Introduction and Background

A representation on a 1-dimensional vector space canonically corresponds to a character of G, that is, a group homomorphism $\chi : G \longrightarrow \mathbb{C}^*$. From now on we assume that G is a finite abelian group. The dual group $\widehat{G}$ can then be defined as the set of characters $\chi : G \longrightarrow \mathbb{C}^*$ such that $\chi(ab) = \chi(a)\chi(b)$ for all $g \in G$.

Let $f : G \longrightarrow \mathbb{C}^*$ be an arbitrary function. Then its Fourier transform $\widehat{f} : \widehat{G} \longrightarrow \mathbb{C}^*$ is given by

$$\widehat{f}(\chi) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\chi(x)} \quad \forall x \in \widehat{G}.$$

The classical uncertainty principle for finite abelian groups states that for a non-zero function $f : G \longrightarrow \mathbb{C}$ the support of f, i.e $\text{supp}(f) = \{x \in G \| \ f(x) \neq 0\}$ is related to the support of its transform by the formula

$$|\text{supp}(f)||\text{supp}(\widehat{f}| \geqslant |G|.$$

Tao [1] showed that when G is the cyclic group $\mathbb{Z}/p\mathbb{Z}$ of prime order p this can be improved to

$$|\text{supp}(f)| + |\text{supp}(\widehat{f}| \geqslant p + 1.$$

Furthermore, Murty and Whang [2] showed that if G is the cyclic group $\mathbb{Z}/m\mathbb{Z}$ where $m > 1$ is an integer, then

$$|\text{supp}(f)| + |\text{supp}(\widehat{f}| \geqslant m + 1$$

whenever $P(\text{supp}(f))$ or $P(\text{supp}(\widehat{f}))$ where $P(X)$ for $X \subseteq \mathbb{Z}/m\mathbb{Z}$ denotes the property that $X = \{k_1, \ldots, k_n\}$ satisfying

$$\frac{\prod_{1 \leqslant i < j \leqslant n} |k_i - k_j|}{\prod_{1 \leqslant i < j \leqslant n} (i - j)} \in \mathbb{N}p_1 + \ldots + \mathbb{N}p_r$$

where $p_1, \ldots, p_r$ are the prime factors of m.

Proposition. *Let $\tilde{A} = \{\tilde{g_1}, \ldots, \tilde{g_k}\} \subseteq \{g_1, ..., g_{|G|}\}$ then if for every minor of the matrix of character values, i.e $\left(\chi_i(g_j)\right)_{1 \leqslant i,j \leqslant |G|}$, with columns in the collection $\tilde{A}$ has a non-zero determinant then for some function*

$$|\text{supp}(f)| + |\text{supp}(\widehat{f}| \geqslant |G| + 1$$

.

Proof. The linear transformation defined by $Tf = \widehat{f}|_{\tilde{A}}$ is invertible since it corresponds to the product of the non-zero minor in question and the values of the function. Suppose now that the statement is false. Then you can find a non-zero function $f : G \longrightarrow \mathbb{C}$ such that $\tilde{A} \cap \text{supp}(\widehat{f}) = \varnothing$ and $|\tilde{A}| = |\text{supp}(f)|$. Then $Tf = 0$ but $f \neq 0$. □

In the case considered in [1], Chebotarev's theorem guarantees the invertibility of these minors (Later on, the equivalence of these statement is proved) and the result in [2] is based on invertibility of certain minors.

Theorem 1. *Let p be a prime number. If $f : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}$ is a non-zero function, then*

$$|\operatorname{supp}(f)| + |\operatorname{supp}(\widehat{f})| \geqslant p + 1$$

Theorem 2. *All the minors of the exponential basis matrix $\left(e^{2\pi i x\xi/p}\right)_{1\leqslant x,\xi\leqslant p}$ where p is a prime number are non-zero.*

Proposition 1. *Theorem 1 $\Rightarrow$ Theorem 2.*

Proof. Consider the smaller square matrix formed by rows $i_1, i_2, ..., i_n$ and columns $j_1, j_2, ..., j_n$ of the exponential basis matrix, where $1 \leqslant i_k, j_k \leqslant p$ for $1 \leqslant k \leqslant n$. Now let us suppose that its determinant is zero. Then let $f : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}$ be a function such that, for $x = j_k$, the values of f represent a non-trivial solution to the system of linear equations represented by the smaller square matrix and f vanishes everywhere else. By definition we have

$$\widehat{f}(\xi) = \frac{1}{p} \sum_{1\leqslant x\leqslant p} e^{2\pi i x\xi/p} f(x)$$

then by construction of f it follows that $\widehat{f}$ vanishes whenever $\xi = i_k$. Therefore $|\operatorname{supp}(\widehat{f})| \leqslant p - n$, and the latter implies that

$$|\operatorname{supp}(f)| + |\operatorname{supp}(\widehat{f})| \leqslant n + (p - n) = p$$

which contradicts Theorem 1. □

This proof can be illustrated by the following where the exponential basis matrix is represented as the matrix with a_{ij} entries.

$$\begin{bmatrix} \widehat{f}(1) \\ \widehat{f}(2) \\ \vdots \\ \vdots \\ \widehat{f}(p) \end{bmatrix} = \frac{1}{p} \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1p} \\ a_{21} & a_{i_1j_1} & \cdots & a_{i_1j_n} & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & a_{i_nj_1} & \cdots & a_{i_nj_n} & \vdots \\ a_{p1} & \cdots & \cdots & \cdots & a_{pp} \end{bmatrix} \begin{bmatrix} 0 \\ f(j_i) \\ \vdots \\ f(j_n) \\ \vdots \\ 0 \end{bmatrix}$$

Proposition 1. *Theorem 2 $\Rightarrow$ Theorem 1.*

Proof. Consider the linear transformation $T : l^2(A) \to l^2(\tilde{A})$, where $A, \tilde{A}$ are non-empty subsets $\mathbb{Z}/p\mathbb{Z}$ such that $|A| = |\tilde{A}|$, defined by its coefficient matrix $\frac{1}{p}\left(e^{2\pi i x\xi/p}\right)_{1\leqslant x,\xi\leqslant p}$, i.e $Tf = \widehat{f}|_{\tilde{A}}$, which is invertible by assumption.

Let us suppose by contradiction that we have a non-zero function f such that $|\operatorname{supp}(f)|+|\operatorname{supp}(\widehat{f})| \leqslant p$. Then if we write $A := \operatorname{supp}(f)$, then we can find a set $\tilde{A}$ in $\mathbb{Z}/p\mathbb{Z}$ such that $\tilde{A} \cap \operatorname{supp}(\widehat{f}) = \varnothing$ and $|\tilde{A}| = |A|$. We obtain a contradiction since $Tf = 0$ but $f \neq 0$. □

Moreover, it is important to note that in [2] paper the authors use representation theory of the unitary group and Weyl's character formula to derive their result. To start off our investigation we look if a similar result can be achieved when G is $\mathbb{Z}_p \oplus \mathbb{Z}_p$ where p is prime.

lemma 1. *Let G_1 and G_2 be two finite abelian group and $\chi_1 : G_1 \longrightarrow \mathbb{C}^*$ and $\chi_2 : G_2 \longrightarrow \mathbb{C}^*$ be characters of these groups respectively. Then the function $\chi : G_1 \times G_2 \longrightarrow \mathbb{C}^*$ where $(g_1, g_2) \longmapsto \chi_1(g_1)\chi_2(g_2)$ is a character.*

Proof. Let (g_1, g_2), $(\tilde{g_1}, \tilde{g_2}) \in G_1 \times G_2$. Then $\chi\big((g_1, g_2)(\tilde{g_1}, \tilde{g_2})\big) = \chi(g_1\tilde{g_1}, g_2\tilde{g_2})$ $= \chi_1(g_1\tilde{g_1})\chi_2(g_2, \tilde{g_2}) = \chi_1(g_1)\chi_1(\tilde{g_1})\chi_2(g_1)\chi_2(\tilde{g_2})$, since χ_1 and χ_2 are characters, $= \chi_1(g_1)\chi_2(g_2)\chi_1(\tilde{g_1})\chi_2(\tilde{g_2}) = \chi(g_1, g_2)\chi(\tilde{g_1}, \tilde{g_2})$. □

lemma 2. *Every character of $G_1 \times G_2$ is of the previous form.*

Proof. Define $\chi_1 = \chi|_{G_1 \times \{e\}}$ and $\chi_2 = \chi|_{\{e\} \times G_2}$, where e is the identity element, then $\chi(g_1, g_2) = \chi(g_1)\chi(g_2) = \chi(g_1)\chi(e)^2\chi(g_2) = \chi(g_1, e)\chi(e, g_2)$
$= \chi_1(g_1)\chi_2(g_2)$. □

Defining the matrix of character value in our case is then clearly $A = \big(\omega^{a_ik_j + b_il_j}\big)_{1 \leqslant i,j \leqslant n}$ where ω is the pth root of unity and(a_i, b_i) and (k_j, l_j) be distinct elements of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for $1 \leqslant i, j \leqslant n$ where $n \leqslant p$.

1 Special Cases Study

Now let us consider cases when A has a non-vanishing determinant.

Case 1: If $b_1 = b_2 = \ldots = b_n = c$ then the determinant of A can be written as

$$\prod_{1 \leqslant j \leqslant n} \omega^{cl_j} \det\big(\omega^{a_ik_j}\big)_{1 \leqslant i,j \leqslant n}.$$

the claim follows since the determinant of $\big(\omega^{a_ik_j}\big)_{1 \leqslant i,j \leqslant n}$ is non-zero by Theorem 2. Similarly, we obtain the same conclusion if the b_i's are replaced by the a_i's, k_j's or the l_j's.

In the first case,there is $\binom{p}{n}$ ways to choose the (a_i, b_i) since the b_i's are constant and $\binom{p}{n}^2$ ways to choose the (k_j, l_j) since they are arbitrary. Therefore the number of ways this case can happen is the following,

$$4\binom{p}{n}^3.$$

Case 2: For every $0 \leqslant i, j \leqslant n$, $a_i k_j = \gamma b_i l_j$, where $\gamma \in \mathbb{N}^* \setminus \{kp-1 \| \ k \in \mathbb{N}\}$, let $\xi := \omega^{1+\gamma}$ which is also a pth root of unity. Then $\det(A) = \det\left(\xi^{a_i k_j}\right)_{0\leqslant i,j\leqslant n}$ is non-zero by Tao's result.

Let us start by considering case 2 when $\gamma = 1$ and when an ordered pair cannot have only one element that is zero. A consequence of the condition imposed is that the ratio between the elements of each ordered pair has to be constant, e.g for a fixed i then for every $1 \leqslant j \leqslant n$, $\kappa := a_i/b_i = l_j/k_j$ is constant and has $(p-1)$ possible values since it's a non-zero element of $\mathbb{Z}p/\mathbb{Z}$. Moreover, if we assume for one of the ordered pair that there's s of them that are non-zero then there's $\binom{p-1}{s}$ ways to choose it. The following table displays the four sub-cases for s for the ordered pairs,

$(\boldsymbol{a_i}, \boldsymbol{b_i})$	$(\boldsymbol{k_j}, \boldsymbol{l_j})$
n	n
$n-1$	n
n	$n-1$
$n-1$	$n-1$

Then the total number of possibilities is

$$\sum_{0\leqslant i,j\leqslant 1} (p-1)\binom{p-1}{n-i}\binom{p-1}{n-j}.$$

Now if we consider the case when the ordered pairs have at least one element which is zero then κ can only take the value zero and similarly the total number of possibilities is

$$\sum_{0\leqslant i,j\leqslant 1} \binom{p-1}{n-i}\binom{p-1}{n-j}.$$

Finally, there's $(p-1)$ possibilities for γ since it's a non-zero element of $\mathbb{Z}p/\mathbb{Z}$. Therefore, the total number of possibilities for case 2 is

$$\sum_{0\leqslant i,j\leqslant 1} p(p-1)\binom{p-1}{n-i}\binom{p-1}{n-j}.$$

Case 3: For every $0 \leqslant i, j \leqslant n$, $a_i = \gamma b_i$, where $\gamma \in \mathbb{N}^*$ and $m_j := \gamma k_j + l_j$ is a weight of distinct elements of $\mathbb{Z}/p\mathbb{Z}$ then $A = \left(\omega^{a_i m_j}\right)_{0\leqslant i,j\leqslant n}$ which is non-zero by Theorem 2. Similarly if for every $0 \leqslant j \leqslant n$, $k_j = \gamma l_j$ where $\gamma \in \mathbb{N}$ and $n_j := a_i + b_i$ is a weight of distinct elements of $\mathbb{Z}/p\mathbb{Z}$.

Let us consider the sub-case when $\gamma = 1$. There is clearly $\binom{p}{n}$ choices for (a_i, b_i) since picking a_i determines b_i. Moreover, since (l_j, k_j) and m_j have to be distinct we can consider the couple $(l_j, \gamma k_j + l_j)$ since if the second entry is a distinct weight than the two previous objects are also. For that one there is $\binom{p}{n}$ ways to choose it and because there's no restrictions on the first entry there's p^n ways to choose it. Finally, there's $(p-1)$ possibilities for γ since it's a non-zero

element of $\mathbb{Z}p/\mathbb{Z}$. Piecing everything together we obtain the following number of possibilities for Case 3,

$$2(p-1)p^n\binom{p}{n}^2$$

Comment 1: $a_ik_j+b_il_j = (a_i \mp \gamma b_i)k_j + b_i(\gamma k_j \pm l_j) = (a_i \mp \gamma l_j)k_j + (b_i \pm \gamma k_j)l_j$, where $\gamma \in \mathbb{N}$, then the above cases can yield the same result if we replace a_i with $(a_i \mp \gamma b_i)$ and l_j with $(\gamma k_j \pm l_j)$ or if we replace a_i with $(a_i \mp \gamma l_j)$ and b_i with $(b_i \pm \gamma k_j)$.

Comment 2: If for every $0 \leqslant i, j \leqslant n$, both i or both j dependent element is equal to p then A is a matrix of all 1's and in particular has zero determinant. This shows that not all of the submatrices of A have non-zero determinant.

2 Representation of the Symmetric Group

A partition of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_l$ and $\lambda_1 + \ldots + \lambda_l = n$.

If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n, then the Young diagram of λ consists of n boxes placed into l rows where the ith row has λ_i boxes. This definition is best illustrated with an example. If $\lambda = (3, 1)$, then the Young diagram is as follows.



A Young tableau of shape λ or a λ-tableau on the other hand is an array t of integers by placing $1, \ldots, n$ into the boxes of the Young diagram for λ. The concept is again best illustrated with an example. Suppose that $\lambda = (3, 2, 1)$. Then some λ-tableaux are as follows.

1	2	3
4	5	
6		

3	2	1
6	5	
4		

3	2	6
1	5	
4		

The group S_n acts transitively on the set of λ-tableau by applying $\sigma \in S_n$ to the entries of the boxes. The result of applying $\sigma \in S_n$ to a Young tableau t is denoted σt. For example, if t is the following

1	3	4
2		

and $\sigma = (1\,3\,2)$, then σt is the following

3	2	4
1		

The column stablizer of a Young tableaux t is the subgroup of S_n preserving the columns of t. It is denoted C_t. Now let us define an equivalence relation $\sim$ on the set of λ-tableaux by putting two tableaux to be equivalent if they have the same entries in each row.

A $\sim$-equivalence class of λ-tableaux is called a λ-tabloid or a tabloid of shape λ. The tabloid of a tableaux t is denoted $[t]$. The set of all tabloids of shape λ is denoted T^λ. There is a well-defined action of S_n on T^λ given by putting $\sigma[t] = [\sigma t]$ for t a λ-tableaux and $\sigma \in S_n$.

Let λ be a partition of n and t be a λ-tableau then define the polytabloid associated to t as the following

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi[t]$$

Let S^λ to be the subspace of $M^\lambda = \mathbb{C}T^\lambda$ spanned by the polytabloids e_t and $\psi^\lambda : S_n \longrightarrow GL(S^\lambda)$ be the corresponding subrepresentation. These representation called the Specht representations form a complete set of irreducible representations of S_n.

References

[1] Terence Tao *An Uncertainty Principle for Cyclic Groups of Prime Order.* 1991.

[2] M. Ram Murty, Junho Peter Whang *The Uncertainty Principle and a Generalization of a Theorem of Tao.* 2012.

[3] Benjamin Steinberg *Representation Theory of Finite Groups, an Introductory Approach*,Springer-Verlag, 2012.