SUPERCONFORMAL CURRENT ALGEBRA

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1. Basic Definitions

Definition 1. Algebra
A vector space A over a field F along with a bilinear map

$$p : A \times A \longrightarrow A$$

is called an algebra over the field F.

Note that we neither demand that the algebra be associative, nor that it have a unit element.

Definition 2. Lie Algebra
An algebra is called a Lie algebra if the product map p satisfies the following conditions:

- $p(a, a) = 0, \quad \forall\, a \in A.$
- Jacobi identity:
  $p(a, p(b, c)) + p(b, p(c, a)) + p(c, p(a, b)) = 0, \quad \forall\, a, b, c \in A.$

For simplicity of notation, the element $p(a, b)$ is denoted as ab or $[a, b]$ when A is an algebra or a Lie algebra respectively.

The first condition for being a Lie algebra is equivalent to saying

$$[a, b] = -[b, a], \quad \forall\, a, b \in A,$$

provided the characteristic of F is not 2.

A Lie algebra is an example of an algebra which is, in general, neither associative nor unital.

Definition 3. Lie Sub-algebra, Ideal and Centre
A subset I of a Lie algebra L closed under the Lie Bracket $[\cdot, \cdot]$ is called a Lie subalgebra of L.

I is said to be an ideal of L if

$$[x, y] \in I, \quad \forall\, x \in I,\ y \in L.$$

The centre of L is defined to be the set

$$Z = \{x \in L \mid [x, y] = 0, \forall\, y \in L\}.$$

Note that Z is always an ideal of L.

Definition 4. General linear Lie algebra
Given a vector space V, we define its general linear Lie algebra, denoted by $\mathrm{gl}(V)$ to be the vector space of all linear transformations of V, along with the Lie bracket

$$[f, g] := f \circ g - g \circ f, \quad \forall\, f, g \in \mathrm{gl}(V)$$

Definition 5. Derivation
Given an algebra A over the field F, a derivation is an F-linear map $D : A \longrightarrow A$ such that

$$D(ab) = D(a)b + aD(b), \quad \forall\, a, b \in A.$$

Note that the set of all derivations of A, denoted by $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathrm{gl}(A)$ with the Lie bracket given by

$$[D, E] = D \circ E - E \circ D, \quad \forall\, D, E \in \mathrm{Der}(A) \tag{1.1}$$

Result 1. If I, J are ideals of L then so are the following:

- $I \cap J$.
- $I + J = \{i + j \mid i \in I,\, j \in J\}$.
- $[I, J] = \mathrm{span}\{\, [i, j] \mid i \in I,\, j \in J\}$.

Definition 6. Smooth Manifold
A topological space M such that:

- it is Housdorff,
- it is second countable,
- it is locally Euclidean, i.e. every point in M has a neighbourhood which is diffeomorphic to some open subset of $\mathbb{R}^n$, for some fixed $n \in \mathbb{N}$,
- if ϕ, ψ are any two of the diffeomorphism maps, then they are C^∞ compatible, i.e. $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are C^∞ over the suitable domain.

is called a smooth manifold.

Definition 7. Tangent space
A tangent vector at a point p in a smooth manifold M is a derivation at p, i.e. it is a linear map $D : C_p^\infty(M) \longrightarrow \mathbb{R}$ such that

$$D(fg) = D(f)g(p) + f(p)D(g), \quad \forall\, f, g \in C_p^\infty,$$

where C_p^∞ denotes the algebra of germs of C^∞ functions at p.

The tangent vectors at p form a vector space $T_p\,M$, called the tangent space of M at p.

Definition 8. Tangent bundle
The tangent bundle of M is the union of all tangent spaces of M:

$$TM = \bigcup_{p \in M} T_p\,M$$

It is known that TM has a manifold structure, derived from M.

Definition 9. Vector field
A vector field on M is a smooth map $v : M \longrightarrow TM$ such that, $v(p) \in T_p\,M$ for every $p \in M$.

Note that the space of all vector fields on M, denoted by $\mathrm{Vect}(M)$, forms a vector space over $\mathbb{R}$.

Result 2. Let M be a smooth manifold and $p \in M$.

(1) $C_p^\infty(M)$, the space of germs of all smooth functions defined at p is an algebra under pointwise multiplication, for every $p \in M$. It is also a Lie algebra on $\mathbb{R}$ with the Lie bracket

$$[f, g] := f \circ g - g \circ f, \quad \forall\, f, g \in C_p^\infty(M).$$

(2) There exists an isomorphism of vector spaces $\phi : \mathrm{Vect}(M) \longrightarrow \mathrm{Der}(C_p^\infty)$, given by

$$\phi(v)f(p) = v_p f, \quad \forall\, v \in \mathrm{Vect}(M),\ f \in C_p^\infty,\ p \in M$$

where $v_p := v(p)$.

(3) For a fixed point $p \in M$, using this isomorphism ϕ and the fact that $\mathrm{Der}(C_p^\infty)$ is a Lie algebra, $\mathrm{Vect}(M)$ can be given a Lie algebra structure with the Lie bracket defined as

$$[v, w](p) = [\phi(v), \phi(w)].$$

Definition 10. Lie group
A Lie group is a smooth manifold G with a group structure such that the maps

$$p : G \times G \longrightarrow G, \quad \text{defined by} \quad p(g,h) = gh$$

and

$$i : G \longrightarrow G, \quad \text{defined by} \quad i(g) = g^{-1}$$

are smooth.

Every tangent vector of a smooth manifold can be associated to a point derivation on that manifold. Also, as shown in Result 2-2, every point derivation on a manifold is associated to a vector field. Using this, one can define the Lie algebra of a Lie group.

Definition 11. Lie algebra of a Lie group
The algebra of a Lie group G is defined to be $\text{Lie}(G) := T_1G$, equipped with the Lie bracket

$$[\cdot,\cdot] : \text{Lie}(G) \times \text{Lie}(G) \longrightarrow \text{Lie}(G)$$

defined by

$$[x,y] := [v_x, v_y] \,, \quad \forall\, x,y \in \text{Lie}(G)$$

where v_x, v_y are vector fields on G corresponding to x, y respectively.

References

[1] K Erdmann, MJ Wildon, *Introduction to Lie Algebras*. Springer. *Chapters 1,2,3.*
[2] Loring W. Tu, *An Introduction to Manifolds*. 2nd. edition, Springer. *Chapters 2,3,4.*

2. Witt algebra

From Result 2-2, we note that any element of $\text{Vect}(\mathbb{S}^1)$ is of the form $f(\theta)\dfrac{d}{d\theta}$, where f is a smooth, real-valued 2π-periodic function on $\mathbb{S}^1$.

The Lie bracket, as given by Equation (1.1) is

$$\left[\, f(\theta)\frac{d}{d\theta} \,,\, g(\theta)\frac{d}{d\theta} \,\right] = (fg' - f'g)\frac{d}{d\theta}, \quad \forall\, f,g \in C^\infty(\mathbb{T}). \tag{2.1}$$

Definition 12. Complex polynomial vector fields
For $n \in \mathbb{Z}$, define $d_n \in \mathrm{Vect}^{\mathbb{C}}(\mathbb{S}^1) := \mathrm{Vect}(\mathbb{S}^1) \otimes \mathbb{C}$ as

$$d_n = ie^{in\theta}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz}, \tag{2.2}$$

where $z = e^{i\theta}$ to be the n-th complex polynomial vector field.

Definition 13. Witt algebra
The Witt algebra is defined as $\mathfrak{d} := \mathrm{span}_{\mathbb{C}}\{d_n \mid n \in \mathbb{Z}\}$. It is a Lie algebra with the Lie bracket given as

$$[d_m, d_n] = (m-n)d_{m+n}, \quad \forall\, m, n \in \mathbb{Z}. \tag{2.3}$$

It is a known result that $\mathrm{Vect}(\mathbb{S}^1) \cong \mathrm{Lie}(\mathrm{Diff}_+\mathbb{S}^1)$, where $\mathrm{Diff}_+\mathbb{S}^1$ denotes the group of orientation-preserving diffeomorphisms of $\mathbb{S}^1$ under the operation of composition of functions.

Let $\omega : \mathfrak{d} \longrightarrow \mathfrak{d}$ be defined such that

$$\omega(d_n) = d_{-n}, \quad \forall\, n \in \mathbb{Z} \tag{2.4}$$

and

$$\omega(\lambda\, x) = \overline{\lambda}\, x, \quad \forall\, \lambda \in \mathbb{C},\ x \in \mathfrak{d}. \tag{2.5}$$

Then, ω is an anti-linear (conjugate linear) anti-involution (also called an involution) such that

$$\omega([x, y]) = -[\omega(x), \omega(y)]\,, \quad \forall\, x, y \in \mathfrak{d}.$$

Also, $\mathfrak{d} \cap \mathrm{Vect}(\mathbb{S}^1)$ contains precisely those elements of $\mathfrak{d}$ which are fixed under $-\omega$.

Definition 14. Representation of a Lie algebra
Given a Lie algebra L over the field F, we say that the pair (V, ϕ) (or more commonly simply V) is a representation of L if V is a finite-dimensional vector space over F and ϕ is a Lie algebra homomorphism, i.e.

$$\phi : L \longrightarrow \mathrm{gl}(V),$$

such that

$$\phi([x, y]) = [\phi(x)\,|\,\phi(y)]\,, \quad \forall\, x, y \in L.$$

Definition 15. Contravariant and Unitary representations
Given a representation V of $\mathfrak{d}$, we say that a Hermitian form $\langle \cdot, \cdot \rangle$ on V is contravariant if

$$\langle x(u)\,|\,v\rangle = \langle u\,|\,\omega(x)(v)\rangle, \quad \forall\, x \in \mathfrak{d},\ u, v \in V.$$

When $\langle \cdot, \cdot \rangle$ is nondegenerate, this means that

$$x^\dagger = \omega(x), \quad \forall\, x \in \mathfrak{d}.$$

Furthermore, V is said to be a unitary representation if in addition

$$\langle v\,|\,v\rangle > 0, \quad \forall\, v \in V \backslash \{0\}.$$

A representation V of a Lie algebra L can also be viewed as the action of L on V, i.e. a map

$$L \times V \longrightarrow V \quad (x, v) \longrightarrow x \cdot v$$

satisfying

$$\begin{aligned}(\lambda x + \mu y) \cdot v &= \lambda(x \cdot v) + \mu(y \cdot v),\\ x \cdot (\lambda v + \mu w) &= \lambda(x \cdot v) + \mu(x \cdot w),\\ [x, y] \cdot v &= x \cdot (y \cdot v) y \cdot (x \cdot v).\end{aligned}$$

Definition 16. Representations $V_{\alpha,\beta}$ of $\mathfrak{d}$
Define $v_k := z^{k+\alpha}(dz)^\beta, \quad \forall\ k \in \mathbb{Z}$. Define $V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{v_k\}$ to be a space of densities.

By defining the action of $\mathfrak{d}$ on $V_{\alpha,\beta}$ as –

$$d_n(v_k) = -(k\alpha + \beta + \beta n)v_{n+k}, \quad \forall\ n \in \mathbb{N},\ k \in \mathbb{Z}, \tag{2.6}$$

$V_{\alpha,\beta}$ becomes a two-parameter family of representations of $\mathfrak{d}$.

Definition 17. Energy operator
The operator d_0 as defined in Equation (2.6) is called the Energy operator.

Theorem 1. *The representation $V_{\alpha,\beta}$ of $\mathfrak{d}$ is reducible iff either*

(1) $\alpha \in \mathbb{Z},\ \beta = 0$*; or*

(2) $\alpha \in \mathbb{Z},\ \beta = 1$.

Proof. Suppose $V_{\alpha,\beta}$ is reducible and U is an irreducible subrepresentation. We can divide it into two cases.

Case 1: $V_{\alpha,\beta}\backslash U$ has atleast two vectors $v_m, v_n\ (m \neq n)$.

Then

$$d_{m-k}(v_k) = -(k + \alpha + \beta + \beta(m - k))v_m$$

and

$$d_{n-k}(v_k) = -(k + \alpha + \beta + \beta(n - k))v_n.$$

Since $v_k \in U$, the left-hand sides of both equations are elements in U. This forces

$$k + \alpha + \beta + \beta(m - k) = 0 = k + \alpha + \beta + \beta(m - k)$$

i.e.,

$$\beta = 0 \quad \text{and} \quad k + \alpha = 0,$$

thus implying condition (1).

Moreover, by a shift of the basis by k elements, we may set $\alpha = 0$. Since in $V_{0,0}$,

$$d_n(v_0) = 0, \quad \forall\, n \in \mathbb{Z},$$

the irreducibility of U results in $U = \mathbb{C}v_0$. Since any v_i $(i \neq 0)$ can be obtained by the action of d_{i-n} on v_n, $V'_{0,0} = V_{0,0}/\mathbb{C}v_0$ is an irreducible representation of $\mathfrak{d}$.

Case 2: $V_{\alpha,\beta}\backslash U$ has exactly one vector, v_m say.

Then every v_i $(i \neq m)$ is in U. Choose $v_k, v_l \in U$ $(k \neq l \neq m)$.

Then,

$$d_{m-k}(v_k) = -(k + \alpha + \beta + \beta(m-k))v_m,$$
$$d_{m-l}(v_l) = -(l + \alpha + \beta + \beta(m-l))v_m$$

forces

$$k + \alpha + \beta + \beta(m-k) = 0 = k + \alpha + \beta + \beta(m-k)$$

i.e.,

$$\beta = 1 \quad \text{and} \quad \alpha + m + 1 = 0,$$

thus implying condition (2).

Moreover, by setting $\alpha = 0$, and thus $m = -1$, we get that U is the linear span of all v_k, $k \in \mathbb{Z}\backslash\{-1\}$. Therefore, $V'_{0,1} = U$ is an irreducible representation fo $\mathfrak{d}$.

The converse is trivial. □

References

[1] VG Kac, AK Raina, N Rozhkovskaya, *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*. World Scientific. *Chapter 1*.
[2] K Erdmann, MJ Wildon, *Introduction to Lie Algebras*. Springer. *Chapter 7*.

3. Virasoro Algebra

Definition 18. Given the Witt algebra $\mathfrak{d}$ and an element C of an arbitrary vector space over the field $\mathbb{C}$, define the central extension of $\mathfrak{d}$, called the Virasoro Algebra, to be the Lie algebra defined as

$$\hat{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{C}C,$$

along with the commutation relations

$$[d_m, d_n] = (m-n)d_{m+n} + \tfrac{m^3-m}{12}\delta_{m+n}C, \quad \forall\, m, n \in \mathbb{Z} \tag{3.1}$$

and

$$[d_m, C] = 0, \quad \forall\, m \in \mathbb{Z}. \tag{3.2}$$

where $\delta_x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{o.w.} \end{cases}, \quad \forall\, x \in \mathbb{C}.$

Note that Equation (3.2) implies that C is in the centre of the Witt algebra as well as the Virasoro algebra (hence the name – central extension).

We are interested in finding unitary representationf for Vir. For the case $C = 0$, Vir reduces to W, the Witt Algebra, for which we know the representations $V_{\alpha,\beta}$. For the case when $C = 1$, we use bosonic oscillators.

4. Oscillator (Heisenberg) Algebra

Definition 19. Oscillator Algebra
We define the Oscillator Algebra, denoted by $\mathcal{A}$, to be a complex Lie algebra with the basis $\{a_n, n \in \mathbb{Z}; \mathbb{1}\}$ and the Lie bracket

$$[\mathbb{1}, a_n] = 0, \quad \forall\, n \in \mathbb{Z},$$
$$[a_m, a_n] = m\delta_{m+n}\mathbb{1}, \quad \forall\, m, n \in \mathbb{Z}.$$

Note that both $\mathbb{1}$ and a_0 lie in the centre of (A).

Definition 20. Representation of $\mathcal{A}$
Define the representation of $\mathcal{A}$ on the Fock space of polynomials in infinite variables, $B = \mathbb{C}[x_1, x_2, \ldots]$ as

$$a_n = \epsilon_n \frac{\partial}{\partial x_n}, \quad \forall\, n \in \mathbb{N}, \tag{4.1}$$
$$a_{-n} = \epsilon_n^{-1} n x_n, \quad \forall\, n \in \mathbb{N}, \tag{4.2}$$
$$a_0 = \mu\, I, \tag{4.3}$$
$$\mathbb{1} = I. \tag{4.4}$$

where $\epsilon_n,\, \mu \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}.$

5. Constructing the Superconformal Algebra

Let G be a compact Lie group. Let $\mathfrak{g}$ be its finite-dimensional Lie algebra. Let $(\cdot,\cdot)$ be a Killing form on $\mathfrak{g}$.

Let $\widetilde{\mathfrak{g}}$ denote the super loop algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1};\theta]$ with the Lie bracket,

$$[x\otimes P, y\otimes Q] = [x,y]\otimes PQ, \quad \forall\, x,y\in\mathfrak{g} \text{ and } P,Q\in\mathbb{C}[t,t^{-1};\theta].$$

We introduce a gradation on $\widetilde{\mathfrak{g}}$ by setting

$$\deg(\mathfrak{g}) = 0\,, \quad \deg(t^n) = n\,, \quad \deg(\theta) = k,$$

$\forall\, n\in\mathbb{Z}$ and for some choice of $k\in\dfrac{1}{2}\mathbb{Z}$.

We denote the resulting graded super loop algebra by $\widetilde{\mathfrak{g}}_k$.

We denote by $\widehat{\mathfrak{g}}_k$ the even central extension of $\widetilde{\mathfrak{g}}_k$ by adding the co-cycle

$$\psi(x\otimes P, y\otimes Q) = (x,\ y) f\big((\mathrm{d}P)Q\big),$$

where f is a linear functional on the $1-$forms of $\mathbb{C}[t,t^{-1};\theta]$ which vanishes on odd (in θ) and exact forms.

We consider derivations of $\widetilde{\mathfrak{g}}_k$ and extend them to derivations of $\widehat{\mathfrak{g}}_k$. We make this extension by requiring that these derivations kill the cocycle. Enofrcing the Liebnitz rule, we obtain, for $\epsilon = 0, 1$

$$\begin{aligned}
D^\epsilon\big(\ [x\otimes P,\ y\otimes Q]\ \big) \ &= \ [x\otimes D^\epsilon P,\ y\otimes Q] \ + \ [x\otimes (P_0+(-1)^\epsilon\theta P_1),\ y\otimes D^\epsilon Q] \\
&= \ [x,\ y]\otimes (D^\epsilon P)Q \ + \ [x,\ y]\otimes (P_0+(-1)^\epsilon\theta P_1)D^\epsilon Q \\
&\qquad + \ (x,\ y) f\big(\mathrm{d}(D^\epsilon P)Q\big)C \\
&\qquad + \ (x,\ y) f\big((P_0+(-1)^\epsilon\theta P_1)\mathrm{d}(D^\epsilon Q)\big)C \\
&= \ [x,\ y]\otimes D^\epsilon(PQ) \ + \ (x,\ y)\Big(f\big(\mathrm{d}(D^\epsilon P)Q\big) \\
&\qquad + \ f\big((P_0+(-1)^\epsilon\theta P_1)\mathrm{d}(D^\epsilon Q)\big)\Big)C
\end{aligned}$$

We wish for the coefficient of the central term in the last equation to be zero. Hence we look for the most general odd and even differentiations $D^\epsilon(\epsilon = 1, 0)$ which satisfy

$$f\Big(\mathrm{d}\big(D^\epsilon(P_0+\theta P_1)\big)Q\Big) \ + \ f\Big(\mathrm{d}\big(P_0+(-1)^\epsilon\theta P_1\big)D^\epsilon Q\Big) \ = \ 0, \tag{5.1}$$

where $Q = Q_0+\theta Q_1$ and $P_0, P_1, Q_0, Q_1\in\mathbb{C}[t,t^{-1}]$.

Here, d acts on $\mathbb{C}[t,t^{-1};\theta]$ as –

$$\mathrm{d}P(t,\theta) = \frac{\partial P(t,\theta)}{\partial t}\mathrm{d}t + \frac{\partial P(t,\theta)}{\partial\theta}\mathrm{d}\theta, \quad \forall\, P(t,\theta)\in\mathbb{C}[t,t^{-1};\theta].$$

Since $\theta^2 = 0$, we have

$$\begin{aligned}
\mathrm{d}(\theta^2) \ &= \ 0 \\
\Rightarrow\ (\mathrm{d}\theta)\theta &= -\theta\mathrm{d}\theta.
\end{aligned}$$

Let $P = t^{a_0}+\theta t^{a_1}$ and $Q = t^{b_0}+\theta t^{b_1}$, where $a_0, a_1, b_0, b_1\in\mathbb{Z}$.

Case 1: $\epsilon = 0$

Let $D^0 = \sum_{n\in\mathbb{Z}} D^0_n$ where

$$D^0_n = -\alpha_n t^{n+1}\frac{\partial}{\partial t} + \beta_n \theta t^n \frac{\partial}{\partial \theta}.$$

Evaluating the first term of Equation (5.1), we get

$$\begin{aligned}
f\big((\mathrm{d}D^0P)Q\big) &= f\Big(\mathrm{d}\big(D^0(t^{a_0}+\theta t^{a_1})\big)Q\Big)\\
&= f\left(\mathrm{d}\left(\sum_{n\in\mathbb{Z}} D^0_n\left(t^{a_0}+\theta t^{a_1}\right)\right)Q\right)\\
&= f\left(\mathrm{d}\left(\sum_{n\in\mathbb{Z}}\left(-\alpha_n a_0 t^{n+a_0} \;-\; \alpha_n a_1 t^{n+a_1}\theta \;+\; \beta_n t^{n+a_1}\theta\right)\right)Q\right)\\
&= f\Bigg(\sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0(n+a_0)t^{n+a_0-1}\mathrm{d}t \;-\; \alpha_n a_1(n+a_1)t^{n+a_1-1}\theta\mathrm{d}t\\
&\qquad\qquad -\; \alpha_n a_1 t^{n+a_1}\mathrm{d}\theta + \beta_n(n+a_1)t^{n+a_1-1}\theta\mathrm{d}t \;+\; \beta_n t^{n+a_1}\mathrm{d}\theta\big)\\
&\qquad\qquad \cdot\left(t^{b_0} \;+\; \theta t^{b_1}\right)\Bigg).
\end{aligned}$$

We simplify this by using the linearity of f, the fact that $\theta^2 = 0$ and also ignoring the terms which are odd in θ.

$$\begin{aligned}
f\big((\mathrm{d}D^0P)Q\big) &= f\Bigg(\sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0(n+a_0)t^{n+a_0+b_0-1}\mathrm{d}t \;+\; \alpha_n a_1 t^{n+a_1+b_1}\theta\mathrm{d}\theta\\
&\qquad\qquad -\;\; \beta_n t^{n+a_1+b_1}\theta\mathrm{d}\theta\big)\Bigg)\\
&= \sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0(n+a_0)f(t^{n+a_0+b_0-1}\mathrm{d}t) \;+\; \alpha_n a_1 f(t^{n+a_1+b_1}\theta\mathrm{d}\theta)\\
&\qquad\qquad -\;\; \beta_n f(t^{n+a_1+b_1}\theta\mathrm{d}\theta)\big).
\end{aligned}$$

Evaluating the second term of Equation (5.1), we get

$$\begin{aligned} f\big((\mathrm{d}P)(D^0Q)\big) \;&=\; f\left(\mathrm{d}(t^{a_0}+\theta t^{a_1})\sum_{n\in\mathbb{Z}} D^0_n(t^{b_0}+\theta t^{b_1})\right) \\ &=\; f\Bigg(\big(a_0t^{a_0-1}\mathrm{d}t \;+\; a_1t^{a_1-1}\theta\mathrm{d}t \;+\; t^{a_1}\mathrm{d}\theta\big)\sum_{n\in\mathbb{Z}}\big(-\alpha_n b_1 t^{n+b_1}\theta \\ &\qquad\qquad -\;\; \alpha_n b_0 t^{n+b_0} \;+\; \beta_n t^{n+b_1}\theta\big)\Bigg) \\ &=\; f\Bigg(\sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0 b_0 t^{n+a_0+b_0-1}\mathrm{d}t \;+\; \alpha_n b_1 t^{n+a_1+b_1}\theta\mathrm{d}\theta \\ &\qquad\qquad -\;\; \beta_n t^{n+a_1+b_1}\theta\mathrm{d}\theta\big)\Bigg) \\ &=\; \sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0 b_0 f(t^{n+a_0+b_0-1}\mathrm{d}t) \;+\; \alpha_n b_1 f(t^{n+a_1+b_1}\theta\mathrm{d}\theta) \\ &\qquad\qquad -\;\; \beta_n f(t^{n+a_1+b_1}\theta\mathrm{d}\theta)\big)\,. \end{aligned}$$

For every $k\in\frac{1}{2}\mathbb{Z}$, we make the Kac-Todorov choice for f –

$$\begin{cases} f_k\left(t^{-1}\mathrm{d}t\right) &= \alpha \\ f_k(t^{-2k}\theta\mathrm{d}\theta) &= \beta \\ f_k\left(t^n\mathrm{d}t\right) &= 0 \quad \text{if } n\neq -1 \\ f_k\left(t^n\theta\mathrm{d}\theta\right) &= 0 \quad \text{if } n\neq -2k \end{cases}, \qquad \text{where} \qquad \alpha,\beta\in\mathbb{C}\backslash\{0\}.$$

Making this choice of f in the first term, we get

$$\begin{aligned} f_k\big((\mathrm{d}D^0P)Q\big) \;&=\; \sum_{n\in\mathbb{Z}}(-\alpha_n a_0(n+a_0)\alpha\delta_{n+a_0+b_0} \;+\; \alpha_n a_1\beta\delta_{n+a_1+b_1,-2k} \\ &\qquad\qquad -\;\; \beta_n\beta\delta_{n+a_1+b_1,-2k}) \\ &=\; \alpha_{-a_0-b_0}a_0b_0\alpha \;+\; \alpha_{-2k-a_1-b_1}a_1\beta \;-\; \beta_{-2k-a_1-b_1}\beta \end{aligned}$$

Making this choice of f in the second term, we get

$$\begin{aligned} f_k\big((\mathrm{d}P(D^0Q)\big) \;&=\; \sum_{n\in\mathbb{Z}}(-\alpha_n a_0 b_0\alpha\delta_{n+a_0+b_0} \;+\; \alpha_n b_1\beta\delta_{n+a_1+b_1,-2k} \\ &\qquad\qquad -\;\; \beta_n\beta\delta_{n+a_1+b_1,-2k}) \\ &=\; -\alpha_{-a_0-b_0}a_0b_0\alpha \;+\; \alpha_{-2k-a_1-b_1}b_1\beta \;-\; \beta_{-2k-a_1-b_1}\beta \end{aligned}$$

Enforcing Equation (5.1) on the sum of the two terms, we get

$$\alpha_{-2k-a_1-b_1}(a_1+b_1)\beta \;-\; 2\beta_{-2k-a_1-b_1}\beta \;=\; 0.$$

But $\beta \neq 0$ implies,

$$\alpha_{-2k-a_1-b_1}\,(a_1+b_1) \;=\; 2\beta_{-2k-a_1-b_1}.$$

This must hold $\forall\ a_1, b_1 \in \mathbb{Z}$. Letting $n = -2k-a_1-b_1$, this must also hold $\forall\ n \in \mathbb{Z}$ i.e.,

$$\beta_n \;=\; -\alpha_n \frac{2k+n}{2}.$$

Setting $\alpha_n = 1,\ \forall\ n \in \mathbb{Z}$, we have

$$D_n^0 \;=\; -t^n \left(t\frac{\partial}{\partial t} \;+\; \left(\frac{n}{2}+k\right)\theta\frac{\partial}{\partial\theta}\right).$$

Case 2: $\epsilon = 1$

Let $D^1 = \sum_{n\in\mathbb{Z}} D_n^1$ where

$$D_n^1 \;=\; -\alpha_n t^{n+1}\theta\frac{\partial}{\partial t} \;+\; \beta_n t^n \frac{\partial}{\partial\theta}.$$

Evaluating the first term of Equation (5.1), we get

$$\begin{aligned}
f\big(\mathrm{d}(D^1P)Q\big) \;&=\; f\Big(\mathrm{d}\big(D^1(t^{a_0}+\theta t^{a_1})\big)Q\Big) \\
&=\; f\left(\mathrm{d}\sum_{n\in\mathbb{Z}}\left(-\alpha_n a_0 t^{n+a_0}\theta \;+\; \beta_n t^{n+a_1}\right)Q\right) \\
&=\; f\Bigg(\sum_{n\in\mathbb{Z}}\big(-\alpha_n a_0(n+a_0)t^{n+a_0-1}\theta\mathrm{d}t \;-\; \alpha_n a_0 t^{n+a_0}\mathrm{d}\theta \\
&\qquad\qquad +\; \beta_n(n+a_1)t^{n+a_1-1}\mathrm{d}t\big)\cdot\left(t^{b_0} \;+\; \theta t^{b_1}\right)\Bigg) \\
&=\; f\left(\sum_{n\in\mathbb{Z}}\left(\alpha_n a_0 t^{n+a_0+b_1}\theta\mathrm{d}\theta \;+\; \beta_n(n+a_1)t^{n+a_1+b_0-1}\mathrm{d}t\right)\right) \\
&=\; \sum_{n\in\mathbb{Z}}\Big(\alpha_n a_0 f\left(t^{n+a_0+b_1}\theta\mathrm{d}\theta\right) \;+\; \beta_n(n+a_1)f\left(t^{n+a_1+b_0-1}\mathrm{d}t\right)\Big).
\end{aligned}$$

Evaluating the second term of Equation (5.1), we get

$$\begin{aligned} f\big(\mathrm{d}(P_0-\theta P_1)D^1(Q)\big) &= f\left(\mathrm{d}\left(t^{a_0}-\theta t^{a_1}\right)\sum_{n\in\mathbb{Z}} D^1_n\left(t^{b_0}+\theta t^{b_1}\right)\right) \\ &= f\Bigg(\left(a_0t^{a_0-1}\mathrm{d}t \;-\; a_1t^{a_1-1}\theta\mathrm{d}t \;-\; t^{a_1}\mathrm{d}\theta\right)\sum_{n\in\mathbb{Z}}\big(-\alpha_n b_0 t^{n+b_0}\theta \\ &\qquad\qquad +\;\beta_n t^{n+b_1}\big)\Bigg) \\ &= f\left(\sum_{n\in\mathbb{Z}}\left(\beta_n a_0 t^{n+a_0+b_1-1}\mathrm{d}t \;-\; \alpha_n b_0 t^{n+a_1+b_0}\theta\mathrm{d}\theta\right)\right) \\ &= \sum_{n\in\mathbb{Z}}\Big(\beta_n a_0 f\left(t^{n+a_0+b_1-1}\mathrm{d}t\right) \;-\; \alpha_n b_0\left(t^{n+a_1+b_0}\theta\mathrm{d}\theta\right)\Big). \end{aligned}$$

Making the Kac-Todorov choice for f in the first term, we get

$$\begin{aligned} f_k\big(\mathrm{d}(D^1P)Q\big) &= \sum_{n\in\mathbb{Z}}\left(\alpha_n a_0\beta\delta_{n+a_0+b_1,-2k} \;+\; \beta_n(n+a_1)\alpha\delta_{n+a_1+b_0}\right) \\ &= \alpha_{-2k-a_0-b_1}a_0\beta \;-\; \beta_{-a_1-b_0}b_0\alpha. \end{aligned}$$

Making the choice of f in the second term, we get

$$\begin{aligned} f_k\big(\mathrm{d}(P_0-\theta P_1)D^1(Q)\big) &= \sum_{n\in\mathbb{Z}}\left(\beta_n a_0\alpha\delta_{n+a_0+b_1} \;-\; \alpha_n b_0\beta\delta_{n+a_1+b_0,-2k}\right) \\ &= \beta_{-a_0-b_1}a_0\alpha \;-\; \alpha_{-2k-a_1-b_0}b_0\beta. \end{aligned}$$

Enforcing Equation (5.1) on the sum of the two terms, we get

$$\alpha_{-2k-a_0-b_1}a_0\beta \;-\; \beta_{-a_1-b_0}b_0\alpha \;+\; \beta_{-a_0-b_1}a_0\alpha \;-\; \alpha_{-2k-a_1-b_0}b_0\beta \;=\; 0 \qquad (5.2)$$

Since this equation is valid for all choices of $a_0, a_1, b_0, b_1 \in \mathbb{Z}$, we look at the case when $a_1 + b_0 = a_0 + b_1$. Let

$$n := -2k - a_0 - b_1.$$

This gives

$$\alpha_n a_0\beta \;-\; \beta_{n+2k}b_0\alpha \;+\; \beta_{n+2k}a_0\alpha \;-\; \alpha_n b_0\beta \;=\; 0$$

$$\Rightarrow \quad \alpha_n\beta(a_0-b_0) + \beta_{n+2k}\alpha(a_0-b_0) = 0.$$

Since $a_0 \neq b_0$,

$$\beta_{n+2k}\alpha \;=\; -\alpha_n\beta.$$

Plugging this back into Equation (5.2), we get

$$\alpha_{-2k-a_0-b_1}a_0\beta \ + \ \alpha_{-2k-a_1-b_0}b_0\beta \ - \ \alpha_{-2k-a_0-b_1}a_0\beta \ - \ \alpha_{-2k-a_1-b_0}b_0\beta \ = \ 0,$$

which is an identity. Thus, there is no more information to be extracted from Equation (5.2).

Let $\alpha_n = \sqrt{\dfrac{\alpha}{\beta}}, \quad \forall\, n \in \mathbb{Z}$. Then, $\beta_{n+2k} = -\sqrt{\dfrac{\beta}{\alpha}}$.

We redefine,

$$\begin{aligned} D^1_{n+k} \ &:= \ -\alpha_n t^{n+1}\theta\frac{\partial}{\partial t} \ - \ \beta_{n+2k}t^{n+2k}\frac{\partial}{\partial\theta} \\ &= \ -\sqrt{\frac{\alpha}{\beta}}t^{n+1}\theta\frac{\partial}{\partial t} \ + \ \sqrt{\frac{\beta}{\alpha}}t^{n+2k}\frac{\partial}{\partial\theta} \\ &= \ t^n\left(\sqrt{\frac{\beta}{\alpha}}t^{2k}\frac{\partial}{\partial\theta} \ - \ t\sqrt{\frac{\alpha}{\beta}}\theta\frac{\partial}{\partial t}\right) \end{aligned}$$

Here, the subscript $n+k$ refers to the degree of the right hand side given our choice of gradation.

Notation: $[\ \cdot\ ,\ \cdot\ ]_+$ denotes the anti-commutator bracket. Also, ∂_t and ∂_θ is short-hand notation for $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial\theta}$ respectively.

Computing the Anti-commutator of two of the odd derivations, we get

$$\begin{aligned} [D^1_{n-k}, D^1_k]_+ \ &= \ [D^1_{(n-2k)+k}\ ,\ D^1_k]_+ \\ &= \ \left[\sqrt{\frac{\beta}{\alpha}}t^n\partial_\theta \ - \ \sqrt{\frac{\alpha}{\beta}}t^{n+1-2k}\theta\partial_t\ ,\ \sqrt{\frac{\beta}{\alpha}}t^2k\partial_\theta \ - \ \sqrt{\frac{\alpha}{\beta}}t\theta\partial_t\right]_+ \\ &= \ \frac{\beta}{\alpha}[t^n\partial_\theta,\ t^{2k}\partial_\theta]_+ \ - \ [t^{n+1-2k}\theta\partial_t,\ t^{2k}\partial_\theta]_+ \ - \ [t^n\partial_\theta,\ t\theta\partial_t]_+ \\ &\qquad\qquad + \ \frac{\alpha}{\beta}[t^{n+1-2k}\theta\partial_t,\ t\theta\partial_t]_+ \\ &= \ \frac{\beta}{\alpha}t^{n+2k}[\partial_\theta,\ \partial_\theta]_+ \ - \ t^{n+1-2k}[\theta\partial_t,\ t^{2k}\partial_\theta]_+ \ - \ t[t^n\partial_\theta,\ \theta\partial_t]_+ \\ &\qquad\qquad + \ \frac{\alpha}{\beta}\theta^2[t^{n+1-2k}\partial_t,\ t\partial_t]_+ \\ &= \ -t^{n+1-2k}[\theta\partial_t,\ t^{2k}\partial_\theta]_+ \ - \ t[t^n\partial_\theta,\ \theta\partial_t]_+, \end{aligned}$$

because $\theta^2 = 0$ and $[\partial_\theta,\ \partial_\theta]_+ = 2\partial_{\theta\theta} = 2\dfrac{\partial^2}{\partial\theta^2} = 0$.

Since,

$$\begin{aligned}[\theta\partial_t,\ t^n\partial_\theta]_+ &= \theta\partial_t(t^n\partial_\theta)\ +\ t^n\partial_\theta(\theta\partial_t)\\ &= t^n\theta\partial_{t\theta}\ +\ nt^{n-1}\theta\partial_\theta\ -\ t^n\theta\partial_{\theta t}\ +\ t^n\partial_t\\ &= nt^{n-1}\theta\partial_\theta\ +\ t^n\partial_t\ .\end{aligned}$$

Substituting into the previous calculation, we get,

$$\begin{aligned}[D^1_{n-k}, D^1_k]_+ &= -t^{n+1-2k}(2kt^{2k-1}\theta\partial_\theta\ +\ t^{2k}\partial_t)\ -\ t(nt^{n-1}\theta\partial_\theta\ +\ t^n\partial_t)\\ &= -t^n(2t\partial_t\ +\ (n+2k)\theta\partial_\theta)\\ &= 2D^0_n.\end{aligned}$$

Therefore,

$$D^0_n = \frac{1}{2}[D^1_{n-k},\ D^1_k]_+.$$

We consider the universal central extension of the space of all derivations which kill the cocycle ψ, given by the basis D^0_n, D^1_{n+k}, where $n \in \mathbb{Z}$. We denote this central extension by SV_k, the super Virasoro algebra. Also, relabelling

$$\begin{aligned}L_n &:= D^0_n\\ G_{n+k} &:= D^1_{n+k}, \quad \forall\ n \in \mathbb{Z},\end{aligned}$$

and setting $\frac{\alpha}{\beta} = 1$, the commutation relations for SV_k are –

$$\begin{aligned}[L_n,\ L_m] &= (n-m)L_{n+m}\ +\ \frac{c}{12}n(n^2-1)\delta_{n+m}\\ [G_{m+k},\ L_n] &= \left(m+k-\frac{n}{2}\right)G_{m+n+k}\\ [G_{m+k},\ G_{n-k}] &= 2L_{m+n}\ +\ \frac{c}{3}\left((k+m)^2-\frac{1}{4}\right)\delta_{m+n}\end{aligned}$$

, $\forall\ m, n \in \mathbb{Z}$ and $k = 0$ or $\frac{1}{2}$

Since G is a compact Lie group, we know that $\mathfrak{g}$ is finite dimensional, with dimension d_G, say. Let $x_1, x_2, \ldots, x_{d_G}$ be a basis for $\mathfrak{g}$ and let C_2 denote the eigenvalue of the Casimir operator for the adjoint representation of G, i.e.

$$\sum_{s=1}^{d_G}\sum_{t=1}^{d_G} f_{sat}f_{sbt}\ =\ C_2\delta_{ab}\ ,\quad \forall\ a, b \in \{1, 2, \ldots, d_G\},$$

where f_{abc} represents the structure constant given by,

$$[x_a,\ x_b]\ =\ \sum_{c=1}^{d_G} f_{abc}x_c,\quad \forall\ a, b \in \{1, 2, \ldots, d_G\}.$$

Also, from the definition of the Casimir operator, we have

$$(x_a,\ x_b)\ =\ -C_2\delta_{ab},\quad \forall\ a, b \in \{1, 2, \ldots, d_G\}.$$

Hence, we define a basis for the super-extended Kac-Moody Lie algebra $\widehat{g}_k$ by

$$\begin{aligned} Q^a_n &= ix_a \otimes t^n, \\ h^a_{n+k} &= ix_a \otimes t^n \theta. \end{aligned}$$

obeying the commutation relations –

$$\begin{aligned} [h^a_{n+k},\ h^b_{m-k}]_+ &= \frac{\lambda}{2}\delta_{m+n}\delta_{ab}, \\ [Q^a_n,\ h^b_{m+k}] &= if_{abc}h^c_{n+m+k}, \\ [Q^a_n,\ Q^b_m] &= if_{abc}Q^c_{n+m} + \frac{\lambda}{2}n\delta_{n+m}\delta_{ab}, \end{aligned}$$

where $\alpha = \beta = \dfrac{\lambda}{2C_2}$.

We can now define $S_k(G)$ to be the superconformal current algebra, obtained as the semidirect sum of the superalgebra SV_k and $\widehat{\mathfrak{g}}_k$ i.e.,

$$S_k(G) := \widehat{\mathfrak{g}}_k \ltimes SV_k.$$

with the Lie bracket defined as –

$$[x \otimes P + \mathfrak{D}_1,\ y \otimes Q + \mathfrak{D}_2] = [x \otimes P,\ y \otimes Q] + y \otimes \mathfrak{D}_1 Q - x \otimes \mathfrak{D}_2 P, \quad (5.3)$$

for every $x, y \in \mathfrak{g}$; $\quad P, Q \in \mathbb{C}[t, t^{-1}; \theta]$; $\quad \mathfrak{D}_1, \mathfrak{D}_2 \in SV_k$.

This allows us to evaluate the Lie bracket of the basis of $\widehat{\mathfrak{g}}_k$ with that of SV_k, i.e.

$$\begin{aligned} [h^a_{m+k},\ L_n] &= \left(m + k + \frac{n}{2}\right) h^a_{m+n+k}, \\ [Q^a_m,\ L_n] &= mQ^a_{m+n}, \\ [h^a_{m+k},\ G_{n-k}]_+ &= Q^a_{m+n}, \\ [Q^a_m,\ G_{n+k}] &= mh^a_{m+n+k}, \end{aligned}$$

$\forall\ m, n \in \mathbb{Z}$; $\quad a \in \{1, 2, \ldots, d_G\}$; $\quad k = 0$ or $\frac{1}{2}$.

6. Unitary Representation of the Superconformal Algebra

Recall that

$$[h^a_r, h^b_s]_+ = \delta_{a,b}\delta_{r+s},$$

where $r, s \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ and $a, b \in \{1, 2, \ldots, d_G\}$.

We define the normal ordering –

$$: h^a_r h^b_s : = \begin{cases} h^a_r h^b_s \,, & \text{if} \quad r < s \\ -h^b_s h^a_r \,, & \text{if} \quad r > s \\ \frac{1}{2}[h^a_r, h^b_s]_+ \,, & \text{if} \quad r = s. \end{cases}$$

Define $H^a(z) := \sum_r h^a_r z^{-r}$, where $r \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$.

Also, define

$$\begin{aligned} Q^a(z) &= \sum_n Q^a_n(z) z^{-n} := \frac{-i}{2} f_{abc} : H^b(z) H^c(z) : \\ &= \frac{-i}{2} f_{abc} : \sum_{r,s} h^b_r h^c_s z^{-r-s} : , \quad \text{where } r, s \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2} \\ &= \frac{-i}{2} f_{abc} \sum_{r,s} : h^b_r h^c_s : z^{-r-s} \\ &= \frac{-i}{2} f_{abc} \sum_r \left(\left(\sum_{s>r} h^b_r h^c_s \; - \sum_{s<r} h^c_s h^b_r \right) z^{-r-s} \; + \; \frac{1}{2} [h^b_r, h^c_r]_+ z^{-2r} \right) \\ &= \frac{-i}{2} f_{abc} \sum_r \left(\left(\sum_{s>r} h^b_r h^c_s \; - \sum_{s<r} h^c_s h^b_r \right) z^{-r-s} \; + \; \frac{1}{2} \delta_{b,c} \delta_{2r} z^{-2r} \right) \\ &= \frac{-i}{2} f_{abc} \left(\sum_r \sum_{s>r} h^b_r h^c_s \; - \sum_s \sum_{r>s} h^c_s h^b_r \right) z^{-r-s} \\ &= \frac{-i}{2} f_{abc} \left(\sum_r \sum_{s>r} h^b_r h^c_s \; - \sum_r \sum_{s>r} h^c_r h^b_s \right) z^{-r-s} \\ &= \frac{-i}{2} f_{abc} \left(\sum_r \sum_{s>r} h^b_r h^c_s \; + \sum_r \sum_{s>r} h^b_r h^c_s \right) z^{-r-s} \\ &= -i f_{abc} \sum_r \sum_{s>r} h^b_r h^c_s z^{-r-s} \\ &= -i f_{abc} \sum_r \sum_{n>2r} h^b_r h^c_{n-r} z^{-n} \\ &= -i f_{abc} \sum_n \sum_{r<\frac{n}{2}} h^b_r h^c_{n-r} z^{-n} \end{aligned}$$

Therefore,

$$Q^a_n = -i f_{abc} \sum_{r<\frac{n}{2}} h^b_r h^c_{n-r}.$$

Note that $n \in \mathbb{Z}$, since $n = r + s$.