Representations of the Lie Algebra $\mathfrak{sl}(2,\mathbb{C})$ Over Polynomial Spaces

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Introduction:

Let $\mathfrak{sl}(2,\mathbb{C})$ denote the Lie algebra of 2×2 complex matrices with zero trace. Let $\{e^+, e^-, h\}$ be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$ defined as follows:

$$e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have that $\mathfrak{sl}(2,\mathbb{C})$ is defined as a Lie algebra with the following commutator relations:

$$[e^+, e^-] = h \qquad [h, e^+] = 2e^+ \qquad [h, e^-] = -2e^-$$

Let $\mathfrak{gl}(V)$ denote the space of linear transfromations from a vector space V to itself. In general, we define a representation to be a linear map, say ϕ, such that $\phi : L \to \mathfrak{gl}(V)$, and such that ϕ preserves the commutator relation, that is, $\phi([x,y]) = [\phi(x), \phi(y)]$ $\forall x, y \in L$. Such representations are said to be L-modules on V. Furthermore we say that any subspace of a module that is itself a module is a submodule of the respective module. We say that a module is irreducible if it contains no non-zero proper submodules. For general background information on Lie Algebras the textbook (Erdmann and Wildon,*Introduction to Lie Algebras*; Verlag, 2006) [2] was consulted.

This paper is comprised of two sections with the first section presenting a decomposition of $\mathbb{C}[x]$ into a direct sum of irreducible submodules given by a certain representation w. The second section extends the representation w via tensor products into a representation of $\mathbb{C}[x,y]$ and a decomposition of the module $\mathbb{C}[x,y]$ into a direct sum of irreducible submodules is demonstrated. The following proposition will be very useful for the purposes of this paper.

Proposition 1: A submodule V over a Lie algebra L is irreducible if $\forall v \in V$ $v \neq 0$, the submodule generated by v over the action of L is V. (This is equivalent to the condition that $\forall$ $x, y \in V$ $x \neq 0$ $y \neq 0$, linear combinations of L-actions on x can send x to y)

Proof: Suppose V is an L-module such that any element of V can be sent to any other element of V via linear combinations of L-actions. Suppose that V contains a non-zero proper submodule, say X. Let x be any non-zero element of X. We have that x can be sent to any element of V via combinations of L-actions, therefore the module X must encompass all of V. Therefore V contains no proper non-zero submodules. Therefore V is irreducible.

Section 1: A specific $\mathfrak{sl}(2,\mathbb{C})$-module on $\mathbb{C}[x]$

We define a specific $\mathfrak{sl}(2,\mathbb{C})$-module on $\mathbb{C}[x]$ as follows:

$$\begin{aligned} &w : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[x]) \\ &w(h) = x\frac{d}{dx} + \frac{1}{2} \\ &w(e^+) = \frac{i}{2}x^2 \\ &w(e^-) = \frac{i}{2}\frac{d^2}{dx^2} \end{aligned}$$

Proposition 2: w defines a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}[x]$.

Proof: Since w is linear by construction, we need only show that w preserves the commutator relations.

Let $\{1, x, x^2, ...\} = \{x^n\}_{n\geq 0}$ be the standard basis of $\mathbb{C}[x]$. As a consequence of the linearity of vector spaces, we need only show that the commutator relations hold for all of the basis vectors:

$$\begin{aligned}
[w(e^+), w(e^-)](x^n) &= w(e^+)(w(e^-)(x^n)) - w(e^-)(w(e^+)(x^n)) \\
&= w(e^+)(\frac{i}{2}n(n-1)x^{n-2}) - w(e^-)(\frac{i}{2}x^{n+2}) \\
&= (\frac{i}{2})^2(n(n-1)x^n - (n+2)(n+1)x^n) \\
&= x^n(\frac{-1}{4}(n^2 - n - n^2 - 3n - 2) \\
&= x^n(\frac{-1}{4})(-4n-2) = x^n(n+\frac{1}{2}) \\
&= nx^n + \frac{1}{2}x^n = w(h)(x^n) \\
&= w([e^+, e^-])(x^n)
\end{aligned}$$

$$\begin{aligned}
[w(h), w(e^+)](x^n) &= w(h)(w(e^+)(x^n)) - w(e^+)(w(h)(x^n)) \\
&= w(h)(\frac{i}{2}x^{n+2}) - w(e^+(nx^n + \frac{i}{2}x^n)) \\
&= \frac{i}{2}((n+2+\frac{1}{2}) - (n+\frac{1}{2})x^{n+2}) \\
&= \frac{i}{2}(2)x^{n+2} = ix^{n+2} = 2(\frac{i}{2}x^{n+2}) = 2w(e^+)(x^n) \\
&= 2w([h, e^+])(x^n)
\end{aligned}$$

$$\begin{aligned}
[w(h), w(e^-)](x^n) &= w(h)(w(e^-)(x^n)) - w(e^-)(w(h)(x^n)) \\
&= w(h)(\frac{i}{2}n(n-1)x^{n-2}) - w(e^-)((n+\frac{1}{2})x^n) \\
&= \frac{i}{2}((n-2+\frac{1}{2})n(n-1)x^{n-2} - n(n-1)(n+\frac{1}{2})x^{n-2}) \\
&= (\frac{i}{2})(n-1)n(n-2+\frac{1}{2}-n-\frac{1}{2})x^{n-2} \\
&= \frac{i}{2}n(n-1)(-2)x^{n-2} = -2w(e^-)(x^n) \\
&= -2w([h, e^-])(x^n)
\end{aligned}$$

It follows that w defines a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}[x]$. This representation was obtained from the book (Roger Howe, Eng Chye Tan *Non-Abelian Harmonic Analysis*; Springer-Verlag, 1992) [1] .

Let the set of elements of $\mathbb{C}[x]$ spanned by the odd powers of x be denoted by $S^{odd} = Span\{x^{2n+1}\}_{n\geq 0}$ Let the set of elements of $\mathbb{C}[x]$ spanned by even powers of x be denoted by $S^{even} = Span\{x^{2n}\}_{n\geq 0}$.

Proposition 3: S^{even} and S^{odd} are each irreducible subrepresentations of $C[x]$ such that $\mathbb{C}[x] = S^{even} \oplus S^{odd}$.

Proof: Since the bases of S^{even} and S^{odd}, $\{x^{2n}\}_{n\geq 0}$ and $\{x^{2n+1}\}_{n\geq 0}$ respectively, form a 2-partition of the standard basis of $\mathbb{C}[x]$ (the even and odd basis elements, respectively), therefore $\mathbb{C}[x] = S^{even} \oplus S^{odd}$.

Now we show that S^{even} and S^{odd} are invariant under the action of $w(\mathfrak{sl}(2,\mathbb{C}))$, that is, S^{even} and S^{odd} are subrepresentations of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}[x]$:

We demonstrate invariance for the basis elements of $w(\mathfrak{sl}(2,\mathbb{C}))$, and for the basis elements of S^{even} and S^{odd}, respectively, and thereby demonstrate that S^{even} and S^{odd} are invariant,

$w(h)(x^{2n}) = (2n + \frac{1}{2})x^{2n} \in S^{even}$

$w(e^+)(x^{2n}) = \frac{i}{2}x^{2n+2} = \frac{i}{2}x^{2(n+1)} \in S^{even}$

$w(e^-)(x^{2n}) = \frac{i}{2}(2n)(2n-1)x^{2n-2} = \frac{i}{2}(2n)(2n-1)x^{2(n-1)} \in S^{even}$

$w(h)(x^{2n+1}) = (2n + 1 + \frac{1}{2})x^{2n+1} \in S^{odd}$

$w(e^+)(x^{2n+1}) = \frac{i}{2}x^{2n+1+2} = \frac{i}{2}x^{2(n+1)+1} \in S^{odd}$

$w(e^-)(x^{2n+1}) = \frac{i}{2}(2n+1)(2n)x^{2n-1} = \frac{i}{2}(2n+1)(2n)x^{2(n-1)+1} \in S^{odd}$

It follows that S^{even} and S^{odd} are invariant under the action of $w(\mathfrak{sl}(2,\mathbb{C}))$, that is, S^{even} and S^{odd} are sub-modules of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}[x]$

To show that S^{even} and S^{odd} are irreducible as $\mathfrak{sl}(2,\mathbb{C})$-modules we use Proposition 1,

First we require that, by $\mathfrak{sl}(2,\mathbb{C})$ actions, each basis vector of S^{even} and S^{odd}, respectively, can be sent to any other basis vector of S^{even} and S^{odd}, respectively:

We show the proof for S^{even} only, the proof for S^{odd} is similar.

Let $v = x^{2n} \in S^{even}$ $vneq0$, $u = x^{2p} \in S^{even}$ $u \neq 0$, and $u \neq v$.

There are two possibilities, either $p > n$ or $p < n$,

suppose $p > n \Rightarrow p = n + t, t > 0 \Rightarrow u = x^{2(n+t)}$

$\Rightarrow (w(e^+))^t(x^{2n}) = (\frac{i}{2})^t x^{2n+2t}$

$\Rightarrow (\frac{i}{2})^{-t}(w(e^+))^t(x^{2n}) = x^{2n+2t} = x^{2p}$

suppose that $p < n \Rightarrow p = n - t$, $n \geq t > 0 \Rightarrow u = x^{2(n-t)}$ and $n > 1$

$\Rightarrow (w(e^-))^t(x^{2n}) = (\frac{i}{2})^t \frac{(2n)!}{(2n-2t)!}x^{2n-2t}$

$\Rightarrow (\frac{i}{2})^{-t}\frac{(2n-2t)!}{(2n)!}(w(e^-))^t(x^{2n}) = x^{2(n-t)} = x^{2p}$

Next we show that any element of S^{even} can be sent to a basis vector of S^{even} and similarly for S^{odd}:

Suppose $v \in S^{even}$ and $v \neq 0 \Rightarrow v = \sum_{j=0}^{t} a_j x^{2j}$ and $a_t \neq 0$ suppose $t > 0 \Rightarrow (w(e^-))^t(v) = (\frac{i}{2})^t \frac{(2t)!}{(2t-2t)!}a_t x^{2t-2t} = (\frac{i}{2})^t(2t)!x^2 \Rightarrow (\frac{1}{a_t}\frac{i}{2})^{-t}\frac{1}{(2t)!}(w(e^-))^t(v) = 1 \Rightarrow$ any $v \in S^{even}$ can be sent to 1 which in turn can be sent to all basis vectors in $S^{even} \Rightarrow v$ can be sent to any other element in S^{even}.

Suppose $v \in S^{odd}$ and $v \neq 0 \Rightarrow v = \sum_{j=0}^{t} a_j x^{2j+1}$ and $a_t \neq 0$ suppose $t > 0$

$\Rightarrow (w(e^-))^t(v) = (\frac{i}{2})^t \frac{(2t+1)!}{(2t-2t+1)!}a_t x^{2t-2t+1} = (\frac{i}{2})^t \frac{(2t+1)!}{1!}a_t x \Rightarrow (\frac{1}{a_t}\frac{i}{2})^{-t}\frac{2}{(2t)!}(w(e^-))^t(v) = x$. Therefore any $0 \neq v \in S^{odd}$ can be sent to x, which in turn can be sent to all basis vectors in S^{odd}, therefore v can be sent to any other element in S^{odd}. Therefore by Proposition 1 we have that Proposition 3 is proven.

Section 2: A specific $\mathfrak{sl}(2,\mathbb{C})$-module of $\mathbb{C}[x,y]$

For the purposes of this paper, a tensor product of two vector spaces, V and Q, with $\{v_i\}$ a basis of V and $\{q_j\}$ a basis of Q, is a vector space with a bilinear map $\bigotimes : V \times Q \to V \otimes Q$ with $V \otimes Q$ being the vector space generated by the image of $\bigotimes$ on $V \times Q$ with basis $\{v_i \otimes q_j\}$.

Proposition 4: Given a Lie algebra L and two L-modules, m_1 on V, and m_2 on Q, we can form an L-module m on the tensor product $V \otimes Q$ with m defined by $m(x)(v \otimes q) = (m_1(x) \cdot v) \otimes q + v \otimes (m_2(x) \cdot q)$ $\forall x \in L, v \in V, q \in Q$

Proof: The map m, as a composition of linear maps, is linear by construction. Therefore we need only check that the commutator relations hold. Let $x \in L, v \in V, q \in Q$, we have,

$$
\begin{aligned}
m([x,y])\cdot v\otimes q & \\
&= (m_1([x,y])\cdot v)\otimes q + v\otimes (m_2([x,y])\cdot q)\\
&= (m_1(x)m_2(y)\cdot v - m_1(y)m_2(x)\cdot v)\otimes q\\
&\quad + v\otimes (m_2(x)m_2(y)\cdot w - m_2(y)m_2(x)\cdot q)\\
&= (m_1(x)m_1(y)\cdot v)\otimes q - (m_1(y)m_1(x)\cdot v)\otimes q\\
&\quad + v\otimes (m_2(x)m_2(y)\cdot q) - v\otimes (m_2(y)m_2(x)\cdot q)
\end{aligned}
$$

$$
\begin{aligned}
[m(x),m(y)]\cdot v\otimes q & \\
&= m(x)m(y)\cdot v\otimes q - m(y)m(x)\cdot v\otimes q\\
&= m(x)\cdot ((m_1(y)\cdot v)\otimes q + v\otimes (m_2(y)\cdot q))\\
&\quad -m(y)\cdot ((m_1(x)\cdot v)\otimes q + v\otimes (m_2(x)\cdot q))\\
&= (m_1(y)\cdot v)\otimes (m_2(x)q)\\
&\quad +(m_1(x)\cdot v)\otimes (m_2(y)\cdot q) + v\otimes (m_2(x)m_2(y)\cdot q)\\
&\quad -(m_1(y)\cdot (m_1(x)\cdot v))\otimes q - (m_1(x)\cdot v)\otimes (m_2(y)\cdot q)\\
&\quad -(m_1(y)\cdot v)\otimes (m_2(x)\cdot q) - v\otimes (m_2(y)m_2(x)\cdot q)\\
&= m([x,y])
\end{aligned}
$$

We can define the space of polynomials in two variables as the tensor product of the space of polynomials in one variable, with itself. In particular, we define the space of polynomials in two variables with the following tensor product relation: $p_x \otimes p_y = (p_x)(p_y)$ for all p_x in $\mathbb{C}[x]$ and all p_y in $\mathbb{C}[y]$.

Using the method outlined in Proposition 4 we can form a representation of $\mathfrak{sl}(2,\mathbb{C})$ on the tensor product space formed from two copies of $\mathbb{C}[x]$. For this purpose, we use two copies of the representation w defined previously (Section 1) over $\mathbb{C}[x]$ yielding the following representation on $\mathbb{C}[x,y]$:

$$m(h) = x\frac{d}{dx} + y\frac{d}{dy} + 1 \qquad m(e^+) = \frac{i}{2}(x^2+y^2) \qquad m(e^-) = \frac{i}{2}(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}) = \frac{i}{2}\triangle$$

First we note that $\mathbb{C}[x,y]$ is an infinite direct sum of its homogenous subspaces of degree k. Our next objective will be to find a decomposition of $\mathbb{C}[x,y]$ into a direct sum of irreducible submodules as per the preceding representation.

For any h-eigenvector of an $\mathfrak{sl}(2,\mathbb{C})$-module we define the weight of that vector to be its eigenvalue with respect to the action of h. For any submodule of an $\mathfrak{sl}(2,\mathbb{C})$-module we define the lowest weight vector to be the element of that subspace with lowest weight and we refer to the submodule as being a lowest weight submodule/subspace. We define the lowest weight subspaces in terms of their respective lowest weight vectors, each of which serves as a generating vector. We will demonstrate a decomposition of $\mathbb{C}[x,y]$ into lowest weight submodules such that our objective is satisfied. For each lowest weight seen in this paper there are two lowest weight vectors and two corresponding lowest weight submodules within $\mathbb{C}[x,y]$. For lowest weight n, we denote the corresponding lowest weight vectors as $b^-_{n-1} = (x-iy)^{n-1}$ and $b^+_{n-1} = (x+iy)^{n-1}$ with corresponding lowest weight subspaces denoted $V^-_n = Span\{\{(m(e^+))^s(b^-_{n-1})\}_{s\geq 0}\}$ and $V^+_n = Span\{\{(m(e^+))^s(b^+_{n-1})\}_{s\geq 0}\}$. That the proposed lowest weight vectors are in fact lowest weight vectors will be verified in the proofs of Propositions 9 and 10. Clearly the sets $\{(m(e^+))^s(b^-_{n-1})\}_{s\geq 0} = \{(\frac{i}{2})^s(x^2+y^2)^s(x-iy)^{n-1}\}_{s\geq 0}$ and $\{(m(e^+))^s(b^+_{n-1})\}_{s\geq 0} = \{(\frac{i}{2})^s(x^2+y^2)^s(x+iy)^{n-1}\}_{s\geq 0}$ are bases of the V^-_n and V^+_n respectively, since the elements of the respective sets are each homogeneous of different degree and hence are linearly independent.

We need to verify that the lowest weight vectors defined previously, b^-_n and b^+_n, are null/vacuum vectors for $m(e^-)$. The following two propositions (Proposition 5, Proposition 6) will be used in the proofs of two propositions, Proposition 7 and Proposition 8.

Proposition 5: $m(e^-)(b^+_n) = 0$ and $m(e^-)(b^-_n) = 0$ for all $n \geq 0$

Proof: For $n = 0$ or $n = 1$ the proposition is trivially true since the second derivatives are equal to zero. Supposing that n is larger than one, we have:

$$
\begin{aligned}
&m(e^-)(b_n^+) \\
&\qquad = m(e^-)(b_n^+) = m(e^-)((x+iy)^n) = \frac{i}{2}(\frac{d^2}{dx^2}(x+iy)^n + \frac{d^2}{dy^2}(x+iy)^n) \\
&\qquad = \frac{i}{2}(n)(n-1)((x+iy)^{n-2} + (i)^2(x+iy)^{n-2}) = 0 \\
&m(e^-)(b_n^-) \\
&\qquad = m(e^-)(b_n^-) = m(e^-)((x-iy)^n) = \frac{i}{2}(\frac{d^2}{dx^2}(x-iy)^n + \frac{d^2}{dy^2}(x-iy)^n) \\
&\qquad = \frac{i}{2}(n)(n-1)((x-iy)^{n-2} + (-i)^2(x-iy)^{n-2}) = 0
\end{aligned}
$$

For the following proposition we use polar coordinates with $r^2 = x^2+y^2$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$, therefore $b_n^+ = r^n(\cos(\theta) + i\sin(\theta))^n$ and $b_n^- = r^n(\cos(\theta) - i\sin(\theta))^n$

Proposition 6: For all $n \geq 1$, $\frac{d}{dr}(b_n^+) = \frac{n}{r}b_n^+$ and $\frac{d}{dr}(b_n^-) = \frac{n}{r}b_n^-$. For $n = 0$, $\frac{d}{dr}(b_n^+) = \frac{d}{dr}(b_n^-) = 0$.

Proof:

$$
\begin{aligned}
&\tfrac{d}{dr}(b_n^+) \\
&\qquad = \frac{d}{dr}(r^n(\cos(\theta) + i\sin(\theta))^n) = nr^{n-1}(\cos(\theta) + i\sin(\theta))^n = \frac{n}{r}b_n^+ \\
&\tfrac{d}{dr}(b_n^-) \\
&\qquad = \frac{d}{dr}(r^n(\cos(\theta) - i\sin(\theta))^n) = nr^{n-1}(\cos(\theta) - i\sin(\theta))^n = \frac{n}{r}b_n^-
\end{aligned}
$$

For the following analysis we need not distinguish between b_n^- and b_n^+, nor between V_n^- and V_n^+, hence we will refer to both simultaneously as b_n, and V_n, respectively. We will now show that, after multiplication by a scalar, the action of $m(e^-)$ on the basis vectors of V_n reverses the action of $m(e^+)$. First we prove this is the case for all V_n such that $n \geq 1$ and then we prove it for the special case of V_0. We use polar coordinates to solve this problem. Suppose $n \geq 1$, let $p_n = (\frac{i}{2})^s b_n$ which implies that $(m(e^+))^s(b_n) = (x^2+y^2)^s(\frac{i}{2})^s b_n = (x^2+y^2)^s p_n$. In polar coordinates this is equivalent to $(m(e^+))^s(b_n) = r^{2s}p_n$. We now note that $m(e^-) = \frac{i}{2}\triangle = \frac{i}{2}(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2})$ which is a scalar multiple of the Laplacian.

Proposition 7: $m(e^-)(r^{2s}p_n) = 4(ns+s^2)r^{2(s-1)}p_n$ for $n \geq 1$ and $s \geq 1$.

Proof:

$$
\begin{aligned}
&m(e^-)(r^{2s}p_n) \\
&\qquad = \frac{d^2}{dr^2}(r^{2s}p_n) + \frac{1}{r}\frac{d}{dr}(r^{2s}p_n) + \frac{1}{r^2}\frac{d^2}{d\theta^2}(r^{2s}p_n) \\
&\qquad = (\frac{d^2}{dr^2}p_n)r^{2s} + 2(\frac{d}{dr}p_n)(\frac{d}{dr}r^{2s}) + (\frac{d^2}{dr^2}r^{2s})p_n + \frac{1}{r}(\frac{d}{dr}p_n)r^{2s} + \frac{1}{r}(\frac{d}{dr}r^{2s})p_n + \\
&\qquad \frac{1}{r^2}(\frac{d^2}{d\theta^2}p_n)r^{2s} + \frac{2}{r^2}(\frac{d}{d\theta}p_n)(\frac{d}{d\theta}r^{2s}) + \frac{1}{r^2}(\frac{d^2}{d\theta^2}r^{2s})p_n \\
&\qquad = r^{2s}(\frac{d^2}{dr^2}p_n + \frac{1}{r}(\frac{d}{dr}p_n) + \frac{1}{r^2}(\frac{d^2}{d\theta^2}p_n)) + 2(\frac{d}{dr}p_n)(\frac{d}{dr}r^{2s}) + p_n(\frac{d^2}{dr^2}r^{2s}) + \frac{1}{r}p_n(\frac{d}{dr}r^{2s}) \\
&\qquad = r^{2s}(2i)(m(e^-)(p_n)) + 2(2s)r^{2s-1}(\frac{d}{dr}p_n) + 2s(2s-1)r^{2s-2}p_n + 2sr^{2s-2}p_n \\
&\qquad = 0 + 2(2s)r^{2s-1}(\frac{n}{r}p_n) + 2s(2s-1)r^{2s-2}p_n + 2sr^{2s-2}p_n = (4s + 2s(2s-1) + 2s)r^{2s-2}p_n \\
&\qquad = 4(ns+s^2)r^{2(s-1)}p_n
\end{aligned}
$$

For $n = 0$ we have that $b_0 = 1$ and that $\frac{d}{dr}b_0 = 0$ and we let $p = (\frac{i}{2})^s b_0$ which implies the following:

Proposition 8: $m(e^-)(r^{2s}p) = 4s^2r^{2(s-1)}p$ $p = (\frac{i}{2})^s b_0$ and $s \geq 1$.

Proof:

$$\begin{aligned} m(e^-)(r^{2s}p) & \\ &= \frac{d^2}{dr^2}(r^{2s}p) + \frac{1}{r}\frac{d}{dr}(r^{2s}p) + \frac{1}{r^2}\frac{d^2}{d\theta^2}(r^{2s}p) \\ &= r^{2s}(2i)(m(e^-)(p)) + 2(2s)r^{2s-1}(\frac{d}{dr}p) + 2s(2s-1)r^{2s-2}p + 2sr^{2s-2}p \\ &= 0 + 0 + 2s(2s-1)r^{2s-2}p + 2sr^{2s-2}p = (2s(2s-1)+2s)r^{2s-2}p \\ &= 4s^2r^{2(s-1)}p \end{aligned}$$

We are now almost ready to prove that the V_n are submodules. By the construction of V_n we have that the action of $m(e^+)$ sends basis elements of V_n to scalar multiples of other basis elements of V_n. From the two propositions above (Proposition 7, Proposition 8) we have that $m(e^-)$ sends basis elements of V_n to scalar multiples of basis elements of V_n. It now remains to show that $m(h)$ sends basis elements of V_n to scalar multiples of basis elements of V_n.

For ease of notation let $p^-_{n,s} = (\frac{i}{2})^s(x^2+y^2)^s(x-iy)^n$ and $p^+_{n,s} = (\frac{i}{2})^s(x^2+y^2)^s(x+iy)^n$, then we have that, for each n, $\{p^-_{n,s}\}_{s\geq 0}$ is a basis of V^-_n and $\{p^+\}_{s\geq 0}$ is a basis of V^+_n. We need to verify that each element of the respective bases (each p) is sent by h to another element of the submodule spanned by the respective basis. The proofs of the following two propositions use the results from Propositions 6 and 7.

Proposition 9: $m(h)(p^+_{n,s}) = (n+2s+1)p^+_{n,s}$ for all $p^+_{n,s} = (\frac{i}{2})^s(x^2+y^2)^s(x+iy)^n$ $n \geq 0$ and $s \geq 0$.

Proof:

$$\begin{aligned} m(h)(p^+_{n,s}) & \\ &= x\frac{d}{dx}p^+_{n,s} + y\frac{d}{dy}p^+_{n,s} + p^+_{n,s} \\ &= (\frac{i}{2})^s(x(n(x^2+y^2)^s(x+iy)^{n-1} + 2xs(x^2+y^2)^{s-1}(x+iy)^n) \\ &+y(ni(x^2+y^2)^s(x+iy)^{n-1} + 2ys(x^2+y^2)^{s-1}(x+iy)^n) + (x^2+y^2)^s(x+iy)^n) \\ &= (\frac{i}{2})^s(n(x^2+y^2)^s(x+iy)^n + 2s(x^2+y^2)^s(x+iy)^n + (x^2+y^2)^s(x+iy)^n) \\ &= (n+2s+1)(\frac{i}{2})^s(x^2+y^2)^s(x+iy)^n \end{aligned}$$

Proposition 10: $m(h)(p^-_{n,s}) = (n+2s+1)p^-_{n,s}$ for all $p^-_{n,s} = (\frac{i}{2})^s(x^2+y^2)^s(x-iy)^n$ $n \geq 0$ and $s \geq 0$.

Proof:

$$\begin{aligned} m(h)(p^-_{n,s}) & \\ &= x\frac{d}{dx}p^-_{n,s} + y\frac{d}{dy}p^-_{n,s} + p^-_{n,s} \\ &= (\frac{i}{2})^s(x(n(x^2+y^2)^s(x-iy)^{n-1} + 2xs(x^2+y^2)^{s-1}(x-iy)^n) \\ &+y(-ni(x^2+y^2)^s(x+iy)^{n-1} + 2ys(x^2+y^2)^{s-1}(x-iy)^n) + (x^2+y^2)^s(x+iy)^n) \\ &= (\frac{i}{2})^s(n(x^2+y^2)^s(x-iy)^n + 2s(x^2+y^2)^s(x-iy)^n + (x^2+y^2)^s(x-iy)^n) \\ &= (n+2s+1)(\frac{i}{2})^s(x^2+y^2)^s(x-iy)^n \end{aligned}$$

Since we have shown that the actions of each basis element of $m(\mathfrak{sl}(2,\mathbb{C}))$ sends basis elements of V_n to other basis elements of V_n, we now have that the V_n are submodules of $\mathbb{C}[x,y]$.

Next we prove that the V_n are irreducible. We note that, from the basis given previously, any element $v \neq 0$ of V_n can be expressed as $v = \sum_{j=0}^{k} a_j m(e^+)^j(b_n)$ with $a_k \neq 0$. We then apply Proposition 8 k times to yield

the result $(m(e^-))^k(v) = \alpha(b_n)$ for some non-zero scalar α. We can then repeatedly apply $m(e^+)$ to $\alpha(b_n)$ to obtain non-zero scalar multiples of any basis elements of V_n. We then take linear combinations of the resulting basis elements to obtain any element of V_n. Hence we have shown that any element of the submodule V_n can be sent to any other element of V_n and hence, by Proposition 1, V_n is irreducible for all $n \geq 0$.

We now need to prove that $\mathbb{C}[x, y]$ is the direct sum of the V_n. Since the V_n are distinct submodules it follows that $V_i \cap V_j = 0$ for all i, j, $i \neq j, i \geq 0, j \geq 0$. It remains to show that the sum of all the V_n is equivalent to $\mathbb{C}[x, y]$. We prove this using induction. For notation let $Z_n = Span\{x^{n-k}y^k\}_{k=n}^n$ denote the space of homogeneous polynomials of degree n in two variables. First we prove that, given a basis of Z_n satisfying a certain property, we can form a basis of Z_{n+2} by applying $m(e^+)$ to the original basis and then augmenting the result with the elements b_{n+2}^+ and b_{n+2}^-.

Proposition 11: Given a basis, $\{t_j\}_{j=0}^n$, of Z_n, such that $m(e^-)m(e^+)(t_j) = a_j t_j$ for all $t_j \in \{t_j\}_{j=0}^n$ for some non-zero scalar a_j, then the set $\{b_{n+2}^+, b_{n+2}^-, \{m(e^+)(t_j)\}_{j=0}^n\}$ is a basis for Z_{n+2}.

Proof: We note that $m(e^+)$ sends non-zero elements of Z_n to non-zero elements of Z_{n+2}. Suppose we have a basis, $\{t_j\}_{j=0}^n$, of Z_n, such that $m(e^-)m(e^+)(t_j) = a_j t_j$ for all $t_j \in \{t_j\}_{j=0}^n$ for some non-zero scalar a_j. We show that the set $\{m(e^+)(t_j)\}_{j=0}^n$ is linearly independent. We have that $\{t_j\}_{j=0}^n$ is a basis and is therefore linearly independent. Suppose that $\{m(e^+)(t_j)\}_{j=0}^n$ is not linearly independent. That implies that there exists a linear combination $\sum_{j=0}^n c_j m(e^+)(t_j) = 0$ with at least one of the scalars $\{c_j\}_{j=0}^n$ not equal to zero. From the specified property of the basis we have that $\sum_{j=0}^n d_j t_j = \sum_{j=0}^n c_j a_j t_j = \sum_{j=0}^n m(e^-)(c_j m(e^+)(t_j)) = 0$ for some set of scalars $\{d_j\}_{j=0}^n = \{c_j a_j\}_{j=0}^n$ and with at least one such d_j not equal to zero. But that would imply that the set $\{t_j\}_{j=0}^n$ is linearly dependent which contradicts the fact that it is a basis. Therefore, we have a proof by contradiction that the set $\{m(e^+)(t_j)\}_{j=0}^n$ is linearly independent.

Next we prove that b_{n+2}^+ and b_{n+2}^- are linearly independent. Since proving that b_n^+ and b_n^- are linearly independent for all $n \geq 1$ implies that b_{n+2}^+ and b_{n+2}^- are linearly independent, we need only prove the former. We take an arbitrary linear combination of b_n^+ and b_n^-, set it equal to zero, and use the first two terms of the binomial expansions to prove that the coefficients in the linear combination must both be zero and hence b_n^+ and b_n^- are linearly independent. Let $Ab_n^+ + Bb_n^- = 0$, that implies that $A\sum_{k=0}^n \binom{n}{k} x^{n-k}(iy)^k + B\sum_{k=0}^n \binom{n}{k} x^{n-k}(-iy)^k = 0$ which, by equating the first two coefficients, implies that $A = -B$ and $nA = nB$, which implies that $A = B = 0$, therefore b_n^+ and b_n^- are linearly independent for all $n \geq 1$.

Next we prove that any non-zero linear combination of b_{n+2}^+ and b_{n+2}^- is not contained in the span of $\{m(e^+)(t_j)\}_{j=0}^n$ and hence the set $\{b_{n+2}^+, b_{n+2}^-, \{m(e^+)(t_j)\}_{j=0}^n\} = \delta$ is a linearly independent set of elements in Z_{n+2}. First we note that, since b_{n+2}^+ and b_{n+2}^- are vacuum vectors of $m(e^-)$ and because of the linearity of $m(e^-)$, $m(e^-)(Ab_{n+2}^+ + Bb_{n+2}^-) = 0$ for all scalars A and B. Suppose that the set, $\{Ab_{n+2}^+ + Bb_{n+2}^-\}$, of linear combinations of b_{n+2}^+ and b_{n+2}^- contains a non-zero element in the span of $\{m(e^+)(t_j)\}_{j=0}^n$. That would imply that there exists a set of scalars $\{f_j\}$ with at least one nonzero element such that $\sum_{j=0}^n f_j m(e^+)(t_j) = Ab_{n+2}^+ + Bb_{n+2}^-$ for some A and B. We then apply $m(e^-)$ to both sides and attain the following $m(e^-)(\sum_{j=0}^n f_j m(e^+)(t_j)) = \sum_{j=0}^n f_j a_j t_j = \sum_{j=0}^n g_j t_j = m(e^-)(Ab_{n+2}^+ + Bb_{n+2}^-) = 0$ for some set of scalars $\{g_j\} = \{f_j a_j\}$ with some g_j not equal to zero. This implies that the set $\{t_j\}$ is linearly dependent which contradicts the fact that it is a basis, therefore we have a proof by contradiction that no element of the set of linear combinations of b_{n+2}^+ and b_{n+2}^- is in the span of the set $\{m(e^+)(t_j)\}_{j=0}^n$. Therefore, $\{b_{n+2}^+, b_{n+2}^-, \{m(e^+)(t_j)\}_{j=0}^n\} = \delta$ is a linearly independent set in Z_{n+2} with $n+2$ elements, hence we have that δ is a basis of Z_{n+2}.

Proposition 12: $\mathbb{C}[x, y] = \bigoplus_{n\geq 0} V_n = \bigoplus_{n\geq 0} V_n^+ \oplus \bigoplus_{n\geq 0} V_n^-$ with each $V_n^- = Span\{(m(e^+))^s(b_{n-1}^-)\}_{s\geq 0} = Span\{(\frac{i}{2})^s(x^2+y^2)^s(x-iy)^{n-1}\}_{s\geq 0}$ and $V_n^+ = Span\{(m(e^+))^s(b_{n-1}^+)\}_{s\geq 0} = Span\{(\frac{i}{2})^s(x^2+y^2)^s(x+iy)^{n-1}\}_{s\geq 0}$ being irreducible submodules of $\mathbb{C}[x, y]$

Proof: We have previously demonstrated that the V_n are irreducible submodules of $\mathbb{C}[x, y]$ and that $V_i \cap V_j = 0$ for all i, j, $i \neq j, i \geq 0, j \geq 0$. We now need only show that the sum of all the V_n is equivalent to $\mathbb{C}[x, y]$. To do this, we recall that $\mathbb{C}[x, y] = \bigoplus_{n\geq 0} Z_n$. We repeatedly apply the inductive step specified by Proposition 11 to the single element set $\{b_0\}$ to generate bases for all Z_n with n even, and repeatedly apply the same inductive

argument to the set $\{b_1^+, b_1^-\}$ to obtain bases for all the Z_n with n odd. Note that we are able to apply Proposition 11 in this case since the bases formed at each inductive step satisfiy the requirement specified in Proposition 11. The requirement is that the given basis, $\{t_j\}_{j=0}^n$, of Z_n, satisfies the relation $m(e^-)m(e^+)(t_j) = a_j t_j$ for all $t_j \in \{t_j\}_{j=0}^n$ for some non-zero scalar a_j. We know this requirement is satisfied by $\{b_0\}$ and $\{b_1^+, b_1^-\}$ and by all elements of $\{V_n\}_{n\geq 0}$ given the results contained in Propositions 7 and 8. The bases obtained from the repeated application of Proposition 11 to $\{b_0\}$ and $\{b_1^+, b_1^-\}$ by construction consist of elements of the lowest weight submodules, $\{V_n\}$. Since $\mathbb{C}[x, y]$ is the direct sum of its homogeneous subspaces, $\{Z_n\}$, therefore we have demonstrated that $\mathbb{C}[x, y]$ is also the direct sum of the lowest weight spaces, $\{V_n\}$.

References

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