A Tale of Two Matrices

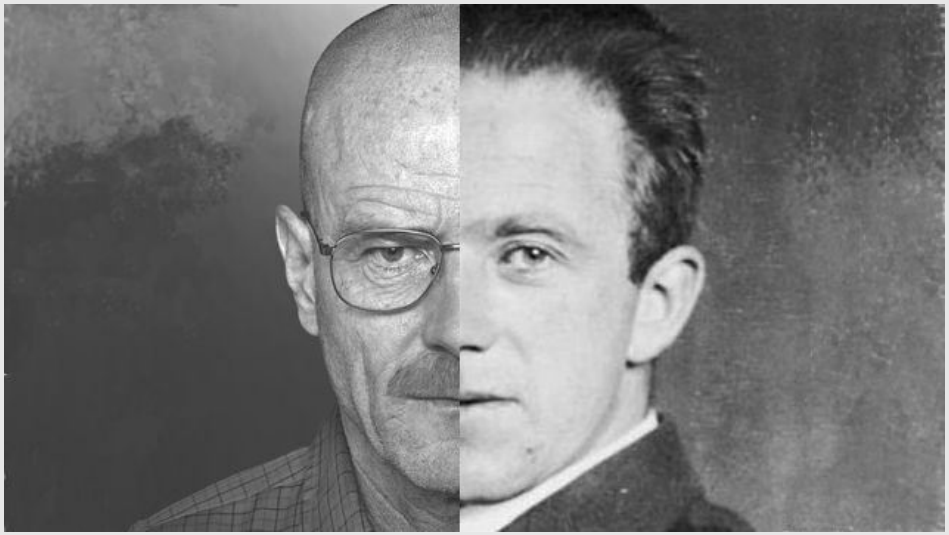
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Supersisor: Hadi Salmasian, Ottawa

Fields Institute

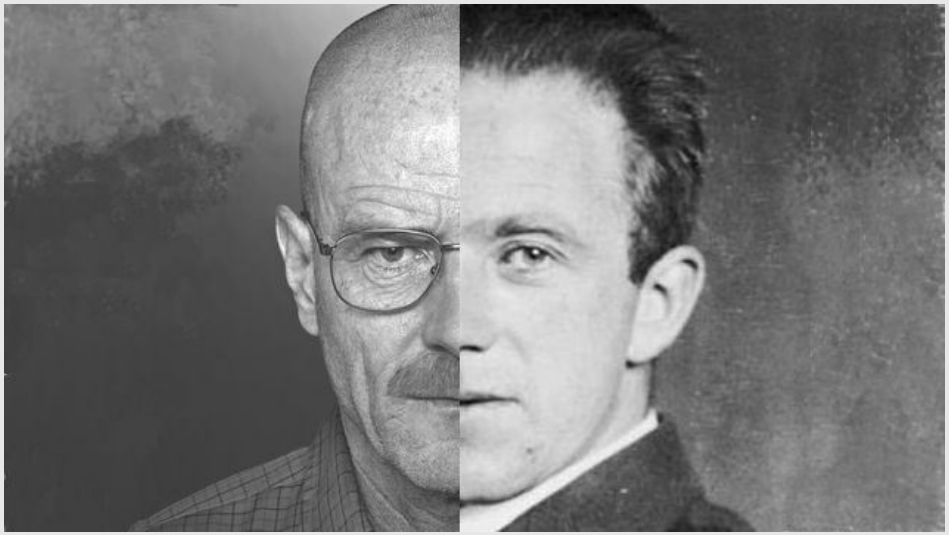
August 29, 2015

What Can Two Matrices Tell Us About...

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Introducing Our Matrices

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Natural Next Step: Multiply

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Why Not Take Linear Combinations?

$$pP + xX + zZ = \begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} := m(p, x, z)$$

where p, x, $z \in \mathbb{R}$

Result: A Lie Algebra

$$\left\{ \begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad p, x, z \in \mathbb{R} \right\} = \{ m(p, x, z) : p, x, z \in \mathbb{R} \}$$

is a **Lie Algebra**.

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  1. $[x, x] = 0$

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  3. $[\cdot, \cdot]$ is linear in both arguments

Examples of Lie Algebras

- The set M_n of all $n \times n$ matrices, with bracket:

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Heisenberg Algebra $\mathfrak{h}$

- The set of all matrices of the form $\begin{pmatrix} 0 & p & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} = m(p, x, z)$ with bracket
$$[A, B] = AB - BA.$$
$$\boxed{[(p, x, z), (p', q', z')] = (0, 0, px' - xp')}$$

Commutation Relations of Heisenberg Algebra $\mathfrak{h}$

Basis Matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$[P, X] = PX - XP = Z$$

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$$[P, X] = PX - XP = Z$$

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Algebras vs. Groups

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Definition: The Exponential Map

Given a matrix Lie algebra defined by $\{A\}$, we obtain a Lie group by the matrix exponential:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots.$$

Let's Exponentiate!

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Take Linear Combinations

$$\mathcal{H} = pP_{\mathcal{H}} + xX_{\mathcal{H}} + zZ_{\mathcal{H}}.$$

Let's Exponentiate!

Result: A Lie Group, $\mathcal{H}$

$$\mathcal{H} = \{\exp(\mathfrak{h})\} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad p, x, z \in \mathbb{R} \right\} := \{M(p, x, z)\}$$

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- TL;DR A Lie group has algebraic and differential structure

Commutation Relations Of Heisenberg Group $\mathcal{H}$

Basis Matrices

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In Summary...

In both $\mathfrak{h}$ and $\mathcal{H}$:

$$[P_{\mathcal{H}}, X_{\mathcal{H}}] = Z_{\mathcal{H}}$$
$$[P_{\mathcal{H}}, Z_{\mathcal{H}}] = 0$$
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What is a representation?

A representation π of a group G is a homomorphism from G to the group $GL(V)$, where $GL(V)$ is the set of all invertible linear maps (operators) $T : V \to V$. That is:

$$\pi : G \to GL(V)$$

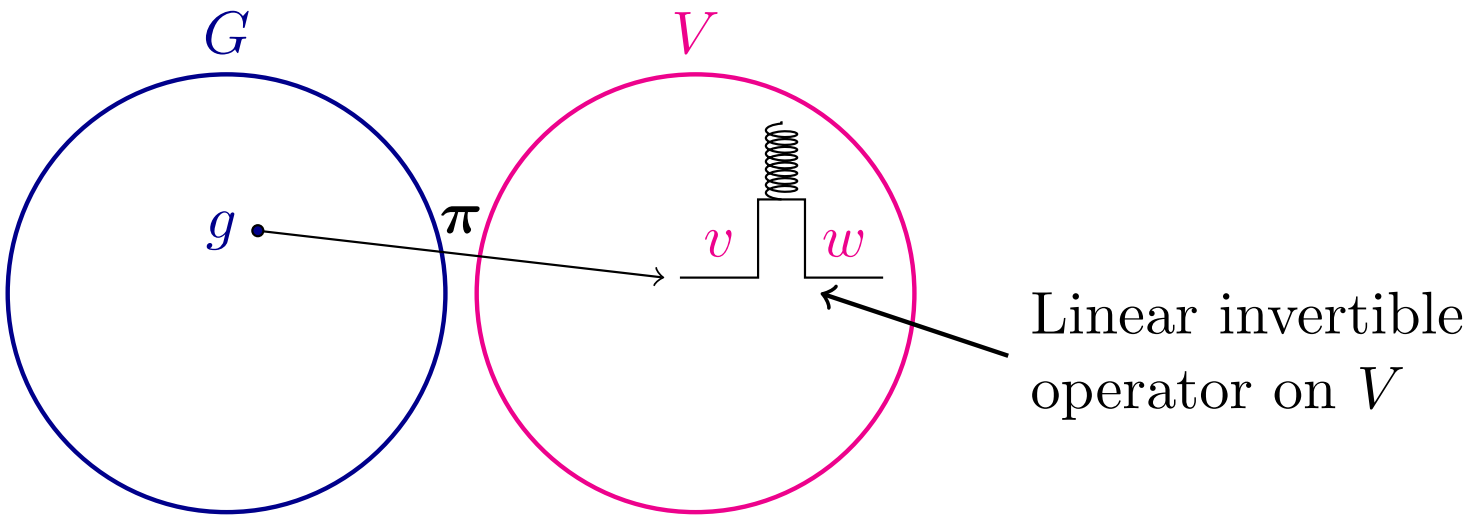
$$\pi(g)(v) = w, \quad \text{for some } v, w \in V, g \in G$$

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$$

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Unitary Representation

The representation π is called *unitary* if for every $g \in G$ the operator $\pi(g)$ is unitary on V, i.e.,

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A closed subspace $W \subset V$ is called *invariant* for π if $\pi(g)W \subset W$ for every $g \in G$.

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Irreducible Representation

The representation π is called *irreducible* if there is no proper (closed) invariant subspace, i.e., the only (closed) invariant subspaces are 0 and V itself.

Heisenberg Group

Let's go back to the Heisenberg group:

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 & p & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \middle| \; p, x, z \in \mathbb{R} \right\}$$

The Center of The Heisenberg Group

The center of $\mathcal{H}$ is

$$Z(\mathcal{H}) := \{(0, 0, z) \mid z \in \mathbb{R}\}$$

Representation Space $L^2(\mathbb{R})$

We will now focus on the representations of the Heisenberg group. As our representation space "V", we choose the infinite-dimensional function space $L^2(\mathbb{R})$ where the functions satisfy the following property:

$$\|f\|_2^2 := \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

Inner Product Space

$L^2(\mathbb{R})$ is an inner product space. That is, it is a vector space provided with an *inner product* which is given for two functions $f, g \in L^2(\mathbb{R})$ as follows:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(\xi)\overline{g(\xi)}\, d\xi$$

Rotation Operator

The rotation operator $R(x)$ for a function $f(\xi) \in L^2(\mathbb{R})$ is defined as follows:

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Remark

The rotation and translation operators are unitary. In addition, they form one-parameter unitary groups as

$$\{R(x)\}_{x\in\mathbb{R}}, \{T(p)\}_{p\in\mathbb{R}}$$

Schrödinger Representation

For $(p, x, z) \in \mathcal{H}$ we define the *unitary representation* $\pi_k(p, x, z)$ on $L^2(\mathbb{R})$ by

$$\pi_k(p, x, z)f(\xi) := e^{i(x\xi+z)k} f(\xi + p).$$

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Notice that

$$\begin{aligned}\pi(p, x, z)f(\xi) &= e^{i(x\xi+z)} f(\xi + p) \\ &= e^{ix\xi} \cdot e^{iz} \cdot f(\xi + p) \\ &= R(x)\chi(z)T(p)\end{aligned}$$

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And now, hold on to your seats... Here is the celebrated theorem!

The Stone-von Neumann Theorem

Theorem: Stone-von Neumann

For $k \neq 0$ the unitary representation π_k is irreducible. Every irreducible unitary representation of $\mathcal{H}$ with central character χ_k is isomorphic to π_k.

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What Now?

- Show that the Schrödinger representations contains the foundations of quantum mechanics
- Bask in our knowledge of the one-ness of the universe

A Closer Look at R and T

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First, setting the scene:

$$\pi(p, 0, 0) = \pi(pP_{\mathcal{H}}) = T(p) \qquad \pi(0, x, 0) = \pi(xX_{\mathcal{H}}) = R(x)$$

$$pP_{\mathcal{H}} = p\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad xX_{\mathcal{H}} = x\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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- We associate the matrices $P_{\mathcal{H}}$ and $X_{\mathcal{H}}$ with the operators $T(p)$ and $R(x)$, respectively.
- **Important:** $P_{\mathcal{H}} \leftrightarrow T(p)$ and $X_{\mathcal{H}} \leftrightarrow R(x)$.

Stone's Theorem (Simplified)

Every group of single-parameter unitary operators $\{U(s)\}_{s\in\mathbb{R}}$ can be uniquely associated with a self-adjoint operator A, such that

$$U(s) = e^{isA} = \sum_{k=0}^{\infty} \frac{(isA)^k}{k!}$$

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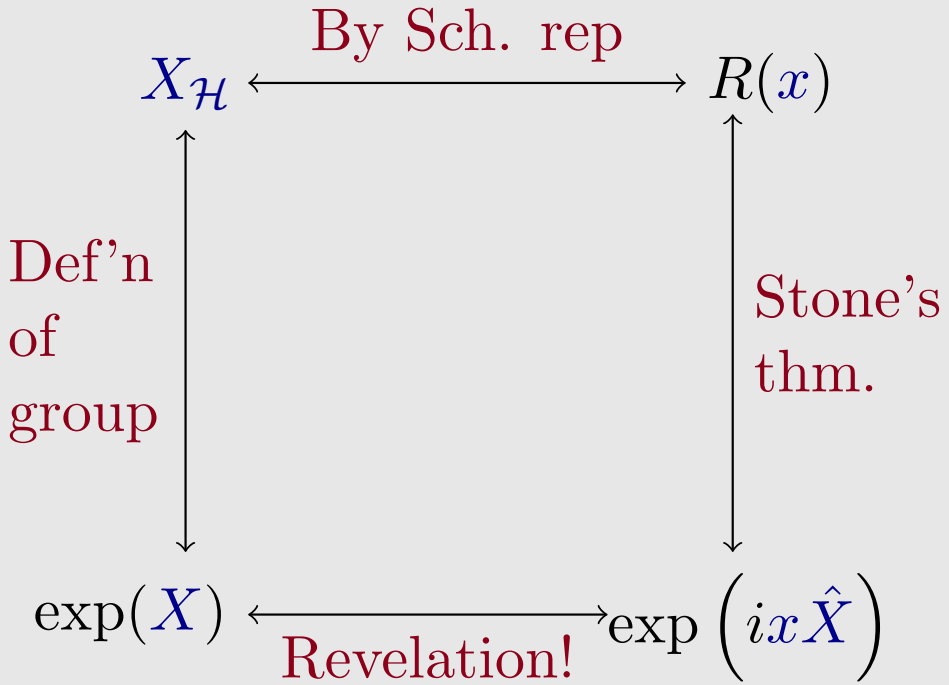
How, Though?

Easy to see for the rotation operator $R(x)$, since

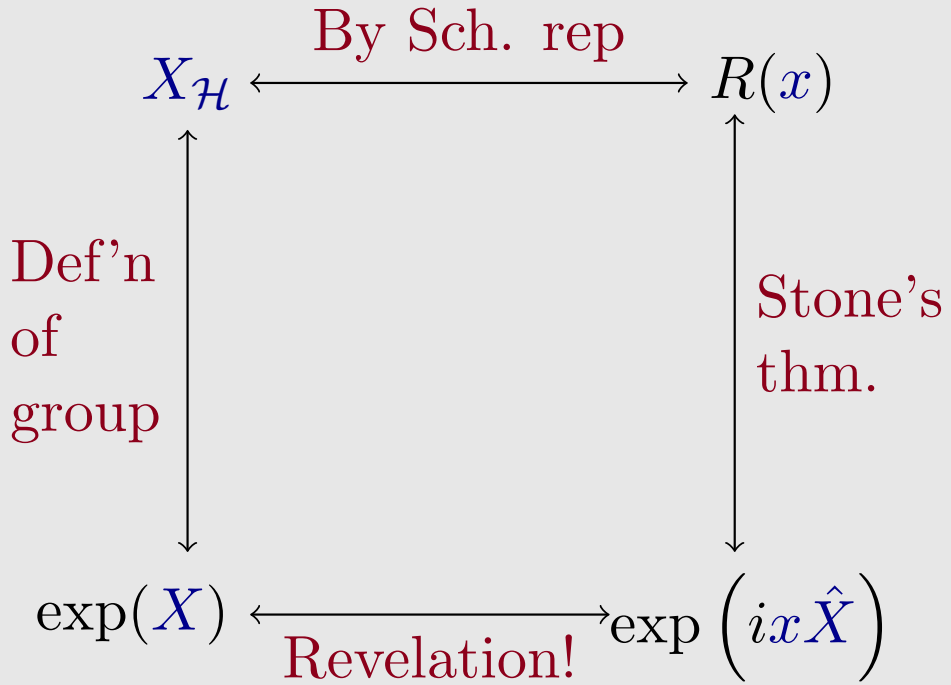
$$R(x)f(\xi) = e^{ix\xi} f(\xi),.$$

So, by Stone's Theorem, $A = \hat{X}$, where $\hat{X}$ is the operator that multiplies a function by its argument, i.e. $\hat{X}f(\xi) = \xi f(\xi).$)

A Revelation (In Pictures)

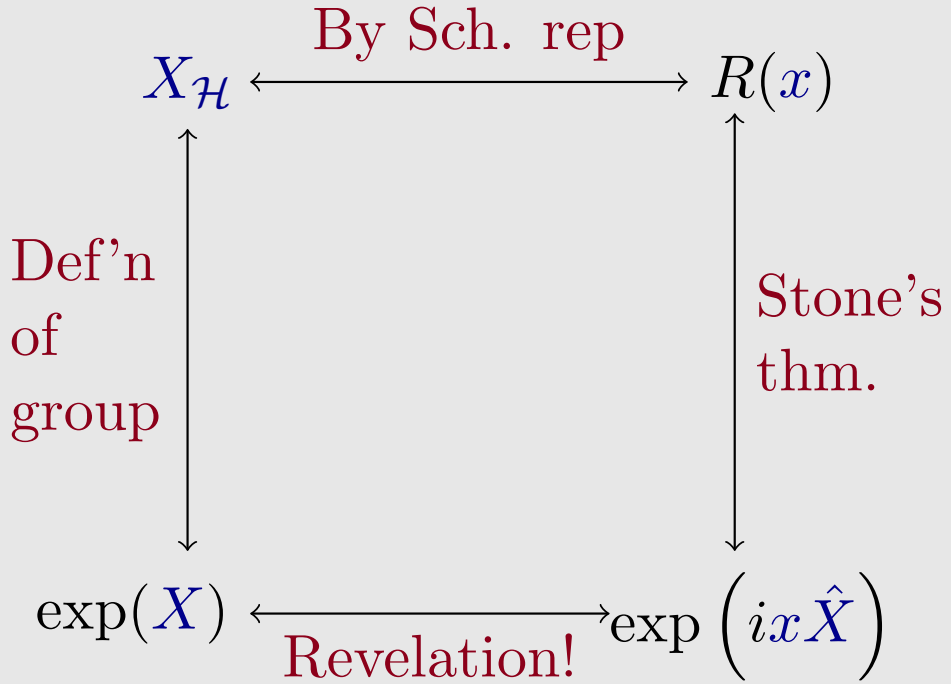


A Revelation (In Pictures)



A Revelation (In Words)

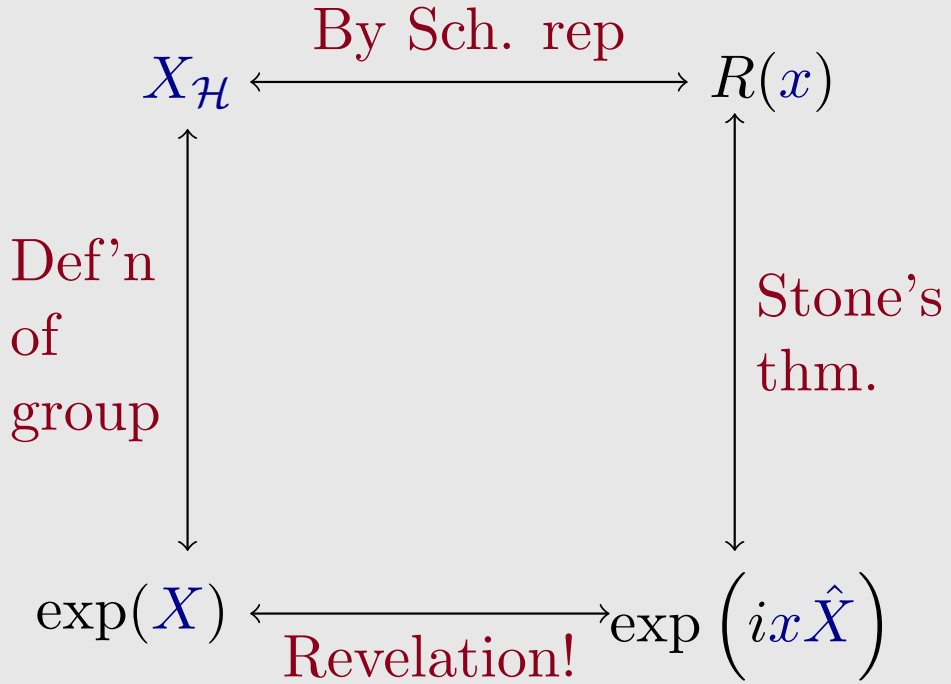
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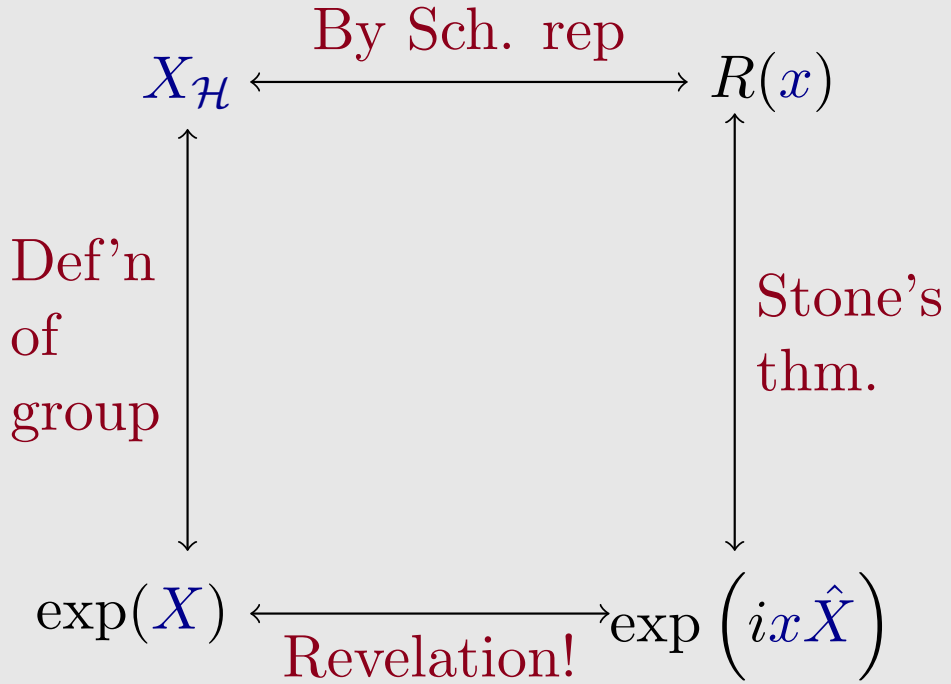
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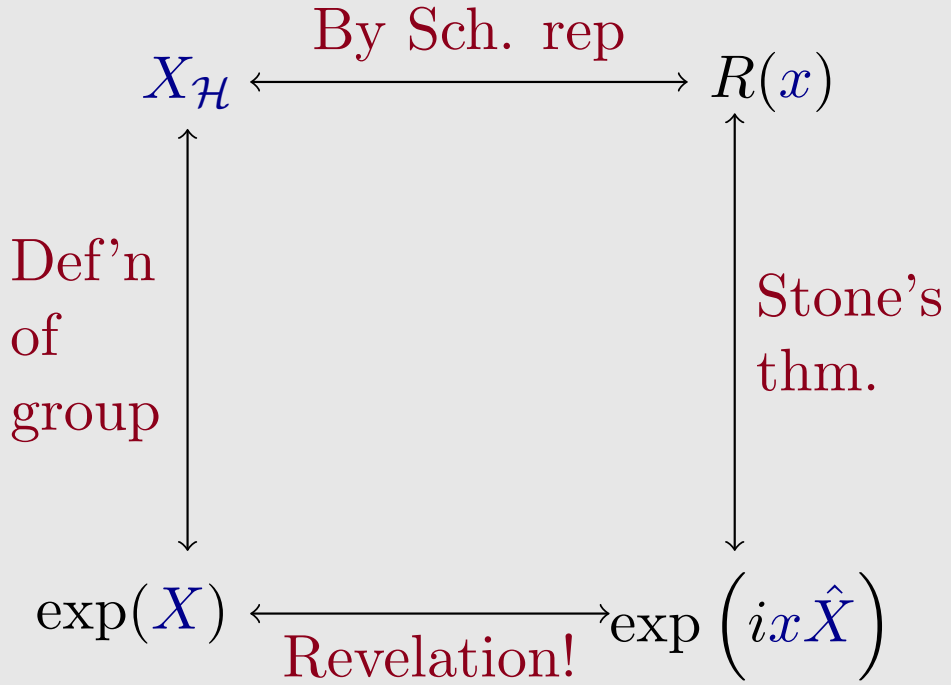
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Repeating For P

It turns out that:

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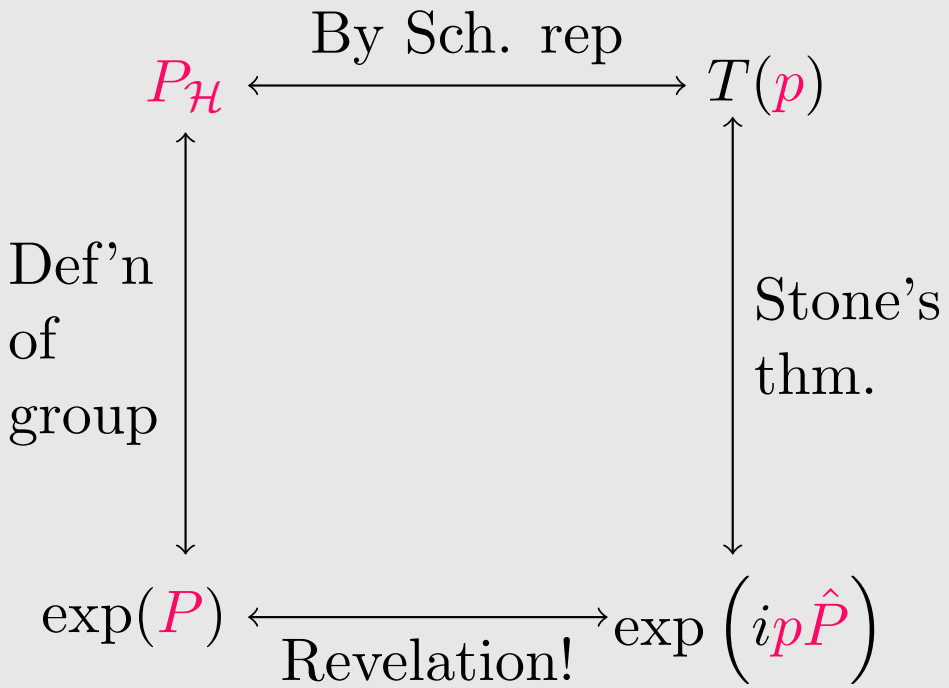
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If You Learn Nothing Else Today...

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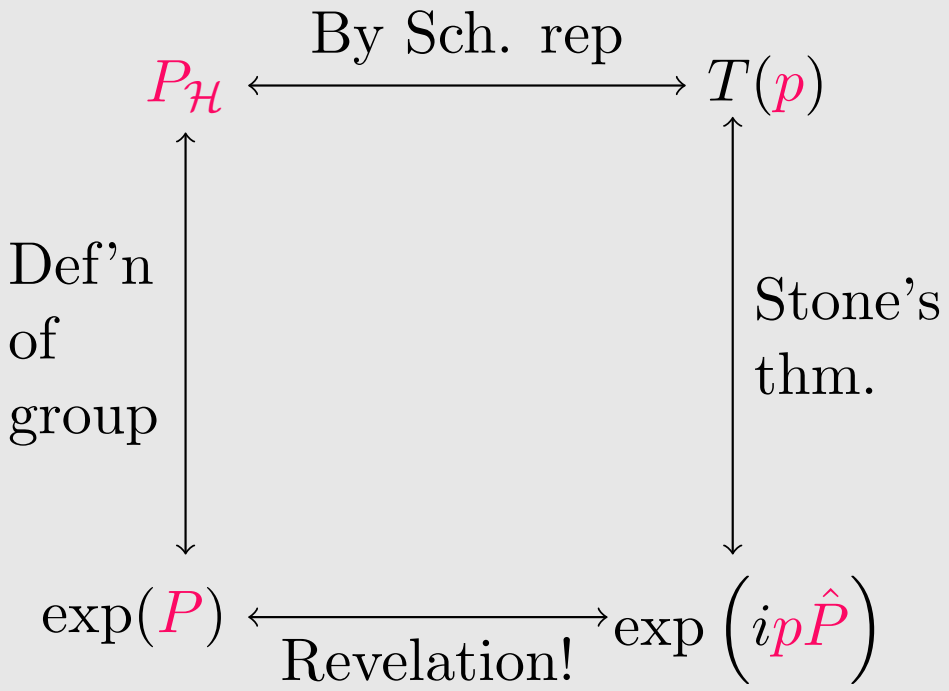
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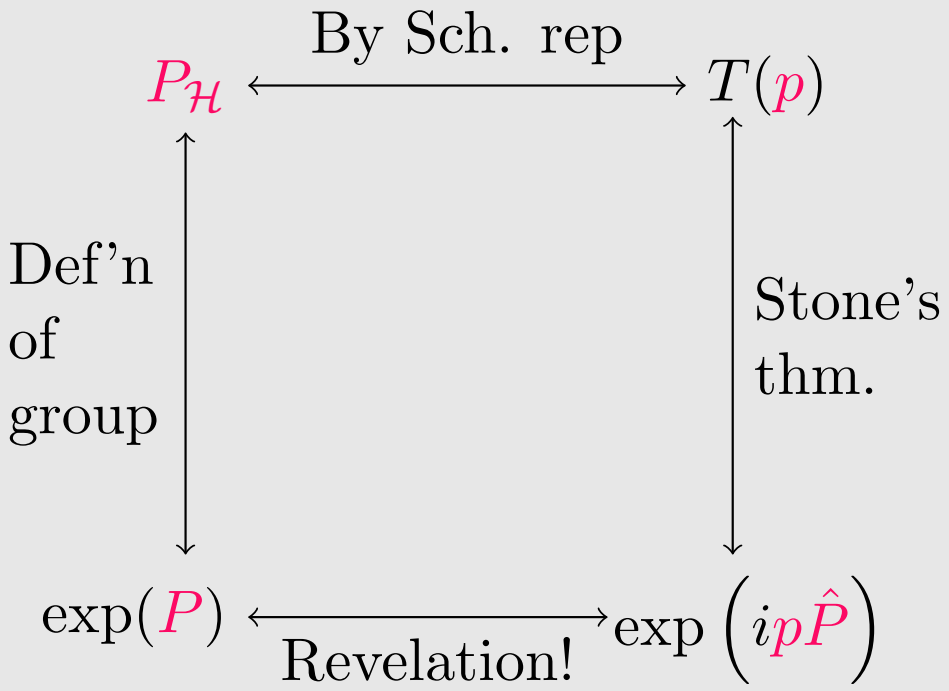
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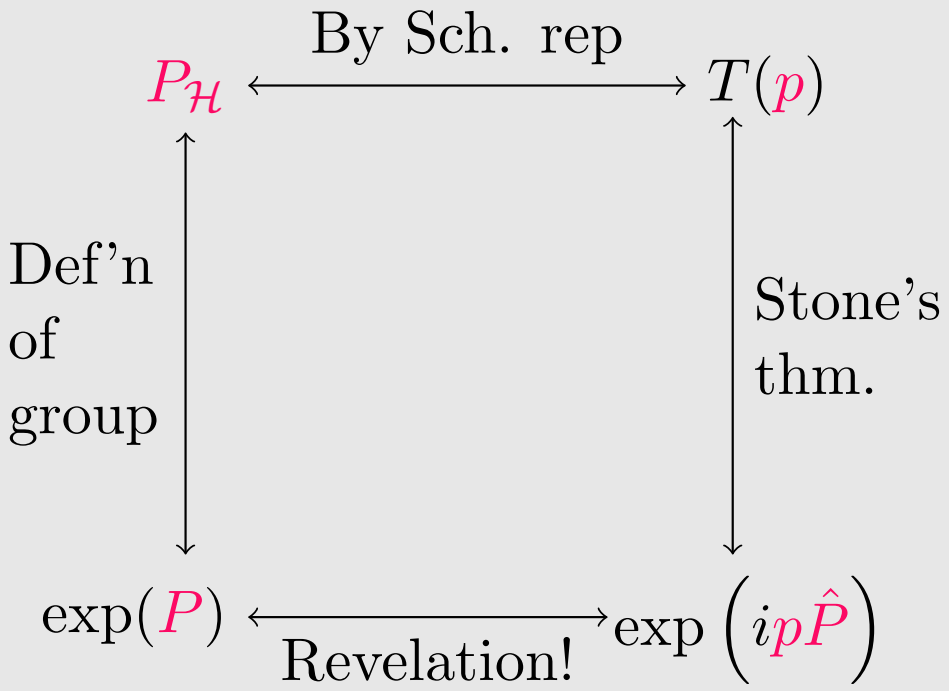


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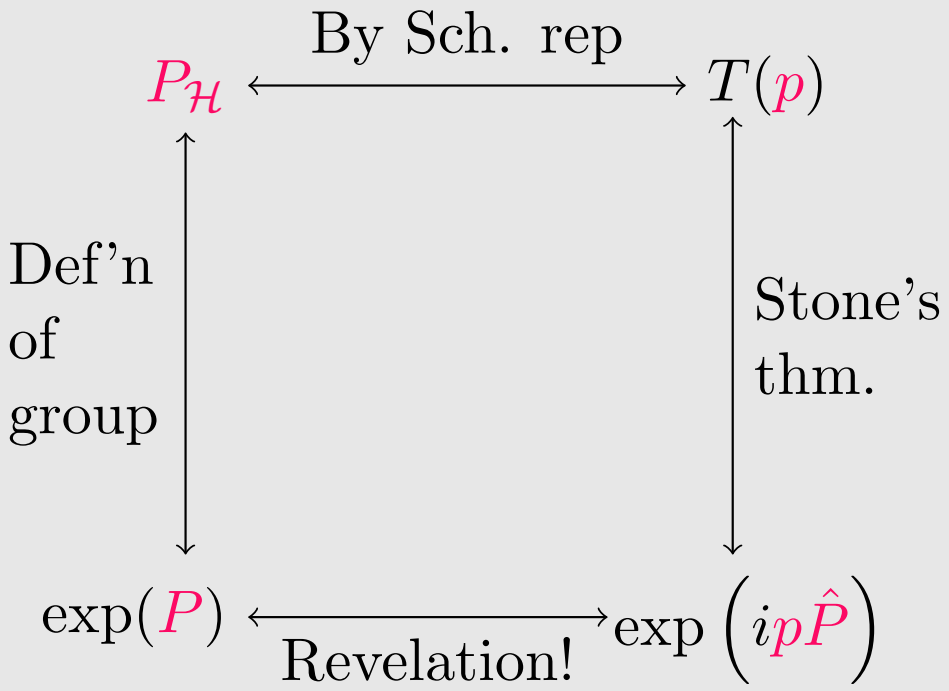


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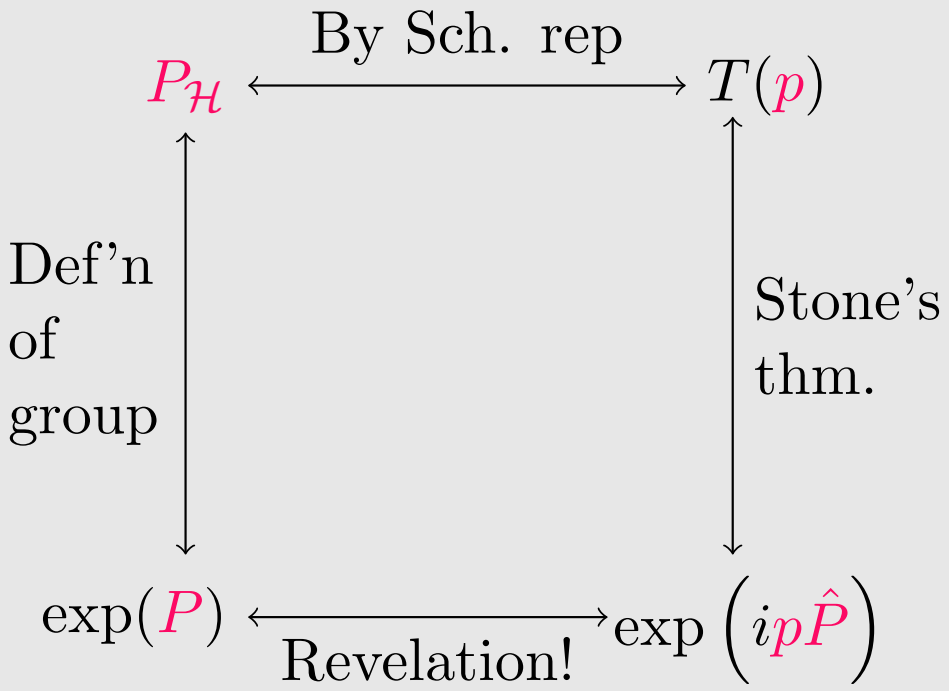


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We often call $\hat{P}$ the **momentum operator** and $\hat{X}$ the **position operator**. We'll find out why shortly!

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Success!

We have now successfully represented our magical matrices as operators on $L^2(\mathbb{R})$. The Stone-von Neumann theorem tells us that this is essentially the only way to construct them.

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In Summary...

$$\left[\hat{P}, \hat{X}\right] = \hat{Z}$$
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The canonical commutation relation lives!

Another Way of Getting the Momentum and Position Operators

Partial Derivative by p

$$\frac{\partial}{\partial p}\pi(p,0,0)\Big|_{p=0} = \frac{\partial}{\partial p}f(\xi+p)\Big|_{p=0} = \lim_{p\to 0}\frac{f(\xi+p)-f(\xi)}{p} = \frac{d}{d\xi}f(\xi)$$

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Partial Derivative by x

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$$\hat{X}(\hat{X}(f)) + \hat{P}(\hat{P}(f))$$

or

$$\left[x^2 - \frac{d^2}{dx^2}\right] f(x)$$

We call this Laplacian the **Hermite Operator**.

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$$Hh_k = \frac{2k+1}{2\pi}h_k$$

Definition: Fourier Transform

Given by

$$\mathcal{F}(f) = \langle f(x), e^{2\pi i \xi x} \rangle = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$$

So this will give me the strength of the function in each linear frequency, k.

Definition: Fourier Transform

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$$\mathcal{F} h_k = (-i)^k h_k$$

Plancherel Theorem

$$\langle f, f \rangle = \langle \mathcal{F}f, \mathcal{F}f \rangle$$

Properties of the Fourier Transform

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Derivative as Multiplication

$$\mathcal{F}\frac{df(x)}{dx} = 2\pi i \xi \{\mathcal{F}f(x)\}(\xi)$$

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$$\mathcal{F}\hat{P} = 2\pi \hat{X}\mathcal{F}$$

This means that the position and momentum operators are called **Fourier Conjugates** of each other.

Where We Are Going: Heisenberg's Uncertainty Principle and Quantum Mechanics

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- This implies that the sum of momentum and position will always be bounded below.

The Heisenberg Uncertainty Principle

$$\|\hat{P}f\|_2^2 + \|\hat{X}f\|_2^2 \geq \frac{1}{2\pi}\|f\|_2^2$$

The Derivation

$$\langle Hf, f\rangle$$

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\end{aligned}$$

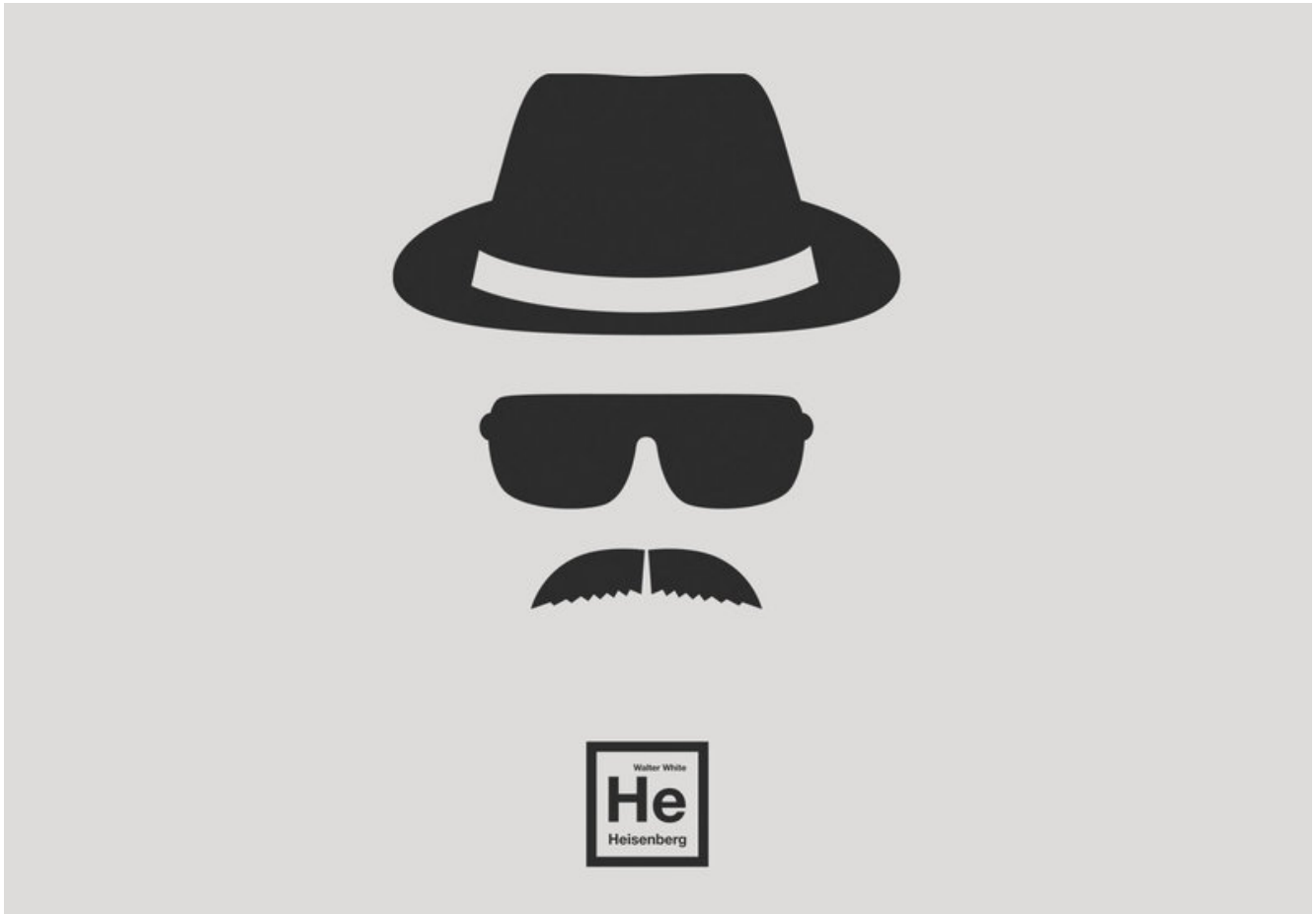
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&\geq \frac{1}{2\pi}\sum_{k=0}^{\infty}|\langle f, h_k\rangle|^2 = \frac{1}{2\pi}\|f\|_2^2
\end{aligned}$$



Thank you!

The Inequality

We attempted to find a shorter proof for the expression

$$\||x|^{\alpha} f(x)\|_p + \||y|^{\beta} \hat{f}(y)\|_q \geq K\|f\|_2$$

which is a known generalization of the Heisenberg Uncertainty Principle.