a}_n$,

Definition 3.9. The symmetric superalgebra $\mathrm{Sym}(V)$ is given by the quotient

$$T(V)/\mathfrak{a}$$

where $T(V)$ and $\mathfrak{a}$ are as define above.

The symmetric algebra one of the two important algebra that can be constructed using the quotient of a tensor product. The other one is the exterior algebra and will be discussed in Section 5.

4 Lie Superalgebras

An example of a super vector space is a Lie superalgebra.

Definition 4.1. Recall that a Lie algebra is a vector space $\mathfrak{g}$ over a field F together with a Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that defines a non-associative bilinear multiplication which satisfies three axioms:

1. Bilinearity: for all scalars $a, b \in F$ and all $x, y, z \in \mathfrak{g}$

   $$[ax + by, z] = a[x, z] + b[y, z]$$

   and

   $$[z, ax + by] = a[z, x] + b[z, y].$$

2. $[x, x] = 0$, $\forall x \in \mathfrak{g}$.

3. The Jacobi identity: for all $x, y, z \in \mathfrak{g}$

   $$[x, [y, z] + [y, [z, x]] + [z, [x, y]] = 0.$$

Definition 4.2. A Lie superalgebra is a super vector space $\mathfrak{g}$ together with a morphism

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

called a super Lie bracket (or simply bracket) which has the following properties:

1. $[x, y] = -(-1)^{|x||y|}[y, x]$, $\forall x, y \in \mathfrak{g}$

2. The super Jacobi identity: for all $x, y, z \in \mathfrak{g}$,

   $$[x, [y, z] + (-1)^{|x|\cdot(|y|+|z|)}[y, [z, x]] + (-1)^{|z|\cdot(|x|+|y|)}[z, [x, y]] = 0$$

Remark 4.3. One can obtain a Lie superalgebra from a superalgebra by taking the bracket:

$$[x, y] = xy - (-1)^{|x||y|}yx$$

defined for any elements x and y of the superalgebra.

Definition 4.4. For a, c, b, d elements of a superalgebra A, the tensor product of a superalgebra is defined as:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

Definition 4.5. The category of SVS admits an inner homomorphism, that we denote $\underline{\mathrm{Hom}}(V, W)$. For super vector spaces V and W, $\underline{\mathrm{Hom}}(V, W)$ consists of all linear maps from V to W. It has a super vector space structure defined by:

$$\begin{aligned} \underline{\mathrm{Hom}}(V, W)_0 &= \{T : V \to W : T \text{ preserves parity}\} \\ \underline{\mathrm{Hom}}(V, W)_1 &= \{T : V \to W : T \text{ reverses parity}\} \end{aligned}$$

Example 4.6. The associative superalgebra $\mathrm{End}(V)$ is the super vector space $\underline{\mathrm{Hom}}(V, V)$ with the composition as a product.

$$\mathrm{End}(V) = \underline{\mathrm{Hom}}(V, V)_0 \oplus \underline{\mathrm{Hom}}(V, V)_1$$

We can verify that it is a super Lie algebra with the bracket

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

5 Exterior Product and Exterior Algebra

The exterior product is the product used when constructing Grassmann algebras. These algebras are useful when trying to express topological objects in an algebraic setting.

5.1 Exterior forms

Definition 5.1. A 1-form is a linear function $\omega : \mathbb{R}^n \to \mathbb{R}$. An exterior form of degree 2 (or a 2-form) is a function of pairs of vectors $\omega^2 : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ which is bilinear and skew-symmetric.

Similarly, an exterior form of degree k is a function of k vectors which is k-linear and antisymmetric. We will now look at the exterior product of two 1-forms.

Remark 5.2. The space of 1-forms on $\mathbb{R}^n$ is the dual space $(\mathbb{R}^n)^*$.

Remark 5.3. If ω^k is a k-form and ω^l is an l-form on $\mathbb{R}^n$, then the exterior product (or wedge product) $\omega^k \wedge \omega^l$ is a $(k + l)$-form.

The exterior product of 1-forms associates to every pair of 1-forms (ω_1, ω_2) on $\mathbb{R}^n$ a 2-form $\omega_1 \wedge \omega_2$ on $\mathbb{R}^n$

Definition 5.4. We define a mapping $\mathbb{R}^n \to \mathbb{R} \otimes \mathbb{R}$ by associating to every $\eta \in \mathbb{R}^n$ the vector $\omega(\eta)$ with components $(\omega_1(\eta), \omega_2(\eta))$ in the plane with coordinates ω_1 and ω_2:

$$\omega : \eta \mapsto (\omega_1(\eta), \omega_2(\eta))$$

Definition 5.5. The exterior product $\omega_1 \wedge \omega_2$ applied on the pair η_1, η_2 is the oriented area of the parallelogram with sides $\omega(\eta_1)$ and $\omega(\eta_2)$ in the ω_1, ω_2-plane:

$$(\omega_1 \wedge \omega_2)(\eta_1, \eta_2) = \begin{vmatrix} \omega_1(\eta_1) & \omega_2(\eta_1) \\ \omega_1(\eta_2) & \omega_2(\eta_2) \end{vmatrix}$$

This definition can be extended to the wedge product of k 1-forms.

Corollary 5.6. Every n-form on $\mathbb{R}^n$ is either the oriented volume of a parallelepiped with some choice of unit volume, or zero.

5.2 Grassmann Algebra

The exterior algebra, also called Grassmann Algebra, is the algebra whose product is the exterior product. Such algebras are graded algebras.

Theorem 5.7. The exterior multiplication of forms is skew-commutative, distributive and associative. For ω_i^k and ω_j^l, $i = 1, 2, 3$, respectively k-forms and l-forms on $\mathbb{R}^n$:

1. $\omega_1^k \wedge \omega_1^l = (-1)^{kl} \omega_1^l \wedge \omega_1^k$
2. $(\lambda_1 \omega_1^k + \lambda_2 \omega_2^k) \wedge \omega_3^l = \lambda_1 \omega_1^k \wedge \omega_3^l + \lambda_2 \omega_2^k \wedge \omega_3^l$
3. $(\omega_1^w \wedge \omega_1^l) \wedge \omega_1^m = \omega_1^k \wedge (\omega_1^l \wedge \omega_1^m)$

Definition 5.8. Let I be the ideal generated by all the elements of the form $x \otimes x$ for $x \in V$. The exterior algebra $\Lambda(V)$ of a vector space V over a field k is the quotient of the tensor algebra by the ideal I:

$$\Lambda(V) := T(V)/I.$$

Consequently, the exterior product is defined by $\alpha \wedge \beta = \alpha \otimes \beta \mod I$.

Definition 5.9. The k^{th} exterior power in V, denoted $\Lambda^k(V)$, is the vector subspace of $\Lambda(V)$ spanned by elements of the form $x_1 \wedge \cdots \wedge x_k$ for $x_i \in V$ and $i = 1, 2, \ldots, k$.

6 Superspaces and Sheafs

6.1 Manifolds

Definition 6.1. A topological space X is a set of points together with open sets satisfying the following axioms:

1. The empty set and X are open.
2. Any union of open sets is open.
3. The intersection of any finite number of open sets is open.

Definition 6.2. Homeomorphisms are the isomorphisms in the category of topological spaces, that is, the mappings that preserve the topological properties of the given topological space. Therefore, a homeomorphism is a continuous function between topological spaces that has a continuous inverse.

Definition 6.3. A manifold is a topological space that locally resembles an Euclidean space. That is, for each point of an n-dimensional manifold, there exist a neighbourhood around that point that is homeomorphic to the Euclidean space of dimension n.

6.2 Sheafs and Ringed Spaces

We now introduce the concept of sheaves. These structures are used in supergeometry to define any supergeometric object in differential of algebraic category.

Definition 6.4. Let M be a differentiable manifold. For $U \subset M$ an open set, let $C^\infty(U)$ be the R-algebra of smooth functions on U. Then $U \mapsto C^\infty(U)$ is called a sheaf. It satisfies the following two conditions:

1. If $U \subset V$ are two sets in M we can define the restriction map on V, which is an algebra morphism: $r_{v,u} : C^\infty(V) \to C^\infty(U)$, $f \mapsto f|_u$ with the following properties:

   i) $r_{v,u} = id$

   ii) $r_{w,u} = r_{v,u} \circ r_{w,v}$

2. Let $\{U_i\}_{i\in I}$ be a covering of U by open sets and let $\{f_i : f_i \in C^\infty(U_i)\}_{i\in I}$ be a family such that the restriction of f_i on the intersection of U_i and U_j is equal to f on $U_i \cap U_j$ $\forall i, j \in I$. Then there exist a unique $f \in C^\infty(U)$ such that $f|_{U_i} = f_i$.

We usually add additional conditions on M such that its topological space is Hansdorff and second countable.

Remark 6.5. A topological space X is a Hausdorff space if any two distinct points of X can be separated by neighbourhoods.

Definition 6.6. A ringed space is a topological space together with a sheaf of rings on it.

Definition 6.7. We say that $a \in \mathcal{O}(U)$ and $b \in \mathcal{O}(V)$ are equivalent if for $x \in X$ and U, V open sets with $x \in U$ and $x \in V$, there exists an open set W with $x \in W \subset U \cap V$ such that a, b have the same restrictions to W.

Definition 6.8. The stalk of a sheaf at x, denoted $\mathcal{O}_x$, is the ring formed by the equivalence classes given in Definition 6.7.

6.3 Superspaces

Definition 6.9. A ringed space is called space if the stalks are all local rings. Similarly, a superspace is a super ringed space such that the stalks are local supercommutative rings.

Remark 6.10. A commutative super ring is said to be local if it has a unique maximal ideal.

Definition 6.11. A supermanifold $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace that is locally isomorphic to $\mathbb{R}^{p|q}$.

Example 6.12. A supermatrix is a $\mathbf{Z}_2$-graded analogue of an ordinary matrix. Therefore, it is a 2×2 block matrix with entries in a superalgebra. The supermatrices of dimensions $(r, s) \times (p, q)$, denoted $M_{(r,s)\times(p,q)}$, are of the form:

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $\dim(A) = r \times p$, $\dim(B) = r \times q$, $\dim(C) = s \times p$ and $\dim(D) = s \times q$. Even matrices will have even entries on the diagonal and odd off-diagonal entries and vice-versa for odd matrices:

$$\text{even} : \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix} \qquad \text{odd} : \begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}$$

Squared supermatrices are matrices of superdimension $(p, q) \times (p, q)$ and are denoted $M_{p|q}$.

Remark 6.13. A homomorphism from a SVS to another must by definition preserve the grading of the vector space. These are only the even matrices since odd matrices will reverse the grading.

Remark 6.14. An ordinary field is simply the purely even part of a commutative superalgebra.

7 The Berezinian

The Berezinian is the generalization of the determinant for the case of supermatrices.

Definition 7.1. The Berezinian, denoted Ber, is uniquely defined by two properties. For X and Y two invertible supermatrices:

1. $\mathrm{Ber}(XY) = \mathrm{Ber}(X) \cdot \mathrm{Ber}(Y)$
2. $\mathrm{Ber}(e^X) = e^{\mathrm{str}(X)}$

where $\mathrm{str}(X)$ is the supertrace of X. The Berezinian is only defined for invertible linear transformations.

Definition 7.2. Let X be a block supermatrix of the form:

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Then the supertrace of X is defined as $\mathrm{str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D)$ where $\mathrm{tr}(E)$ refers to the ordinary trace of a matrix E.

Remark 7.3. Let X and Y be two block supermatrices. The supertrace of XY, denoted $< X, Y >:= \mathrm{str}(XY)$, has the following two properties:

$$\begin{aligned} < Y, X > &= (-1)^{|X||Y|} < X, Y > \\ < X, Y > &= 0 \text{ for } |X| \neq |Y| \end{aligned}$$

Example 7.4. Let X be a supermatrices of the form

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D have even entries and B and C have odd entries. This is an even matrix with entries in a supercommutative algebra R.

We note that X is invertible if and only if both A and D are invertible in the commutative ring R_0, (*i.e.* the even subalgebra of R). In this case, the Berizinian is given by

$$\mathrm{Ber}(X) = \det(A - BD^{-1}C)\det(D)^{-1} = \det(A)\det(D - CA^{-1}B)^{-1}$$

Remark 7.5. The expression $D - CA^{-1}B$ in the example above is called Schur's complement.

Corollary 7.6. *Properties of the Berezinian.* Let X, Y be invertible supermatrices, then:

1. $\mathrm{Ber}(X)$ is always a unit in the ring R_0.
2. $(\mathrm{Ber}(X))^{-1} = \mathrm{Ber}(X^{-1})$
3. $\mathrm{Ber}(X^{\mathrm{st}}) = \mathrm{Ber}(X)$ where X^{st} is the supertranspose of X.
4. $\mathrm{Ber}(X \oplus Y) = \mathrm{Ber}(X) \cdot \mathrm{Ber}(Y)$

8 Universal Enveloping Algebra

Let $\mathfrak{g}$ be a Lie superalgebra. Its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is a structure that preserves the important properties of $\mathfrak{g}$ while providing a unital associative superalgebra that is potentially easier to work with. An important property that is conserved is that the representations of $\mathfrak{g}$ correspond to the modules over $\mathfrak{U}(\mathfrak{g})$ in a one-to-one manner.

Definition 8.1. The universal enveloping algebra of $\mathfrak{g}$ is defined as the quotient of the tensor space $T(\mathfrak{g})$ by the ideal I generated by the elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$:

$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/I$$

The map $l : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is a natural embedding of $\mathfrak{g}$ in $\mathfrak{U}(\mathfrak{g})$ that comes from the natural mapping of the elements of $\mathfrak{g}$ into $T(\mathfrak{g})$. The universal enveloping algebra of $\mathfrak{g}$ is the pair $(l, \mathfrak{U}(\mathfrak{g}))$.

Remark 8.2. The enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is a unital associative algebra (an associative algebra with multiplicative unit 1).

Proposition 8.3. The enveloping algebra $\mathfrak{U}(\mathfrak{g})$ has the universal property that for every Lie superalgebra A and for each Lie superalgebra morphism $\rho : \mathfrak{g} \to A$, there exist a unique associative algebra homomorphism $\tilde{\rho} : \mathfrak{U}(\mathfrak{g}) \to A$ such that $\rho = \tilde{\rho} \circ l$

Theorem 8.4. (PBW Theorem). Let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{U}(\mathfrak{g})$ the universal enveloping superalgebra of $\mathfrak{g}$ with the mapping $l : \mathfrak{g} \mapsto \mathfrak{U}(\mathfrak{g})$. Let $\{X_i, Y_j\}$, for i, j elements of an indexing set I, be a homogeneous basis of $\mathfrak{g}$ with $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_1$. Then

1. $l : \mathfrak{g} \mapsto \mathfrak{U}(\mathfrak{g})$ is injective.
2. The elements $l(X_{i_1}) \cdots l(X_{i_r}) l(Y_{j_1}) \cdots l(Y_{j_s})$ with $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$ form a basis for $\mathfrak{U}(\mathfrak{g})$.

Remark 8.5. The PBW Theorem implies that if $x_1, \ldots, x_m$ are homogeneous elements forming a vector space basis for $\mathfrak{g}$, then the set of all monomials of the form

$$x_1^{\alpha_1}, \ldots, x_m^{\alpha_m}$$

with $\alpha_i \in \mathbb{N}$ if $|x_i| = 0$ and $\alpha_i \in \{0, 1\}$ if $|x_i| = 1$ is a basis for $\mathfrak{U}(\mathfrak{g})$.

9 Hopf Superalgebras

Hopf algebras represent an alternative approach to discuss Lie supergroups. Here we simply discuss the concepts necessary to define Hopf algebras.

Definition 9.1. A coalgebra A (over a field k) is the dual of a unital associative algebra (an associative algebra with multiplicative unit 1). The coalgebra A is associated with two linear maps, the co-multiplication $\Delta : A \to A \otimes A$ and the co-unit $\epsilon : A \to k$ such that:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id}\otimes\epsilon} & A \otimes k \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\cong} \\ A & \xrightarrow{\mathrm{id}} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\epsilon\otimes\mathrm{id}} & k \otimes A \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\cong} \\ A & \xrightarrow{\mathrm{id}} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\Delta\otimes\mathrm{id}} & A \otimes A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\mathrm{id}\otimes\Delta} \\ A & \xrightarrow{\Delta} & A \end{array}$$

Definition 9.2. A super bialgebra A (over a field k) is a structure with the following properties:

1. A is a superalgebra (with multiplication $\mu : A \otimes A \to A$ and unit $i : k \to A$).
2. A is a supercoalgebra, with comultiplication and counit as defined above.
3. The multiplication and the unit are morphisms of supercoalgebras.
4. The comultiplication and the counit are morphisms of superalgebras.

Definition 9.3. A superalgera A is a Hopf superalgebra if it is a bialgebra and if A is equipped with a bilinear map $S : A \to A$ called the *antipode* such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{S\otimes\mathrm{id}} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\mu} \\ A & \xrightarrow{i\cdot\epsilon} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id}\otimes S} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle\mu} \\ A & \xrightarrow{i\cdot\epsilon} & A \end{array}$$

10 Super Harish-Chandra Pairs

Although this will not be proved here, the category of super Harish-Chandra pairs is equivalent to the category of Lie supergroups and are therefore an alternative approach to the study of Lie supergroups.

Definition 10.1. Consider a the pair $(G_0, \mathfrak{g})$ consisting of a Lie group and a super Lie algebra, together with a representation $\sigma : G_0 \to \mathrm{Aut}(\mathfrak{g})$ (where $\mathrm{Aut}(\mathfrak{g})$ is the set of parity-preserving linear automorphisms of $\mathfrak{g}$) such that:

1. $\mathfrak{g}_0 \cong \mathrm{Lie}(G_0)$
2. $\forall x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}$:
$$\frac{d}{dt}\sigma(e^{tx})y\bigg|_{t=0} = [x, y]$$
3. The action of σ on $\mathfrak{g}_0$ is the adjoint representation of G_0 on $\mathrm{Lie}(G_0)$

Then $(G_0, \mathfrak{g})$ (also written $(G_0, \mathfrak{g}, \sigma)$) is called a super Harish-Chandra pair (SHCP).

Definition 10.2. Let $(G_0, \mathfrak{g}, \sigma)$ and $(H_0, \mathfrak{h}, \tau)$ be two SHCP. A morphism between them is a pair (ψ_0, ρ^ψ) such that $\psi_0 : G_0 \to H_0$ and $\rho^\psi : \mathfrak{g} \to \mathfrak{h}$ are compatible Lie group and Lie algebra homomorphisms respectively.

Remark 10.3. We associate, to each Lie supergroup $\mathcal{G}$, a Lie group $\tilde{G}$, called the reduced group and defined with the following morphisms:

$$|\mu| : \tilde{G} \times \tilde{G} \to \tilde{G} \qquad |i| : \tilde{G} \to \tilde{G} \qquad |e| : \mathbb{R}^0 \to \tilde{G}$$

where μ, i, and e are respectively the multiplication, inverse, and unit operation of $\mathcal{G}$.

Proposition 10.4. We define a functor

$$\mathscr{H} : \mathbf{SGrp} \to (shcp)$$

from the category of Lie supergroups to the category of super Harish-Chandra pairs. It can be shown that the category of Lie supergroups is equivalent to the category of super Harish-Chandra pairs.

Remark 10.5. Consequently, SHCP are an alternative approach to study Lie supergroups and their representations without relying on their sheaf structure.

11 Unitary Representations of Lie Supergroups

11.1 Unitary Representations

Unitary representations of Lie supergroups are formulated using super Harish-Chandra pairs. As mentioned previously, the category of Lie supergroups is naturally equivalent to the category of super Harish-Chandra pairs.

Definition 11.1. A representation of a group G on a vector space V is a map

$$\Phi : G \times V \to V$$

such that

1. $\forall g \in G$ the map $\psi(g) : V \to V$, $v \mapsto \Phi(g, v)$ is linear.
2. $\forall g, h \in G$ and $\forall v \in V$
   i) $\Phi(e, v) = v$ where e is the identity element of G;
   ii) $\Phi(g, \Phi(h, v)) = \Phi(gh, v)$ where gh is the product in G.

Definition 11.2. A linear representation of a group G on a vector space V is a map

$$\rho : G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the general linear group on V and such that $\forall g, h \in G$,

$$\rho(gh) = \rho(g)\rho(h).$$

Definition 11.3. A Hilbert space is a vector space H with an inner product and with the property that the space is complete when the norm is defined by the inner product. A super Hilbert space $\mathcal{H}$ will have a decomposition $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$.

Definition 11.4. A unitary representation of a Lie supergroup $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho, \mathcal{H})$ with the properties that:

1. π is an even unitary representation of G_0 in the super Hilbert space $\mathcal{H}$.
2. ρ is a linear map of $\mathfrak{g}_1$ into $C^\infty(\pi)$.
3. ρ satisfies the following properties:
   - i) $\rho(g_0 X) = \pi(g_0)\rho(X)\pi(g_0)^{-1}$ for $X \in \mathfrak{g}_1$ and $g_0 \in G_0$, that is, ρ is compatible with π.
   - ii) $\rho(X)$ with domain $C^\infty(\pi)$ is symmetric for all $X \in \mathfrak{g}_1$.
   - iii) $-i \cdot d\pi([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ for $X, Y \in \mathfrak{g}_1$ on $C^\infty(\pi)$, where $d\pi$ is the infinitesimal representation of π.

11.2 The Poincaré Group

Definition 11.5. The Minkowski spacetime is a combination of Euclidean space and time into a four-dimensional manifold. It is independent of the inertial frame of reference.

Definition 11.6. A Lorentz transformation is a coordinate transformation between two inertial frames of reference. The Lorentz group is the group of all Lorentz transformations of the Minkowski spacetime.

Definition 11.7. The Poincaré group is the semi-direct product of spacetime translations and the Lorentz group.

Remark 11.8. A classical semi-direct product is of the form

$$G_0 = T_0 \times' L_0$$

where T_0 is a vector space of finite dimension over $\mathbb{R}$ (*i.e.* the translation group) and L_0 is a closed unimodular subgroup of $GL(T_0)$ acting on T_0 naturally (*i.e.* the conformal group).

In this context, the Poincaré group can be expressed as

$$\mathrm{so}(3,1) \times' \mathbb{R}^{3|1}.$$

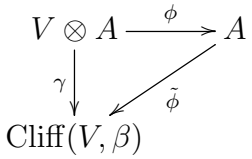
Remark 11.9. The Lie supergroup $(G_0, \mathfrak{g})$ is a super semidirect product if $G_0 = T_0 \times' L_0$ with the added properties that $L_0 \subset \mathrm{SL}(T_0)$ is a closed subgroup, that T_0 acts trivially on $\mathfrak{g}_1$ and that $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{t}_0 := \mathrm{Lie}(T_0)$.

12 Spinors and Clifford Algebras

Let V be a finite-dimensional complex vector space with a symmetric bilinear form β. We will define a Clifford algebra for the pair (V, β).

Definition 12.1. A Clifford algebra for the pair (V, β) is an associative algebra, denoted $\mathrm{Cliff}(V, \beta)$ with unit 1 over $\mathbb{C}$, together with a linear map $\gamma : V \to \mathrm{Cliff}(V, \beta)$ such that:

1. The anticommutator $\{\gamma(x), \gamma(y)\} = \beta(x, y)1$ for $x, y \in V$.
2. $\gamma(V)$ generates $\mathrm{Cliff}(V, B)$ as an algebra.
3. For any complex associative algebra A with unit 1 and a linear map $\psi : V \to A$ such that $\{\phi(x), \phi(y)\} = \beta(x, y)1$, there exist an associative algebra homomorphism $\tilde{\phi} : \mathrm{Cliff}(V, \beta) \to A$ such that $\phi = \tilde{\phi} \circ \gamma$:



Existence and uniqueness of the Clifford algebra $\mathrm{Cliff}(V, \beta)$ can also be shown, but this beyond the scope of this work.

Definition 12.2. A space of spinors for (V, β) is a pair (S, γ) where S is a complex vector space and $\gamma : V \to \mathrm{End}(S)$ is a linear map, such that:

1. The anticommutator $\{\gamma(x), \gamma(y)\} = \beta(x, y)I$, $\forall x, y \in V$
2. The only subspaces of S that are invariant under $\gamma(V)$ are 0 and S.

We note that since $\mathrm{Cliff}(V, \beta)$ is finite-dimensional, then so is the space of spinors for (V, β).

13 Iwasawa Decomposition of Lie Supergroups

Let $\mathcal{G}$ be a Lie supergroup, and $\mathfrak{g}$ its associated Lie superalgebra. The general linear Lie superalgebra $\mathfrak{gl}_{m|2n}(\mathbb{R})$ is the vector space of $(m+2n)\times(m+2n)$ block matrices of the form:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are respectively $(m\times m)$, $(m\times 2n)$, $(2n\times m)$, and $(2n\times 2n)$ matrices. The blocks A and D are even and the blocks B and C are odd. It is a Lie superalgebra with supercommutator:

$$[x,y] := xy - (-1)^{|x||y|}yx.$$

Definition 13.1. The orthosymplectic Lie superalgebra of dimension $m\times 2n$ is defined as follow:

$$\mathfrak{osp}_{m|2n}(\mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ -JB^t & D \end{bmatrix} : A = -A^t \text{ and } JD^tJ = D \right\}$$

where J is a $(2n\times 2n)$ real matrix of the form $J = \begin{bmatrix} J_2 & & 0 \\ & \ddots & \\ 0 & & J_2 \end{bmatrix}$ with $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Remark 13.2. The transformation θ defined below gives an explicit matrix realization of the orthosympletic Lie superalgebra. For A, B, C, and D as defined above, let θ be such that:

$$\theta \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left(\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & -J \end{bmatrix} \right)^{\text{st}} = \begin{bmatrix} -A^t & -C^tJ \\ -JB^t & JD^tJ \end{bmatrix}$$

where the supertranspose of a matrix M is denoted by M^{st}. We see that

$$-C^tJ = -(-JB^t)^tJ = BJ^tJ = BI = B$$

and we obtain the Lie superalgebra $\mathfrak{osp}_{m|2n}(\mathbb{R})$ given in Definition 13.1.

Remark 13.3. Recall that the special orthogonal Lie algebra of dimension m, $\mathfrak{so}_m(\mathbb{R})$ is defined as follows: $\mathfrak{so}_m(\mathbb{R}) = \{A \in \mathbb{M}_{m\times m} : A = -A^t\}$. Let

$$\mathfrak{p} = \{A \in \mathbb{M}_{m\times m} : A = A^t\}$$

be the set of symmetric matrices, then

$$\mathfrak{gl}_m(\mathbb{R}) \cong \mathfrak{so}_m(\mathbb{R}) \oplus \mathfrak{p}.$$

Similarly, the symplectic Lie algebra is $\mathfrak{sp}_{2n}(\mathbb{R}) = \{D : J_2D^t_{ji}J_2 = D_{ij}\}$ and we set

$$\mathfrak{p}' = \{D : J_2D^t_{ji}J_2 = -D_{ij}\}.$$

Therefore,

$$\mathfrak{gl}_{2n}(\mathbb{R}) = \mathfrak{sp}_{2n}(\mathbb{R}) \oplus \mathfrak{p}'.$$

Proposition 13.4. The general linear superalgebra $\mathfrak{gl}_{m|2n}(\mathbb{R})$ can be decomposed into the following direct sum:

$$\mathfrak{gl}_{m|2n}(\mathbb{R}) = \mathfrak{osp}_{m|2n}(\mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here,

$$\mathfrak{a} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{ccc:ccccc} b_1 & & & & & 0 & & \\ & \ddots & & & & & & \\ & & b_m & & & & & \\ \hdashline & & & a_1 & & & & \\ & & & & a_1 & & & \\ & 0 & & & & \ddots & & \\ & & & & & & a_n & \\ & & & & & & & a_n \end{array}\right]$$

where the blocks A, B, C, and D are of the dimensions given previously. Furthermore, B and C are zero matrices while A and D are diagonal matrices with the additional property that every diagonal entry of D is repeated twice. We also define $\mathfrak{n}$ as the set of upper diagonal matrices with zeros on the diagonal for the top left $m \times m$ block and 2×2 blocks of zeros for the lower right $2n \times 2n$ block:

$$\mathfrak{n} = \left[\begin{array}{ccc:ccccc} 0 & & & & & * & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ \hdashline & & & 0_{2\times 2} & & & & \\ & & & & & & & \\ & 0 & & & & \ddots & & \\ & & & & & & 0_{2\times 2} & \end{array}\right]$$

Remark 13.5. We set $\mathfrak{k} := \mathfrak{osp}_{m|2n}(\mathbb{R})$ and $\mathfrak{g} := \mathfrak{gl}_{m|2n}(\mathbb{R})$ and note that :

$$\mathfrak{g} = \mathfrak{gl}_{m|2n}(\mathbb{R}) \supset \mathfrak{k} = \mathfrak{osp}_{m|2n}(\mathbb{R})$$

Remark 13.6. The Iwasawa decomposition KAN of $\mathfrak{g}_0$ gives that $\mathfrak{g}_0 = \mathfrak{k}_0 \otimes \mathfrak{a}_0 \otimes \mathfrak{n}_0$. For the Lie groups G, K, A, and N generated by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$, the Iwasawa decomposition is $G = KAN$.

13.1 Superalgrebra Structure

Let U be an open set. We define

$$\mathfrak{F}(U) := \hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(U)).$$

For $S, T \in \mathfrak{F}(U)$, let us consider a mapping $S * T$ from the Cartesian product of $\hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(U))$ to itself that is defined in the most natural way:

$$S * T : \hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(U)) \times \hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(U)) \to \hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(U))$$

$$(S,T) \longmapsto m \circ S \otimes T \circ \Delta$$

Where $\Delta : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$, $x \mapsto x \otimes 1 + 1 \otimes x$, for $x \in \mathfrak{g}$ is the comultiplication of the univeral envelopping superalgebra $\mathfrak{U}(\mathfrak{g})$. Moreover, $S \otimes T$ is the tensor product of S and T:

$$(S \otimes T)(x_1 \otimes x_2) = (-1)^{|T||x_1|} S(x_1) \otimes T(x_2) \text{ for } x_1, x_2 \in \mathfrak{U}(\mathfrak{g})$$

and m is the point-wise multiplication :

$$m : C^\infty(U) \times C^\infty(U) \to C^\infty(U)$$

such that $m(f,g)(u) = f(u) \cdot g(u)$ for $f, g \in C^\infty(U)$ and $u \in U$.

Proposition 13.7. $\mathfrak{F}(U)$ has the structure of an associative superalgebra.

Proof. It is clearly a superalgebra. We shall simply show associativity, *i.e.* that $\forall\, T \in \mathfrak{F}(U)$,

$$(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3).$$

Let x be in $\mathfrak{g}$. On the right-hand side, we have:

$$\begin{aligned}
((T_1 * T_2) * T_3)(x) &= m \circ [m \circ (T_1 \otimes T_2) \circ \Delta] \otimes T_3 \circ \Delta x \\
&= m \circ [m \circ (T_1 \otimes T_2) \circ \Delta] \otimes T_3(x \otimes 1 + 1 \otimes x) \\
&= m \circ [(m \circ (T_1 \otimes T_2) \circ \Delta x) \otimes T_3 1 + (m \circ (T_1 \otimes T_2) \circ \Delta 1) \otimes T_3 x] \\
&= m \circ [(m \circ (T_1 x \otimes T_2 1) + m \circ (T_1 1 \otimes T_2 x)) \otimes T_3 1 + (m \circ (T_1 1 \otimes T_2 1)) \otimes T_3 x] \\
&= m \circ [m \circ (T_1 x \otimes 1) \otimes 1 + m \circ (1 \otimes T_2 x) \otimes 1 + (m \circ (1 \otimes 1)) \otimes T_3 x] \\
&= m \circ [T_1 x \otimes 1] + m \circ [T_2 x \otimes 1] + m \circ [1 \otimes T_3 x] \\
&= T_1 x + T_2 x + T_3 x
\end{aligned}$$

On the left-hand side:

$$\begin{aligned}
(T_1 * (T_2 * T_3))(x) &= m \circ [T_1 \otimes (m \circ (T_2 \otimes T_3) \circ \Delta] \circ \Delta x \\
&= m \circ [T_1 \otimes (m \circ (T_2 \otimes T_3) \circ \Delta](x \otimes 1 + 1 \otimes x) \\
&= m \circ [T_1 x \otimes (m \circ (T_2 \otimes T_3) \circ \Delta) 1 + T_1 1 \otimes (m \circ (T_2 \otimes T_3) \circ \Delta) x] \\
&= m \circ [T_1 x \otimes (m \circ (T_2 1 \otimes T_3 1)) + T_1 1 \otimes (m \circ (T_2 x \otimes T_3 1) + m \circ (T_2 1 \otimes T_3 x))] \\
&= m \circ [T_1 x \otimes (m \circ (1 \otimes 1)) + 1 \otimes (m \circ (T_2 x \otimes 1)) + 1 \otimes (m \circ (1 \otimes T_3 x))] \\
&= m \circ [T_1 x \otimes 1] + m \circ [1 \otimes T_2 x] + m \circ [1 \otimes T_3 x] \\
&= T_1 x + T_2 x + T_3 x
\end{aligned}$$

Thus $(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3)$ for all $x \in \mathfrak{g}$. This proof can easily be extended to the general case $x \in \mathfrak{U}(\mathfrak{g})$.

□

13.2 Quotient Supermanifold

Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie supergroups, and G and H the usual (even) Lie group. Recall that $G/H = \{gH : g \in G\}$. Define a mapping π, such that:

$$\pi : G \to G/H,\ g \mapsto gH.$$

We note that

$$\mathcal{G}/\mathcal{H} = (G/H, \mathcal{O}_{\mathcal{G}/\mathcal{H}}).$$

Definition 13.8. Let U be an open set in G/H. The sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U)$ is the set of mappings $T \in \hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(U)))$. These mappings are H invariant, that is:

1. $T(h^{-1}x)(gh) = T(x)(g)$ for $h \in \mathcal{H},\ x \in \mathfrak{U}(\mathfrak{g}),$ and $g \in \pi^{-1}(U)$.
2. $T(xy) = 0$ for $x \in \mathfrak{U}(\mathfrak{g})$ and $y \in \mathfrak{h}_1$.

Proposition 13.9. The sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}(U)$ is $\mathfrak{g}_0$ invariant.

Proof. It suffices to show that $S * T$ satisfies the two properties of Definition 13.8 for $S, T \in \hom_{\mathfrak{g}_0}(\mathfrak{U}(g), C^\infty(\pi^{-1}(U)))$. We assume $x \in \mathfrak{g}$ and that $g, h \in \pi^{-1}(U)$. Let us first show that $(S * T)(h^{-1}x)(gh) = (S * T)(x)(g)$. On the left-hand side we have:

$$\begin{aligned}
(S * T)(h^{-1}x)(gh) &= m \circ S \otimes T \circ \Delta[(h^{-1}x)(gh)] \\
&= m \circ (S \otimes T)[h^{-1}x \otimes 1 + 1 \otimes h^{-1}x](gh) \\
&= m \circ [S(h^{-1}x) \otimes T1 + S1 \otimes T(h^{-1}x)](gh) \\
&= S(h^{-1}x)(gh) + T(h^{-1}x)(gh) \\
&= S(x)(g) + T(x)(g)
\end{aligned}$$

On the right-hand side:

$$\begin{aligned}
(S * T)(x)(g) &= m \circ S \otimes T(x \otimes 1 + 1 \otimes x)g \\
&= S(x)(g) + T(x)(g)
\end{aligned}$$

Thus $(S * T)(h^{-1}x)(gh) = (S * T)(x)(g)$. Let us now show that $(S * Y)(xy) = 0$:

$$\begin{aligned}
(S * T)(xy) &= m \circ S \otimes T \circ \Delta(xy) \\
&= m \circ S \otimes T(xy \otimes 1 + 1 \otimes xy) \\
&= m \circ [S(xy) \otimes T1 + S1 \otimes T(xy)] \\
&= m \circ [0 \otimes 1 + 1 \otimes 0] \\
&= 0
\end{aligned}$$

This proof can easily be extended to the general case $x \in \mathfrak{U}(\mathfrak{g})$. □

13.3 Nearly Iwasawa Decomposition

Recall that $\mathfrak{gl}_{m|2n}(\mathbb{R}) = \mathfrak{osp}_{m|2n}(\mathbb{R}) \otimes \mathfrak{a} \otimes \mathfrak{n}$, and that $\mathfrak{k} := \mathfrak{osp}_{m|2n}(\mathbb{R})$. If we consider the Lie groups associated with these Lie algebras, we have:

$$\mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_{2n}(\mathbb{R}) \supset (\mathrm{O}_m(\mathbb{R}) \times \mathrm{Sp}_{2n}(\mathbb{R})) \cdot (A_m \times A_{2n}) \cdot (N_{GL_m}(\mathbb{R}) \times N_{GL_{2n}}(\mathbb{R}))$$

where $\mathrm{O}_m(\mathbb{R})$ is the orthogonal group of dimension m, and:

$$A_m = \exp(\mathfrak{a}_m), \qquad A_{2n} = \exp(\mathfrak{a}_{2n}).$$

With $\mathfrak{a}$ taking the form:

$$\mathfrak{a} \cong \left[\begin{array}{c|c} \mathfrak{a}_m & 0 \\ \hline 0 & \mathfrak{a}_{2n} \end{array}\right]$$

The groups $N(\mathbb{R})$ are defined similarly.

We take the quotient of the Lie supergroup $\mathcal{G}$ by $\mathcal{K}$, that is:

$$\mathcal{G}/\mathcal{K} = (GL_m(\mathbb{R}) \times GL_{2n}(\mathbb{R}))/(O_m(\mathbb{R}) \times \mathrm{Sp}_{2n}(\mathbb{R})).$$

We also define an injective mapping $\sigma : N \times A \hookrightarrow \mathcal{G}$, $\sigma(n, a) \mapsto na$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G} & \xleftarrow{\sigma} & N \times A \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ \mathcal{G}/\mathcal{K} & & \end{array}$$

Definition 13.10. Let W be a subgroup of $\mathcal{G}/\mathcal{K}$, and consider the mapping that takes T in the sheaf

$$\hom_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(W)))$$

to an element $\tilde{T}$ in the sheaf

$$\hom_{\mathfrak{n}_0 \otimes \mathfrak{a}}(\mathfrak{U}(\mathfrak{n}) \otimes \mathfrak{U}(\mathfrak{a}), C^\infty(\sigma^{-1}\pi^{-1}(W))).$$

We note that $\pi^{-1}(W) \subset \mathcal{G}$ and that $\sigma^{-1}\pi^{-1}(W) \subset N \times A$. For $x_N \in \mathfrak{U}(\mathfrak{n})$, $x_A \in \mathfrak{U}(\mathfrak{a})$, and $(\tilde{n}, \tilde{a}) \in \sigma^{-1}\pi^{-1}(W)$, $\tilde{T}$ is defined such that :

$$\tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a}) = T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a}).$$

Remark 13.11. Since $W \subset \mathcal{G}/\mathcal{K}$ and $T : \mathfrak{U}(\mathfrak{g}) \to C^\infty(\pi^{-1}(W))$ then T is $\mathcal{K}$ invariant, so that $\forall k \in K$ and $\forall x_K \in \mathfrak{U}(\mathfrak{k})$ the following two properties hold:

1. $T(x)(nak) = T(kx)(na)$
2. $T(x_N x_A x_K)(na) = 0$

Definition 13.12. Let G be a group and X and Y two sets on which G can act by a left G-action. A map $\psi : X \to Y$ is said to be G-equivariant if $\psi(g \cdot x) = g \cdot \psi(x)$ for all $g \in G$ and all $x \in X$.

We now want to show that $\tilde{T}$ defined in terms of T is $\mathfrak{n}_0 \otimes \mathfrak{a}$ equivariant and that T defined in terms of $\tilde{T}$ is $\mathfrak{g}_0$ equivariant.

Proposition 13.13. Let $\tilde{T}$ be in $\hom(\mathfrak{U}(\mathfrak{n}) \otimes \mathfrak{U}(\mathfrak{a}), C^\infty(\sigma^{-1}\pi^{-1}(W)))$ and defined such that $\tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a}) = T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a})$, then $\tilde{T}$ is $\mathfrak{n}_0 \otimes \mathfrak{a}$ equivariant.

Proof. Let x_0 be an element of $\mathfrak{n}_0$ or of $\mathfrak{a}_0 = \mathfrak{a}$. We want to show that

$$\tilde{T}(x_0 x_N \otimes x_A)(\tilde{n}, \tilde{a}) = x_0 \cdot \tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a}).$$

On the left-hand side, using the assumption that T is $\mathfrak{g}_0$ equivariant, we have

$$\begin{aligned} \tilde{T}(x_0 x_N \otimes x_A)(\tilde{n}, \tilde{a}) &= T(\tilde{a}^{-1} \cdot (x_0 x_N) x_A)(\tilde{n}\tilde{a}) \\ &= T(\tilde{a}^{-1} \cdot x_0 \cdot \tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a}) \\ &= (\tilde{a}^{-1} \cdot x_0) \cdot T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a}) \end{aligned}$$

On the right-hand side, we take the derivative with respect to x_0:

$$\begin{aligned} x_0 \cdot \tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a}) &= \lim_{s \to 0} \frac{1}{s} (\tilde{T}(x_N \otimes x_A)(\tilde{n} e^{s x_0}, \tilde{a}) - \tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a})) \\ &= \lim_{s \to 0} \frac{1}{s} (T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n} e^{s x_0} \tilde{a}) - T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a})) \\ &= \lim_{s \to 0} \frac{1}{s} (T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a} e^{s(\tilde{a}^{-1} \cdot x_0)}) - T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a})) \\ &= (\tilde{a}^{-1} \cdot x_0) \cdot T(\tilde{a}^{-1} \cdot x_N x_A)(\tilde{n}\tilde{a}) \end{aligned}$$

Therefore we have $\tilde{T}(x_0 x_N \otimes x_A)(\tilde{n}, \tilde{a}) = x_0 \cdot \tilde{T}(x_N \otimes x_A)(\tilde{n}, \tilde{a})$ for any $x_0 \in \mathfrak{n}_0$ or $x_0 \in \mathfrak{a}$ and we conclude that $\tilde{T}$ is $\mathfrak{n}_0 \otimes \mathfrak{a}$ equivariant.

□

Proposition 13.14. Let T be in $\hom(\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(W)))$ be $\mathcal{K}$ invariant that is, $\forall k \in K$ and $\forall x_K \in \mathfrak{U}(\mathfrak{k})$ we have:

1. $T(x)(nak) = T(kx)(na)$
2. $T(x_N x_A x_K)(na) = 0$

We define T such that for $na \in \pi^{-1}$, $T(x_N x_A)(na) = \tilde{T}(a x_N \otimes x_A)(n, a)$. Then T is $\mathfrak{g}_0$ equivariant.

Proof. By the Iwasawa decomposition we know that $\mathfrak{g}_0 \cong \mathfrak{k}_0 \otimes \mathfrak{a}_0 \otimes \mathfrak{n}_0$. Therefore, it suffices to show the equivariance for $\mathfrak{k}_0$, $\mathfrak{a}_0$ and $\mathfrak{n}_0$ separately. We want to show that

$$T(x_0 x_N x_A)(na) = x_0 \cdot T(x_N x_A)(na).$$

Case 1: Suppose that $x_0 \in \mathfrak{k}_0$. On the left-hand side we have

$$T(x_0 x_N x_A)(na) = T([x_0, x_N x_A])(na).$$

This occurs since by the $\mathcal{K}$ invariance of T, the term $T(x_N x_A x_0) = 0$. On the right-hand side, by taking the derivative with respect to x_0 and using the $\mathcal{K}$ invariance, we get:

$$\begin{aligned} x_0 \cdot T(x_N x_A)(na) &= \lim_{s\to 0} \frac{1}{s} \left(T(x_N x_A)(nae^{sx_0}) - T(x_N x_A)(na)\right) \\ &= \lim_{s\to 0} \frac{1}{s} (T(e^{sx_0}(x_N x_A))(na) - T(x_N x_A)(na)) \\ &= T([x_0, x_N x_A])(n, a) \end{aligned}$$

Note that the last step utilizes the chain rule of derivation:

$$\begin{aligned} \lim_{s\to 0} \frac{1}{s} (T(e^{sx_0}(x_N x_A))(na) - T(x_N x_A)(na)) &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{s=0} \frac{T(e^{sx_0} x_N x_A e^{-sx_0} - x_N x_A)}{s} \\ &= T(x_0 x_N x_A + x_N x_A(-x_0)) \\ &= T([x_0, x_N x_A]) \end{aligned}$$

Therefore, we have $\mathfrak{k}_0$ equivariance.

Case 2: Suppose that $x_0 \in \mathfrak{n}_0$. By using the definition of T in terms of $\tilde{T}$ and by assuming that $\tilde{T}$ is $\mathfrak{n}_0 \otimes \mathfrak{a}_0$ equivariant, we get:

$$\begin{aligned} T(x_0 x_N x_A)(na) &= \tilde{T}(a \cdot (x_0 x_N) \otimes x_A)(n, a) \\ &= (a \cdot x_0) \cdot \tilde{T}(a \cdot x_N \otimes x_A)(n, a) \\ &= \lim_{s\to 0} \frac{1}{s} (\tilde{T}(a \cdot x_N \otimes x_A)(ne^{sa\cdot x_0}, a) - \tilde{T}(a \cdot x_N \otimes x_A)(n, a)) \\ &= \lim_{s\to 0} \frac{1}{s} (T(x_N x_A)(nae^{sx_0}) - T(x_N x_A)(na)) \\ &= x_0 \cdot T(x_N x_A)(na) \end{aligned}$$

Therefore, we have $\mathfrak{n}_0$ equivariance.

Case 3: Suppose that $x_0 \in \mathfrak{a}_0 = \mathfrak{a}$. On the left-hand side, since x_0 is even, we use the definition of T to obtain:

$$\begin{aligned} T(x_0 x_N x_A)(na) &= T([x_0, x_N] x_A + x_N x_0 x_A)(na) \\ &= \tilde{T}(a \cdot [x_0, x_N] \otimes x_A) + \tilde{T}(a \cdot x_N \otimes x_0 x_A)(n, a) \end{aligned}$$

On the right-hand side, we take the derivative with respect to x_0 and use the chain rule to obtain:

$$\begin{aligned} x_0 \cdot T(x_N x_A)(na) &= \lim_{s\to 0} \frac{1}{s} (T(x_N x_A)(nae^{sx_0}) - T(x_N x_A)(na)) \\ &= \lim_{s\to 0} \frac{1}{s} (\tilde{T}(ae^{sx_0} x_N \otimes x_A)(n, ae^{sx_0}) - \tilde{T}(a x_N \otimes x_A)(n, a)) \\ &= \tilde{T}(a \cdot [x_0, x_N] \otimes x_A) + x_0 \cdot \tilde{T}(a \cdot x_N \otimes x_A)(n, a) \\ &= \tilde{T}(a \cdot [x_0, x_N] \otimes x_A) + \tilde{T}(a \cdot x_N \otimes x_0 x_A)(n, a) \end{aligned}$$

We have shown that $T(x_0 x_N x_A)(na) = x_0 \cdot T(x_N x_A)(na)$ for $x_0 \in \mathfrak{g}_0$ and thus that T is $\mathfrak{g}_0$ equivariant for T defined on $NA \subset G$. We now want to show this for T defined on $NAK = G$. That is, we want to show that for any $x \in \mathfrak{U}(\mathfrak{g})$,

$$T(x_0 x)(nak) = x_0 \cdot T(x)(nak).$$

Since we assume $\mathcal{K}$ invariance we have that $T(x_0 x)(nak) = T(k \cdot (x_0 x))(na)$. There are then three cases, depending whether kx_0 is in $\mathfrak{k}_0$, $\mathfrak{a}_0$, or $\mathfrak{n}_0$.

Case 1: Suppose that $kx_0 \in \mathfrak{k}_0$ (or simply that $x_0 \mathfrak{k}_0$), then using the $\mathfrak{k}_0$ equivariance shown previously the left-hand side becomes:

$$\begin{aligned} T(x_0 x)(nak) &= T(kx_0 kx)(na) \\ &= kx_0 \cdot T(kx)(na) \\ &= \lim_{s \to 0} \frac{1}{s} (T(kx)(nae^{skx_0}) - T(kx)(na)) \\ &= \lim_{s \to 0} \frac{1}{s} (T(e^{skx_0} kx)(na) - T(kx)(na)) \\ &= T(k \cdot [x_0, x])(na) \end{aligned}$$

On the right-hand side, we take the derivative with respect to x_0:

$$\begin{aligned} x_0 \cdot T(x)(nak) &= \lim_{s \to 0} \frac{1}{s} (T(x)(nake^{sx_0}) - T(x)(nak)) \\ &= \lim_{s \to 0} \frac{1}{s} (T(ke^{sx_0} x)(na) - T(kx)(na)) \\ &= T(k \cdot [x_0, x])(na) \end{aligned}$$

Thus $T(x_0 x)(nak) = x_0 \cdot T(x)(nak)$, that is T is x_0-equivariant $\forall x_0$ such that $kx_0 \in \mathfrak{k}_0$.

Case 2: Suppose that $kx_0 \in \mathfrak{a}$. On the left-hand side, using the $\mathfrak{a}$ equivariance shown previously, we get:

$$\begin{aligned} T(x_0 x)(nak) &= kx_0 \cdot T(kx)(na) \\ &= \lim_{s \to 0} \frac{1}{s} (T(kx)(nae^{skx_0}) - T(kx)(na)) \end{aligned}$$

On the right-hand side, we take the derivative with respect to x_0:

$$\begin{aligned} x_0 \cdot T(x)(nak) &= \lim_{s \to 0} \frac{1}{s} (T(x)(nake^{sx_0}) - T(x)(nak)) \\ &= \lim_{s \to 0} \frac{1}{s} (T(x)(nae^{skx_0} k) - T(x)(nak)) \\ &= \lim_{s \to 0} \frac{1}{s} (T(kx)(nae^{skx_0}) - T(kx)(na)) \end{aligned}$$

Thus $T(x_0 x)(nak) = x_0 \cdot T(x)(nak)$, that is T is x_0-equivariant $\forall x_0$ such that $kx_0 \in \mathfrak{a}_0 = \mathfrak{a}$.

Case 3: Suppose that $kx_0 \in \mathfrak{n}_0$. On the left-hand side, using the $\mathfrak{n}_0$ equivariance shown previously, we get:

$$\begin{aligned} T(x_0x)(nak) &= kx_0 \cdot T(kx)(na) \\ &= \lim_{s\to 0} \frac{1}{s}(T(kx)(nae^{skx_0}) - T(kx)(na)) \\ &= \lim_{s\to 0} \frac{1}{s}(T(kx)(ne^{sakx_0}a) - T(kx)(na)) \end{aligned}$$

On the right-hand side, we take the derivative with respect to x_0:

$$\begin{aligned} x_0 \cdot T(x)(nak) &= \lim_{s\to 0} \frac{1}{s}(T(x)(nake^{sx_0}) - T(x)(nak)) \\ &= \lim_{s\to 0} \frac{1}{s}(T(x)(ne^{sakx_0}ak) - T(x)(nak)) \\ &= \lim_{s\to 0} \frac{1}{s}(T(kx)(ne^{sakx_0}a) - T(kx)(na)) \end{aligned}$$

Thus $T(x_0x)(nak) = x_0 \cdot T(x)(nak)$, that is T is x_0-equivariant $\forall x_0$ such that $kx_0 \in \mathfrak{n}_0$.

We conclude that T is $\mathfrak{g}_0$ equivariant.

□

Remark 13.15. The compatibility of the constraints given in Proposition 13.14 follows from the PBW Theorem and the Iwasawa decomposition.

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