Symmetric Polynomials, Combinatorics and Mathematical Physics

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Abstract

Symmetric polynomials have interesting connection with mathemmatical physics. The Jack polynomials are connected to the Hamiltonian of the quantum n body systems. In this work, we present this link between Jack polynomials and the CMS operator. The Kerov's map translates the Jack functions into super Jack polynomials. The supersymmetric version of the Jack polynomials turn out to the eignfunctions of the deformed CMS operators.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Evans Clifford Boadi, 19 May 2016

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1. Introduction

An n-variable polynomial $f(x_1, ..., x_n)$ is called symmetric if it does not change by any permutation of its variables. The symmetric n-variable polynomials form a ring. The ring of symmetric functions plays an important role in mathematics and mathematical physics. It has several application in algebra [Aguiar et al. (2012)], combinatorics [Stanley (1989)], presentation theory of symmetric groups, general linear groups [Green (1955)], and geometry [Helgason, 2001]. For example, a distinguished family of symmetric polynomials called Schur polynomials, that are indexed by combinatorial objects called Young diagrams, describes the character theory of the group S_n (symmetric group on n letters). The Schur polynomials and their generalizations such as Jack and Macdonald polynomials [Macdonald (1995)] are related to geometric objects such as symmetric spaces and flag varieties. They have also found connection with representation theory of super Lie algebras [Sahi and Salmasian (2015), Sergeev and Veselov (2004)]

Besides their connection with representation theory, symmetric functions also have an application to mathematical physics. They are applied in Boson-Femion correspondence which are applied in string theory [Green et al. (2012)] and integrable systems [Miwa et al. (2000)]. There is also an interesting connection to quantum physics: the Jack polynomials are the eigenstates of the Hamiltonian of the quantum n-body problem. For example, in [Sergeev and Veselov (2005)], the Jack polynomials were shown to be the eigenfunctions of the Calogero Moser Schortland operators.

Their application in super symmetry is made possible by a certain homomorphism map called Kerov's map [Kerov et al. (1997)].

The goal of this project is to study the Jack polynomials and their variants. We also discuss the relation of the Jack symmetric functions in the ring of symmetric functions and it extension to the ring of super symmetric functions. We will also study the connection of the Jack symmetric polynomials and mathematical physics; their relation with the CMS operator.

This project is organized as follows: First, we review facts from the theory of symmetric polynomials. Some properties of the symmetric polynomials are studied. We also define the ring of symmetric functions in infinitely many variables, which is a project limit of the ring of symmetric functions in n variables.

In chapter three, we center our study on Jack symmetric polynomials and discuss some of their properties. We study the relationship of the Jack polynomials with the CMS operator. Here we show that they are eigenfunctions of the CMS operators.

In chapter four, our main discussion is based on the super Jack polynomials; the application of Kerov's map from the ring of symmetric functions to the ring of super Jack symmetric polynomials. We mention that the super Jack polynomials are the eigenfunctions of the deformed CMS operators which preserve the algebra.

2. Ring of Symmetric Functions

In this chapter, we review standard facts about symmetric functions and Schur polynomials. Our main reference is Macdonald (1995).

2.1 Preliminaries

2.1.1 Definition. A group homomorphism between two groups G_1 and G_2 is a map $\rho : G_1 \longrightarrow G_2$ satisfying the following property:

$$\rho(g_1.g_2) = \rho(g_1).\rho(g_2)$$

for all $g_1, g_2 \in G_1$.

2.1.2 Definition (Group actions)**.** Let G be a group. A G-action on a set S is a map $G \times S \longrightarrow S$ such that $(g, s) \in G \times S \mapsto g.s \in S$, which satisfies the following properties:

i. $g_1.(g_2.s) = (g_1.g_2).s$, for $g_1, g_2 \in G, s \in S$;

ii. $e.s = s$, for all $s \in S$.

2.1.3 Definition. Let R be a ring. For $a \in R$, a graded ring is a ring R with decomposition $R = \oplus_{n\in\mathbb{Z}} R_n$ such that each R_n is closed under addition, and

$$R_m \cdot R_n \subset R_{m+n}$$

for non-negative integers n, m.

2.1.4 Definition. A ring R is said to be generated by elements $a_1, \ldots, a_n \in R$ over $\mathbb{C}$ if every element can be uniquely written as a noncommutative polynomials in $a_1, \ldots, a_n$ with complex coefficients. So that

$$R = \mathbb{C}[a_1, \ldots, a_n]$$

2.1.5 Definition (Polynomial ring)**.** Let R be a commutative ring with identity. Let $f\colon R \to R$ such that for any indeterminate, $x \in R$ the formal sum $f(x) = a_0 + a_1x + \ldots + a_nx^n$ where $a_0, a_1, \ldots, a_n \in R$ is called a polynomial in x with coefficients in R. Under these operations, the ring $R[x]$ of polynomials in x with coefficients in R is called a polynomial ring with identity.

2.1.6 Definition. Let R and S be rings. A map $f : R \to S$ is called a ring homomorphism if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$ and $f(1_R) = 1_S$.

2.1.7 Definition. A ring homomorphism, f is called an isomorphism if the map $f\colon R \to S$ is a bijection. Then, R and S are isomorphic rings.

We now review some notions of partition of a set.

2.1.8 Definition. A partition is a sequence of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots,)$ of nonnegative intergers and decreasing order $\lambda_i \geq 0$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$$

with finitely many non-zero terms.

The non-zero terms in λ are called parts. The length of λ, denoted by $l(\lambda)$ is the number of nonzero terms.

2.1.9 Definition. The weight of a partition λ denoted $|\lambda|$ is defined as the sum of the parts.

$$|\lambda| = \sum_{i=1}^{\infty} \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \cdots$$

We say that λ is a partition of n if $|\lambda| = n$.

Notation
Let $\mathcal{P}$ be the set of all partitions and denote by $\mathcal{P}_n$, the set of all partitions of n. Then, $\mathcal{P}_0$ consists of a single element, the unique partition of zero.

2.1.10 Definition. The multiplicity of $i \in \lambda$, denoted by m_i is defined by

$$m_i = m_i(\lambda) = \mathsf{card}\{j \in \mathbb{N} \colon \lambda_j = i\}$$

to mean exactly m_i of the parts of λ are equal to i.

2.1.11 Definition. Let λ be a partition, λ' is the conjugate partition of λ and is defined as

$$\lambda'_i = \mathsf{card}\{j \in \mathbb{N} \colon \lambda_j \geq i\}$$

The length, $l(\lambda')$ and the weight, $|\lambda'|$ of λ' are given respectively as

$$l(\lambda') = \lambda_1 \text{ and } |\lambda'| = |\lambda|$$

2.1.1 Example. Let $\lambda = (5, 4, 3, 1)$, then the conjugate of λ is $\lambda' = (4, 3, 3, 2, 1)$.

2.1.1 Lemma. For any set X, the symmetric group is the group S_X of bijections $\sigma \colon X \longrightarrow X$. The group operation is composition of functions.

Elements of the symmetric group are called permutations. The symmetric group on $\{1, \ldots, n\}$ is denoted by $\mathcal{S}_n$.

2.1.12 Definition. A superalgebra or $\mathbb{Z}_2$-graded algebra is a vector superspace or algebra over $\mathbb{C}$ decompose into a direct sum $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ equiped with a bilinear multiplication operator

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \quad \text{such that} \quad \mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}, \quad \forall\, i, j \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

That is

$$\mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}}\mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}}\mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}$$

An element $a \in \mathcal{A}$ is said to be homogeneous if $a \in \mathcal{A}_i$, for $i = 0$ or $i = 1$. We write $|a| = i$. A superalgebra is said to be supercommutative if

$$a \cdot b = (-1)^{|a| \cdot |b|} b \cdot b \quad a, b \in \mathcal{A}, \quad |a| \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

2.2 Symmetric Functions

Let $P_n = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring in n-independent variables, $x_1, \ldots, x_n$, with complex coefficients. The set of symmetric polynomials Λ_n forms a subring of P_n. The symmetric group, $\mathcal{S}_n$ acts on P_n by permuting the variables.

2.2.1 Definition. The action of the symmetric group on a polynomial $f(x_1, \ldots, x_n)$ is $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

2.2.2 Definition. A polynomial function f is symmetric in $x_1, \ldots, x_n$ if

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

for every permutation σ of $\{1, \ldots, n\}$.

The space of all symmetric polynomials in $x_1, \ldots, x_n$ is denoted by Λ_n.

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a set of nonnegative integers for many positive monomials and let the shape of α be the partition defined from α by rearranging its positive entries in decreasing order.

2.2.3 Definition. A homogeneous function f is symmetric if

$$f(x) = \sum_{|\alpha|=n} C_\alpha x^\alpha$$

where $C_\alpha \in \mathbb{C}$ depends only on the shape of α and x^α means $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $\Lambda_n^k \subseteq \Lambda_n$ be a space of homogeneous symmetric polynomials in n variables of degree k. Then Λ_n is a graded ring because

$$\Lambda_n = \bigoplus\nolimits_{k \geq 0} \Lambda_n^k$$

We want to define a projective system of homomorphisms on P_n.

Let m, n be nonnegative integers, such that $m \geq n$. Let $P_m \in \mathbb{C}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]$ and $P_n \in \mathbb{C}[x_1, \ldots, x_n]$, the homomorphism,

$$P_m \to P_n \quad \text{such that} \quad x_i \mapsto \begin{cases} x_i & \forall i = 1, \ldots, n \\ 0 & \forall i = n+1, \ldots, m \end{cases}$$

Restricting the above map to $\Lambda_m^k \subseteq P_m$, we obtain the homomorphism

$$\rho_{m,n}^k : \Lambda_m^k \longrightarrow \Lambda_n^k \text{ is } \begin{cases} \text{surjective} & \forall\ k \geq 0, \quad m \geq n \\ \text{bijective} & \forall\ k \geq m \geq n \end{cases}$$

In particular, a restriction to $\Lambda_m \in P_m$ gives the homomorphism

$$\rho_{m,n} : \Lambda_m \longrightarrow \Lambda_n \tag{2.2.1}$$

Consider the inverse limit on Λ_n, such that

$$\Lambda = \varprojlim \Lambda_n$$

With this we can define the ring of symmetric functions

2.2.4 Definition. The ring of symmetric functions is the graded ring

$$\Lambda = \oplus_{k\geq 0}\Lambda^k$$

The homomorphism $\rho : \Lambda \longrightarrow \Lambda_n$ is surjective.

In the next sections, we determine bases for Λ_n. These allow us to express every polynomials uniquely as linear combination of the bases. Another important property of the ring of symmetric functions is that it can generate other functions and can be expressed as identity.

2.3 Monomial symmetric function

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n) \in \mathbb{N}^n$. Define a monomial x^α as

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ... x_n^{\alpha_n}$$

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{P}$ such that $l(\lambda) \leq n$ and let $\Omega = \{$all permutations α of $\lambda\}$. The monomial symmetric polynomial, m_λ is defined by

$$m_\lambda(x_1, ...x_n) = \sum_{\alpha\in\Omega} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ... x_n^{\alpha_n} \tag{2.3.1}$$

2.3.1 Example. a. $m = 1$ is the monomial symmetric polynomial for the empty partition.

b. The monomial symmetric polynomial for the partition $\lambda = (1)$ is given as

$$m_{(1)} = \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

The monomial symmetric polynomial m_λ form a $\mathbb{Z}$-basis of Λ_n and hence for m_λ such that $l(\lambda) \leq n$ and $|\lambda| = n$ form a $\mathbb{Z}$-basis of Λ_n^k.

2.3.1 Definition. Let m, n be nonnegative integers, such that $m \geq n$, the homomorphism,

$$\mathbb{Z}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m] \longrightarrow \mathbb{Z}[x_1, \ldots, x_n]$$

such that

$$x_i \mapsto \begin{cases} x_i & \forall i = 1, \ldots, n \\ 0 & \forall i = n+1, \ldots, m \end{cases}$$

Let $\Lambda_m \in \mathbb{Z}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]$ and $\Lambda_n \in \mathbb{Z}[x_1, \ldots, x_n]$ we have the homomorphism

$$\rho_{m,n} \colon \Lambda_m \longrightarrow \Lambda_n$$
$$m_\lambda(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \mapsto \rho_{n,m}\left(m_\lambda(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)\right)$$

where

$$\rho_{n,m}\left(m_\lambda(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)\right) = \begin{cases} m_\lambda(x_1, \ldots, x_n) & \text{if } l(\lambda) = n \\ 0 & \text{if } l(\lambda) > n \end{cases}$$

then $\rho_{m,n}$ is surjective.

2.4 Elementary symmetric polynomial

The elementary symmetric polynomial is a special case of the monomial symmetric polynomial for $\lambda = (1^n) = (1, 1, 1, \ldots, 1)$, n times.

2.4.1 Definition. For each integer $r \geq 0$, the rth elementary symmetric polynomial e_r is the sum of all products of n distinct variables x_i,

$$e_r(x_1, x_2, ..., x_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} x_{i_2} x_{i_3} ... x_{i_r} \quad r \geq 0, \quad n \geq 1$$

In particular, $e_0 = 1$ and $e_r = m_{(1^r)}$.

For any elementary polynomial e_r. We have the following definition for the generating function

2.4.2 Definition. For infinite set of variables $x = (x_1, x_2, \ldots)$ and a parameter t, we have the generating function for the elementary symmetric polynomial e_r as

$$E(t) = \sum_{r=0}^{\infty} e_r t^r = \prod_{i=1}^{\infty} (1 + x_i t) \tag{2.4.1}$$

and for finite $x = (x_1, \ldots, x_n)$, we have

$$E(t) = \sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n} (1 + x_i t) \tag{2.4.2}$$

and $e_r = 0$, $\forall \quad r > n$.

2.4.3 Definition. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ be a partition, we define

$$e_\lambda = \prod_{i=1}^{\infty} e_{\lambda_i}$$

as the elementary symmetric function for the partition λ.

2.4.1 Theorem. *The set of functions* $\{e_\lambda \mid l(\lambda) \leq n\}$ *is a basis of* Λ_n.

For the proof, see Macdonald (1995).

Theorem 2.4.1 means that for $\Lambda_n = \mathbb{Z}[e_1, e_2, e_3, ...]$, every polynomial in Λ_n can be written as a linear combination of $e_1, e_2, e_3, ...$ Thus, every element of Λ_n can be expressed uniquely as a polynomial of order n.

Let $P_n(x)$ be a monic polynomial of order n in a single variable. Then

$$P_n(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + ... \pm a_n$$

where $a_i \in \mathbb{C}$. P_n can be expressed in terms of its roots α_i, $i = 1, \ldots, n$ as

$$P_n(x) = \prod_{i=1}^{n} (x - \alpha_i)$$

The elementary symmetric polynomial e_i is related to the coefficients a_i as

$$a_i = e_i(\alpha_1, \alpha_2, ..., \alpha_n, 0, 0, ..., 0), \quad i \in \mathbb{N}$$

2.4.2 Theorem (Macdonald (1995)). *Let λ be a partition and λ', the conjugate of λ. Then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu} m_\mu \tag{2.4.3}$$

where $a_{\lambda,\mu}$ are nonnegative integers and the sum is over all partitions $\mu < \lambda$ in the natural ordering, $\mu_1 + \mu_2 + \ldots + \mu_i \leq \lambda_1 + \lambda_2 + \ldots + \lambda_i$.

Proof. The multiplication of the product $e_{\lambda'} = e_{\lambda'_1} e_{\lambda'_2} e_{\lambda'_3} \cdots$ is a sum of monomials of the form

$$(x_{i_1} x_{i_2} x_{i_3} \cdots)(x_{j_1} x_{j_2} x_{j_3} \cdots) \cdots = x^\alpha$$

where $i_1 < i_2 < i_3 < i_4 < \cdots < i_{\lambda'_1}$, $j_1 < j_2 < j_3 < j_4 < \cdots < j_{\lambda'_2} \cdots$

Entering the numbers $i_1, i_2, i_3, i_4, ..., i_{\lambda'_1}$ in the first column in the order of the diagram λ. Then $j_1, j_2, j_3, j_4, ..., j_{\lambda'_2}$ in the second column in the order along the row and so on. Then for each $r > 1$ all symbols $< r$ so entered the diagram of λ must occur in the top r rows.

Hence $\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_r < \lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_r$ for each $r \geq 1$, we have $\alpha < \lambda$ and

$$e_{\lambda'} = \sum_{\mu \leq \lambda} a_{\lambda,\mu} m_\mu$$

with $\alpha_{\lambda,\mu} \geq 0$ for each $\mu \geq \lambda$, and the argument shows that the monomial x^λ occurs exactly once so that $a_{\lambda\lambda} = 1$ □

The expression of the elementary symmetric functions in terms of the basis of the monomial symmetric functions implies the following theorem

2.4.3 Theorem. *The set $\{e_\lambda \mid |\lambda| = n\}$ form a basis for $\Lambda = \mathbb{Q}[e_1, e_2, \ldots]$*

Proof. Elements of the set $\{e_\lambda \mid |\lambda| = n\}$ consist of all monomials $e_1^{\alpha_1} e_2^{\alpha_2} e_3^{\alpha_3} \cdots$ where $\alpha_i \in \mathbb{N}$, and any element of Λ can be expressed as a linear combination of e_λ. Every element of Λ is uniquely expressed as a polynomial of e_λ and the e_λ are algebraically independent. Also, since the e_λ can be expressed as a linear combination of the monomial symmetric functions m_λ, and the m_λ are algebraically independent, it implies that the e_λ are also algebraically independent. □

2.5 Complete symmetric polynomial

We define another type of basis for the symmetric functions called the complete symmetic polynomial

2.5.1 Definition. The complete symmetric polynomial $h_r(x_1, x_2, ..., x_n)$ of n distinct variables is the sum of all distinct monomials of degree r in n variables, so that

$$h_r = \sum_{|\lambda|=r} m_\lambda \tag{2.5.1}$$

In particular, $h_0 = 1$, $h_1 = e_1$ and $h_r = 0$ for $r < 0$.

2.5.2 Definition. The complete symmetric functions h_r are generated by the polynomial

$$H(t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \tag{2.5.2}$$

The Newton formulas relate the generating functions of the elementary symmertric polynomial, $E(t)$ and the complete symmetric polynomial, $H(t)$ by the following theorem

2.5.1 Theorem.

$$H(t)E(-t) = 1 \tag{2.5.3}$$

and equivalently, we have the relation

$$\sum_{k=0}^{n}(-1)^k e_k h_{n-k} = 0, \qquad \forall \quad n \geq 1 \tag{2.5.4}$$

Proof. Buy using formula (2.5.2) and substituting $-t$ in formula (2.4.1). we have

$$H(t)E(-t) = \prod_{i=1}^{\infty}(1 - x_i t)\prod_{i=1}^{\infty}(1 - x_i t)^{-1} = 1$$

□

Let N be a positive integer, then (2.5.4), we have the determinant identity for the $(N+1)\times(N+1)$ lower triangular matrices

$$H = (h_{i-j})_{0\leq i,j\leq N} \qquad E = \left((-1)^{i-j}\, e_{i-j}\right)_{0\leq i,j\leq N} \tag{2.5.5}$$

Both matrices H and E have unit determinant.

The following definition shows that the complete homogeneous symmetric functions form a basis in the graded ring, Λ.

2.5.3 Definition. Let ρ be a homomorphism of the graded ring Λ such that

$$\rho : \Lambda \longrightarrow \Lambda, \qquad \rho(e_r) = h_r$$

for all $r \geq 0$. We have $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ and the h_r are also algebraically independent over $\mathbb{Z}$ since the e_r are linearly independent [Macdonald (1995), page 22].

For a finite set of n-variables, with $e_r = 0$ for $r > n$, $\Lambda_n = \mathbb{Z}[h_1, h_2, \ldots, h_n]$ with $h_1, h_2, \ldots, h_n$ algebraically independent but $h_{n+1}, h_{n+2}, \ldots, h_m$ are nonzero polynomials in $h_1, h_2, \ldots, h_n$.

We now define an important involution on the ring Λ_n.

2.5.4 Definition. The homomorphism map

$$\omega : \Lambda_n \longrightarrow \Lambda_n \quad \text{such that } \omega(h_r) = e_r$$

satisfies the following properties

a. $\omega(p+q) = \omega(p) + \omega(q)$ and $\omega(p\cdot q) = \omega(p)\cdot\omega(q)$, for all $p, q \in \Lambda_n$

b. $\omega(h_r) = e_r$ and $\omega(e_r) = h_r$

c. $\omega^2 = id.$

2.6 Power Sum symmetric polyomial

The power sum symmetric polynomial is a special case of the monomial symmetric polynomial $m_{(r,0,0,0,\ldots,0)}(x_1, x_2, \ldots,$

2.6.1 Definition. Let $r \geq 1$ be a natural number and $x = (x_1, \ldots, x_n)$. The rth power sum symmetric polynomial is defined by

$$p_r(x_1, x_2, \ldots, x_n) = x_1^r + x_2^r + \ldots + x_n^r = \sum_{i=1}^{n} x_i^r \tag{2.6.1}$$

2.6.2 Definition. The p_r is generated by the polynomial function

$$P(t) = \sum_{r=1}^{\infty} p_r t^{r-1} = \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} x_i^r t^{r-1} \tag{2.6.2}$$

2.6.1 Theorem (Macdonald (1995), Chapter I, Section 2, page 23)**.** *The generating function $P(t)$ can be expressed in terms of the generating functions $H(t)$ and $E(t)$ as*

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}$$

$$P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}$$

Proof. Using the Taylor series expansion for $\log \dfrac{1}{1 - x_i t}$,

$$\begin{aligned} \log H(t) &= \log \left(\prod_{i \geq 1} \frac{1}{1 - x_i t} \right) = \sum_{i \geq 1} \log \left(\frac{1}{1 - x_i t} \right) \\ &= \sum_{i \geq 1} \sum_{r \geq 1} \frac{(x_i t)^r}{r} = \sum_{r \geq 1} \frac{t^r}{r} \sum_{i \geq 1} x_i^r = \sum_{r \geq 1} \frac{t^r}{r} p_r(x) \\ \frac{d}{dt} \log H(t) &= \frac{H'(t)}{H(t)} = \sum_{r \geq 1} t^{r-1} p_r(x) \\ &= P(t) \end{aligned}$$

Similarly, set $t \to -t$ in $H(t)$. Using (2.5.3), we have

$$P(-t) = \frac{d}{dt} \log H(-t) = \frac{E'(t)}{E(t)}$$

□

2.6.3 Definition. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. The power sum symmetric function for λ is given by

$$p_\lambda = \prod_{i \geq 1} p_{\lambda_i}, \quad i \in \mathbb{N} \tag{2.6.3}$$

2.6.1 Proposition. For any partition λ, let $z_\lambda = \prod_{i\geq 1} i^{r_i} r_i!$

$$h_n = \sum_{|\lambda|=n} \frac{1}{z_\lambda} p_\lambda$$

Proof. We show that

$$H(t) = \sum_{r\geq 0} h_r t^r = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda t^{|\lambda|}$$

From the above

$$\begin{aligned}
\log H(t) &= \sum_{r\geq 1} \frac{t^r}{r} p_r(x) \\
H(t) &= \exp\left(\sum_{r\geq 1} \frac{t^r}{r} p_r(x)\right) \\
&= \sum_{k\geq 0} \frac{1}{k!} \left(\sum_{r\geq 1} \frac{t^r}{r} p_r(x)\right)^k \\
&= \sum_{\substack{k\geq 0 \\ m_1+m_2+\ldots=k}} \frac{1}{k!} \binom{k}{m_1, m_2, \ldots} \prod_{i\geq 1} \binom{p_i t^i}{i}^{m_i} \\
&= \sum_{m_1,m_2,\ldots} \frac{1}{m_1! m_2! \cdots} \frac{p_1^{m_1} p_2^{m_2} \cdots}{1^{m_1} m^{m_2} \ldots} t^{m_1+m_2+\ldots} = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda t^{|\lambda|}
\end{aligned}$$

where $\lambda = (\cdots, 3^{m_3}, 2^{m_2}, 1^{m_1})$, $z_\lambda = m_1! m_2! \cdots 1^{m_1} m^{m_2} \ldots = \prod_{i\geq 1} i^{m_i} m_i!$ □

2.6.2 Theorem. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two sets of independent variables. Let $\lambda = (\lambda, \lambda_2, \ldots)$ be a partition. Then*

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y), \quad z_\lambda = \prod_{k\geq 1} m_k! k^{m_k} \tag{2.6.4}$$

Proof. For a single variable y, we have

$$\log \prod_{i,j} (1 - x_i y)^{-1} = \sum_{i\geq 1} \log\left(\frac{1}{1 - x_i y}\right) = \prod_{i\geq 1} \sum_{k\geq 1} \frac{1}{k} (x_i y)^k = \sum_{k\geq 1} \frac{1}{k} p_k(x) y^k$$

$$\prod_{i,j} (1 - x_i y)^{-1} = \exp\left(\sum_{k\geq 1} \frac{1}{k} p_k(x) y^k\right)$$

Hence for $y = (y_1, y_2, \ldots,)$, we have

$$\begin{aligned}
\prod_{i,j} (1 - x_i y_j)^{-1} &= \exp\left(\sum_{k\geq 1} \sum_{j\geq 1} \frac{1}{k} p_k(x) y_j^k\right) = \exp\left(\sum_{k\geq 1} \frac{1}{k} p_k(x) p_k(y)\right) \\
&= \prod_{k\geq 1} \sum_{m_k\geq 0} \frac{(p_k(x))^{m_k} (p_k(y))^{m_k}}{m_k! k^{m_k}} \\
&= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y)
\end{aligned}$$

□

Newton's identity or formula for symmetric polynomials [Macdonald (1995)] relates the power symmetric polynomials p_r, complete symmetric polynomials h_r and the elementary symmetric polynomials e_r in the following proposition

2.6.2 Proposition.

$$rh_r = \sum_{k=1}^{r} p_k h_{r-k} \tag{2.6.5}$$

$$re_r = \sum_{k=1}^{r} (-1)^{k-1} p_k e_{r-k}$$

Newton's formulas (2.6.5) and Proposition 2.6.1 show that h_r can be expressed as a linear combination of p_r. Since $h_r \in \mathbb{Q}[p_1, p_2, \ldots, p_n]$, we have $p_r \in \mathbb{Q}[h_1, h_2, \ldots, h_n]$. Also the h_r are algebraically independent over $\mathbb{Q}[p_1, p_2, \ldots, p_n]$ means that $\mathbb{Q}[p_1, p_2, \ldots, p_n] = \mathbb{Q}[h_1, h_2, \ldots, h_n]$. We have

$$\Lambda_n = \mathbb{Q}[p_1, p_2, \ldots, p_n] \tag{2.6.6}$$

Thus, the p_r form a basis in Λ_n and any polynomial in Λ_n can be written as a linear combination of the p_r. For any partition λ, we have the following theorem

2.6.3 Theorem. *The set* $\{p_\lambda \mid |\lambda| = n\}$ *is a basis for* Λ_n

2.7 Schur Functions

In this section, we define a symmetric function which is a ratio of antisymmetric functions called the Schur functions. The Schur functions in a finite set of variables is called the Schur polynomials. The Schur polynomial relates symmetric polynomials and antisymmetric polynomials.

A permutation can be written as a product of transpositions. If the parity of the number of transpositions in the product is even then the permutation is even otherwise it is odd.

2.7.1 Definition. The sign of a permutation σ, denoted, $sgn(\sigma)$ is defined as

$$sgn(\sigma) = (-1)^\sigma = \begin{cases} 1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases}$$

We define an antisymmetric polynomial.

2.7.2 Definition. Let $\mathcal{S}_n$ be the set of permutations of σ. A polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is antisymmetric or alternating if

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma f(x_1, \ldots, x_n)$$

for all permutations $\sigma \in \mathcal{S}_n$.

We denote by $\mathcal{A}_n$, the space of anti symmetric polynomials in $x_1, x_2, \ldots, x_n$.

2.7.3 Definition. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multiindex of finite n-variables, the monomial antisymmetric polynomial is defined by

$$a_\alpha = a_\alpha(x^{\alpha_1} \ldots x^{\alpha_n}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \ldots x_{\sigma(n)}^{\alpha_n}$$

Using the property that the quotient of antisymmetric functions is symmetric.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of length $\leq n$ and let $\delta = (n-1, n-2, \ldots, 1, 0)$. Then, we write

$$\alpha = \lambda + \delta = (\lambda_1 + n - 1, \ldots, \lambda_{n-1} + 1, \lambda_n + 0)$$

2.7.4 Definition. The alternating polynomial a_α is defined as

$$a_\alpha = a_{\lambda+\delta} = \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma x_\sigma^{\lambda+\delta}$$

It is expressed in determinant form as

$$a_{\lambda+\delta} = \det\left(x_i^{\lambda_i+n-j}\right)_{1\leq i,j\leq n} = \begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_1+n-2} & \cdots & x_1^{\lambda_1} \\ x_2^{\lambda_2+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_n+n-1} & x_n^{\lambda_n+n-2} & \cdots & x_n^{\lambda_n} \end{vmatrix} \tag{2.7.1}$$

The following definition is fundamental definition for alternatng polynomials.

2.7.5 Definition (Vandermonde determinant)**.** For $\delta = (n-1, n-2, ..., 1, 0)$, the Vandermonde determinant a_δ is defined as

$$a_\delta = \det\left(x_i^{n-j}\right)_{1\leq i,j\leq n} = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix} = \prod_{1\leq i<j\leq n} (x_i - x_j) \tag{2.7.2}$$

2.7.1 Remark. A polynomial $f(x)$ in n variables is alternating if and only if it is of the form $f(x) = V(x)g(x)$ with $g(x)$ a symmetric polynomial and $V(x)$, the Vandermonde determinant in n variables.

Definitions 2.7.4 and 2.7.5 allow us to define the Schur polynomial.

2.7.6 Definition. The Schur symmetric function $s_\lambda(x_1, ..., x_n)$ for a partition λ is defined as the quotient

$$s_\lambda(x_1, ..., x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \tag{2.7.3}$$

The Schur polynomial s_λ is a symmetric function since it is a quotient of alternating functions and a polynomial since all alternating polynomials are divisible by $(x_i - x_j)$ $\forall i, j$ and hence their product $\prod_{i\leq i,j\leq n}(x_i - x_j)$ [Zhou (2003)].

Let Λ be the space consisting of all symmetric functions. When $l(\lambda) \leq n$, the polynomials $s_\lambda(x_1, \ldots, x_n)$ form a $\mathbb{Z}$-basis of Λ_n. Also, for any partition λ, the polynomial s_λ of infinite variables define a unique element $s_\lambda \in \Lambda$, homogeneous of degree $|\lambda|$. The s_λ such that $|\lambda| = k$, $k \geq 0$ form a basis of Λ^k

[Macdonald (1995)].

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ be the conjugate of λ. The Schur polynomial s_λ is related to the elementary symmetric polynomial, e_λ and the homogeneous symmetric polynomial, h_λ by the following determinant [Macdonald (1995), Zhou (2003)].

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le n} = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_n-n+2} & \cdots & h_{\lambda_n} \end{vmatrix} \tag{2.7.4}$$

and

$$s_\lambda = \det\left(e_{\lambda'_i + j - i}\right)_{1 \le i,j \le n} = \begin{vmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+n-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_n-n+1} & e_{\lambda'_n-n+2} & \cdots & e_{\lambda'_n} \end{vmatrix} \tag{2.7.5}$$

2.7.1 Generating function for the Schur polynomials. Let λ be a partition and $\delta = (n-1, n-2, ..., 1, 0)$, such that $d = \lambda + \delta$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, ..., y_n)$ be a finite set of independent variables.

Consider the following generating series $s(x_1, ..., x_n, y_1, ..., y_n)$ of n-variables

$$\begin{aligned} s(x_1, ..., x_n, y_1, ..., y_n) &= \sum_{\substack{\lambda_i+n-i\ge 0 \\ 1\le i\le n}} \frac{1}{\Delta(x_1, ..., x_n)} \begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_1+n-2} & \cdots & x_1^{\lambda_1} \\ x_2^{\lambda_2+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_n+n-1} & x_n^{\lambda_n+n-2} & \cdots & x_n^{\lambda_n} \end{vmatrix} y_1^{\lambda_1+n-1} y_1^{\lambda_1+n-2} \cdots y_1^{\lambda_1} \\ &= \sum_{\substack{\lambda_i+n-i\ge 0 \\ 1\le i\le n}} \frac{1}{\Delta(x_1, \ldots, x_n)} \begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_1+n-2} & \cdots & x_1^{\lambda_1} \\ x_2^{\lambda_2+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_n+n-1} & x_n^{\lambda_n+n-2} & \cdots & x_n^{\lambda_n} \end{vmatrix} \prod_j^n y_j^{\lambda_j+n-j} \\ &= \sum_{\substack{\lambda_i+n-i\ge 0 \\ 1\le i\le n}} s_\lambda(x_1, ..., x_n) \prod_j^n y_j^{\lambda_j+n-j} \end{aligned}$$

Multiplying the right hand side by $\dfrac{\Delta(y_1, ..., y_n)}{\Delta(y_1, ..., y_n)}$, we obtain

$$s(x_1, ..., x_n, y_1, ..., y_n) = \Delta(y_1, ..., y_n) \sum_{l(\lambda\le n)} s_\lambda(x) s_\lambda(y) \tag{2.7.6}$$

Where $\Delta(x_1, ..., x_n)$ and $\Delta(y_1, ..., y_n)$ are the Vandermonde determinant for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ respectively.

2.7.1 Theorem.

$$s(x_1, ..., x_n, y_1, ..., y_n) = \frac{\Delta(y_1, ..., y_n)}{\prod_{1\le i,j\le n}(1 - x_i y_j)} \tag{2.7.7}$$

Proof. By using the standard properties of the determinant

$$s(x_1,...,x_n,y_1,...,y_n) = \frac{1}{\Delta(x_1,...,x_n)} \begin{vmatrix} \sum_{d_1\geq 0}(x_1y_1)^{d_1} & \sum_{d_1\geq 0}(x_1y_2)^{d_2} & \cdots & \sum_{d_n\geq 0}(x_1y_n)^{d_n} \\ \sum_{d_1\geq 0}(x_2y_1)^{d_1} & \sum_{d_2\geq 0}(x_2y_2)^{d_2} & \cdots & \sum_{d_n\geq 0}(x_2y_n)^{d_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{d_n\geq 0}(x_ny_1)^{d_1} & \sum_{d_2\geq 0}(x_ny_2)^{d_2} & \cdots & \sum_{d_n\geq 0}(x_ny_n)^{d_n} \end{vmatrix}$$

$$= \frac{1}{\Delta(x_1,...,x_n)} \begin{vmatrix} \frac{1}{(1-x_1y_1)} & \frac{1}{(1-x_1y_2)} & \cdots & \frac{1}{(1-x_1y_n)} \\ \frac{1}{(1-x_2y_1)} & \frac{1}{(1-x_2y_2)} & \cdots & \frac{1}{(1-x_2y_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(1-x_ny_1)} & \frac{1}{(1-x_ny_2)} & \cdots & \frac{1}{(1-x_ny_n)} \end{vmatrix}$$

Let $t_{ij} = 1 - x_iy_j$. The determinant can be evaluated as follows:

We subtract the last row from the i-th row ($i > n$), and use the common denominator. We get

$$\begin{vmatrix} \frac{1}{t_{1,1}} & \frac{1}{t_{1,2}} & \cdots & \frac{1}{t_{1,n}} \\ \frac{1}{t_{2,1}} & \frac{1}{t_{2,2}} & \cdots & \frac{1}{t_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t_{n,1}} & \frac{1}{t_{n,2}} & \cdots & \frac{1}{t_{n,n}} \end{vmatrix} = \begin{vmatrix} \frac{t_{n,1}-t_{1,1}}{t_{n,1}t_{1,1}} & \frac{t_{n,2}-t_{1,2}}{t_{n,2}t_{1,2}} & \cdots & \frac{t_{n,n}-t_{1,n}}{t_{n,n}t_{2,n}} \\ \frac{t_{n,1}-t_{2,1}}{t_{n,1}t_{2,1}} & \frac{t_{n,2}-t_{2,2}}{t_{n,2}t_{2,2}} & \cdots & \frac{t_{n,n}-t_{2,n}}{t_{n,n}t_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t_{n,1}} & \frac{1}{t_{n,2}} & \cdots & \frac{1}{t_{n,n}} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{n-1}(x_i - x_n)}{\prod_{j=1}^{n} t_{n,j}} \begin{vmatrix} \frac{y_1}{t_{1,1}} & \frac{y_2}{t_{1,2}} & \cdots & \frac{y_n}{t_{1,n}} \\ \frac{y_1}{t_{2,1}} & \frac{y_2}{t_{2,2}} & \cdots & \frac{y_n}{t_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y_1}{t_{n-1,1}} & \frac{y_2}{t_{n-1,2}} & \cdots & \frac{y_n}{t_{n-1,n}} \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

Now, subtract the last column from the j-column, use common denominators, and simplify as above. We get:

$$\begin{vmatrix} \frac{1}{t_{1,1}} & \frac{1}{t_{1,2}} & \cdots & \frac{1}{t_{1,n}} \\ \frac{1}{t_{2,1}} & \frac{1}{t_{2,2}} & \cdots & \frac{1}{t_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{t_{n,1}} & \frac{1}{t_{n,2}} & \cdots & \frac{1}{t_{n,n}} \end{vmatrix} = \frac{\prod_{i=1}^{n-1}(x_i - x_n)(y_i - y_n)}{t_{n,n}\prod_{j=1}^{n-1} t_{n,j}t_{j,n}} \begin{vmatrix} \frac{y_1}{t_{1,1}} & \frac{y_2}{t_{1,2}} & \cdots & \frac{y_n}{t_{1,n}} \\ \frac{y_1}{t_{2,1}} & \frac{y_2}{t_{2,2}} & \cdots & \frac{y_n}{t_{2,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y_1}{t_{n-1,1}} & \frac{y_2}{t_{n-1,2}} & \cdots & \frac{y_n}{t_{n-1,n}} \end{vmatrix}$$

Hence, by induction we have

$$\begin{vmatrix} \frac{1}{(1-x_1y_1)} & \frac{1}{(1-x_1y_2)} & \cdots & \frac{1}{(1-x_1y_{n-1})} \\ \frac{1}{(1-x_2y_1)} & \frac{1}{(1-x_2y_2)} & \cdots & \frac{1}{(1-x_2y_{n-1})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(1-x_ny_1)} & \frac{1}{(1-x_ny_2)} & \cdots & \frac{1}{(1-x_ny_n)} \end{vmatrix} = \frac{\Delta(x_1,...,x_n)\Delta(y_1,...,y_n)}{\prod_{i,j=1}^{n}(1-x_iy_j)}$$

□

Using (2.7.6) and Theorem 2.7.1, we obtain the following relation

$$\prod_{i=1}^{n}\frac{1}{(1-x_iy_j)} = \sum_{l(\lambda)\leq n} s_\lambda(x)s_\lambda(y) \tag{2.7.8}$$

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ be a set of variables. x and y can be finite. Denote by $s_\lambda(y)$, $m_\lambda(x)$, $h_\lambda(x)$ the symmetric functions of x's and $s_\lambda(y)$, $m_\lambda(y)$, $e_\lambda(y)$, $h_\lambda(y)$ the symmetric functions of y's. We have the relation

2.7.2 Theorem.

$$\prod_{i,j}\frac{1}{(1-x_iy_j)} = \sum_\lambda h_\lambda(x)m_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y) \tag{2.7.9}$$

Proof. From (2.5.2), we have for a single variable, y

$$\prod_{i\geq 1}\frac{1}{1-x_iy} = \sum_{r\geq 0} h_r(x)y^r$$

Hence for $y = (y_1, y_2, \ldots)$

$$\begin{aligned}\prod_{i,j}\frac{1}{1-x_iy_j} &= \prod_{j\geq 1}\sum_{r_j\geq 0} h_{r_j}(x)y^{r_j} = \sum_{r_j\geq 0} h_{r_j}(x)\prod_{j\geq 1} y^{r_j} \\ &= \sum_{r_j\geq 0} h_{r_j}(x)m_{r_j}(y) \\ &= \sum_\lambda h_\lambda(x)m_\lambda(y)\end{aligned}$$

□

We want to define a scalar product on the space of symmetric functions which is given in terms of the basis (h_λ) and (m_λ)

2.7.7 Definition. Let (h_λ) and (e_λ) be some bases in Λ, A scalar product on Λ is defined by

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}, \quad \delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases} \tag{2.7.10}$$

2.7.8 Definition (Dual basis)**.** For each $n \geq 0$, let (u_λ) and (v_λ) be a $\mathbb{Q}$-basis of Λ^n indexed by the partitions of n. Then the following conditions are equivalent:

a. $\langle u_\lambda, v_\lambda \rangle = \delta_{\lambda\mu}$ for all partitions λ and μ

b. $\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i,j} \dfrac{1}{(1-x_iy_j)}$

where the Kronecker delta, $\delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$

Proof. Let $u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho$ and $v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma$, then

$$\begin{aligned} \langle u_\lambda, v_\mu \rangle &= \sum_\rho a_{\lambda\rho} \sum_\sigma b_{\mu\sigma} \langle h_\lambda, m_\mu \rangle \\ &= \sum_\rho a_{\lambda\rho} \sum_\sigma b_{\mu\sigma} \delta_{\rho\sigma} \\ &= \sum_\rho a_{\lambda\rho} b_{\mu\rho} \end{aligned}$$

so that a. is equivalent to

$$\langle u_\lambda, v_\mu \rangle = \sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu} \tag{2.7.11}$$

Also,

$$\begin{aligned} \sum_\lambda u_\lambda(x)v_\lambda(y) &= \sum_\lambda \sum_\sigma \sum_\rho a_{\lambda\rho} b_{\lambda\sigma} h_\rho(x) m_\sigma(y) \\ &= \sum_\lambda h_\lambda(x) m_\lambda(y), \quad \text{when } \sigma = \rho \\ &= \prod_{i,j} \frac{1}{(1-x_iy_j)} \end{aligned}$$

Since

$$\sum_\lambda a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}$$

□

2.7.2 Skew-Schur Function. The skew Schurs functions are a more general form of Schur functions depending on two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$. Such a Schur function is denoted by $s_{\lambda/\mu}$. We have the following definition.
First recall that every $f \in \Lambda$ can be uniquely expressed in terms of s_λ by a scalar product

$$f = \sum_\lambda \langle f, s_\lambda \rangle s_\lambda \tag{2.7.12}$$

by the fact that s_λ form a orthonormal basis of Λ.

2.7.9 Definition. Let λ, μ be partitions, define a function $s_{\lambda/\mu}$, called skew Schur functions by

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\nu s_\mu \rangle \tag{2.7.13}$$

for all partitions, ν.

In particular, for the zero partition, $s_{\lambda/0} = s_\lambda$. It can be verified that $s_{\lambda/\mu}$ is homogeneous of degree $|\lambda| - |\mu|$ and is zero if $|\lambda| < |\mu|$. Thus, $s_{\lambda/\mu} = 0$ if $\lambda \leq \mu$.

We express the $s_{\lambda/\mu}$ in terms of the monomial symmetric function, m_λ and the homomgeneous symmetric functions h_λ as

$$\begin{aligned}\sum_\lambda s_{\lambda/\mu}(x)s_\lambda(y) &= \sum_{\lambda,\mu} C^\lambda_{\mu\nu} s_{\lambda/\mu}(x)s_\lambda(y) \\ &= \sum_\nu s_\nu(x)s_\mu(y)s_\nu(y) \\ &= s_\mu(y)\sum_\nu s_\nu(x)s_\nu(y) \\ &= s_\mu(y)\sum_\nu h_\nu(x)m_\nu(y) \end{aligned} \tag{2.7.14}$$

Let $y = (y_1, ..., y_n)$ be a set of variables. Let λ and μ be partitions of length $\leq n$ and let λ' and μ' be their respective conjugate partitions The skew Schur polynomial $s_{\lambda/\mu}$ is expressed in determinant form using equation (2.7.4) and (2.7.5) by the following

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\leq i,j\leq n} = \begin{vmatrix} h_{\lambda_1-\mu_1} & h_{\lambda_1-\mu_2+1} & \cdots & h_{\lambda_1-\mu_n+n-1} \\ h_{\lambda_2-\mu_1-1} & h_{\lambda_2-\mu_2} & \cdots & h_{\lambda_2-\mu_n+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-\mu_1-n+1} & h_{\lambda_n-\mu_2-n+2} & \cdots & h_{\lambda_n-\mu_n} \end{vmatrix} \tag{2.7.15}$$

$$s_{\lambda/\mu} = \det\left(e_{\lambda'_i-\mu'_j-i+j}\right)_{1\leq i,j\leq n} = \begin{vmatrix} e_{\lambda'_1-\mu'_1} & e_{\lambda'_1-\mu'_2+1} & \cdots & e_{\lambda'_1-\mu'_n+n-1} \\ e_{\lambda'_2-\mu'_1-1} & e_{\lambda'_2-\mu'_2} & \cdots & e_{\lambda'_2-\mu'_n+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_n-\mu'_1-n+1} & e_{\lambda'_n-\mu'_2-n+2} & \cdots & e_{\lambda'_n-\mu'_n} \end{vmatrix} \tag{2.7.16}$$

where $l(\lambda) \leq n$.
In particular, if $\mu = 0$ then (2.7.15) and (2.7.16) reduce to (2.7.4) and (2.7.5) respectively and we have

$$s_{\lambda/0} = s_\lambda = \det\left(h_{\lambda_i-i+j}\right)_{1\leq i,j\leq n} = \det\left(e_{\lambda'_i-i+j}\right)_{1\leq i,j\leq n}$$

3. Jack Symmetric Functions and Mathematical Physics

In this chapter, we study a class of symmetric functions called the Jack symmetric functions which is a generalization of the Jack symmetric polynomials to infinite set of variables. The Jack symmetric functions depend on a parameter α. Special cases of the Jack symmetric functions are the Schur symmetric functions, s_λ and the Zonal symmetric functions, Z_λ. For more information on Zonal symmetric functions and polynomials see Macdonald (1995).

The Jack symmetric polynomials are characterized by two main properties; orthogonality and triangularity. We begin by defining an inner product of the basis elements of a field, $\mathbb{F} = \mathbb{Q}(\alpha)$ of rational functions. The inner product is defined on symmetric functions that depend on α and a set of variables.

3.1 Some Definitions

3.1.1 Definition. Let λ and μ be any two partitions and α be an indeterminate. Let $\mathbb{F} = \mathbb{C}(\alpha)$ denotes a field of rational functions of a parameter α. Let $\Lambda_{\mathbb{F}} = \Lambda \bigotimes \mathbb{F}$ be a vector space of all symmetric functions with coefficients in $\mathbb{F}$. Define an inner product

$$\langle p_\lambda, p_\mu \rangle = \langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_\lambda^{-1}(\lambda) \alpha^{-l(\lambda)} \tag{3.1.1}$$

where

$$z_\lambda = \prod_{r \geq 0} (r^{m_r} m_r!) \quad \text{and} \quad \delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

We will see later that this belienear form generalizes the one defined in Definition 2.7.7.

Let $\Lambda^k_{\mathbb{F}} = \Lambda^k \bigotimes \mathbb{Q}(\alpha)$ denotes the vector space of homogeneous symmetric functions of degree k with coefficients in $\mathbb{F}$. Then,

$$\Lambda^k = \Lambda \bigotimes \mathbb{Q}(\alpha) = \bigcup_{k \geq 0} \Lambda^k \bigotimes \mathbb{Q}(\alpha)$$

3.1.2 Definition. Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ be any two sets of infinite sequences of independent indeterminates, then we define the product

$$\Pi(x, y; \alpha) = \prod_{i,j} \frac{1}{(1 - x_i y_j)^{\frac{1}{\alpha}}} \tag{3.1.2}$$

A differential operator on $\mathbb{F}$ satisfies the following conditions.

3.1.1 Theorem (Orthogonality, Macdonald (1995))**.** *For each $n \geq 0$, let (u_λ) and (v_λ) be $\mathbb{F}$-basis of $\Lambda^n_{\mathbb{F}}$ indexed by the partitions of n. Then, the following conditions are equivalent:*

a. $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ *for all partitions λ, μ.*

b. $\sum_\lambda u_\lambda(x) u_\lambda(y) = \Pi(x, y; \alpha) = \prod_{i,j} (1 - x_i y_j)^{-\frac{1}{\alpha}}$

Proof. Let $p^*_\lambda = z_\lambda(\alpha)^{-1}\alpha^{-l(\lambda)}p_\lambda$ so that for all partitions λ, μ. $\langle p^*_\lambda, p_\mu\rangle = \delta_{\lambda\mu}$. Let $u_\lambda = \sum_\rho a_{\lambda\rho}p^*_\rho$ and $v_\lambda = \sum_\sigma a_{\mu\sigma}p_\sigma$, then from (a)

$$\begin{aligned}\langle u_\lambda, v_\mu\rangle &= \sum_\rho a_{\lambda\rho}\sum_\sigma a_{\mu\sigma}\langle p^*_\rho, p_\sigma\rangle \\ &= \sum_\rho a_{\lambda\rho}\sum_\sigma a_{\mu\sigma}\delta_{\rho\sigma} = \sum_\rho a_{\lambda\rho}a_{\mu\rho}\end{aligned}$$

Condition (a) is equivalent to $\sum_\rho a_{\lambda\rho}a_{\mu\rho} = \delta_{\rho\sigma}$ and using the fact that $\Pi(x,y;\alpha) = \sum_\rho p^*_\rho(x)p_\rho(y)$ condition (b) is equivalent

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \Pi(x,y;\alpha) = \sum_\rho z_\rho(\alpha)^{-1}\alpha^{-l(\lambda)}p_\rho(x)p_\rho(y) = \sum_\rho p^*_\rho(x)p_\rho(y)$$

therefore $\sum_\lambda a_{\lambda\rho}a_{\lambda\sigma} = \delta_{\rho\sigma}$ □

We define a linear operator on the space of symmetric functions.

3.1.2 Theorem (Linear Operator)**.** *Let $E : \Lambda_{\mathbb{F}} \longrightarrow \Lambda_{\mathbb{F}}$. Then E is an $\mathbb{F}$-linear operator and the following conditions on E are equivalent:*

a. $\langle Ef, g\rangle = \langle f, Eg\rangle$*, for all $f, g \in \Lambda_{\mathbb{F}}$. That is, E is self adjoint.*

b. $E_x\Pi(x,y;\alpha) = E_y\Pi(x,y;\alpha)$

Proof. For any two partitions λ, μ. Let

$$e_{\lambda\mu} = \langle Em_\lambda, m_\mu\rangle \text{ and } e_{\mu\lambda} = \langle m_\lambda, Em_\mu\rangle$$

then from condition (a)

$$e_{\lambda\mu} = e_{\mu\lambda}$$

Let $g_\lambda = g_\lambda(x;\alpha)$ (see Macdonald (1995)) be a basis of $\Lambda_{\mathbb{F}}$ and define

$$Em_\lambda = \sum_\mu e_{\lambda\mu}g_\mu$$

we have

$$\Pi(x,y;\alpha) = \sum_\lambda g_\lambda(x)m_\lambda(y) = \sum_\lambda m_\lambda(y)g_\lambda(x)$$

This implies that (b) is equivalent to

$$\sum_{\lambda\mu} e_{\lambda\mu}g_\mu(x)g_\lambda(y) = \sum_{\lambda\mu} e_{\lambda\mu}g_\mu(y)g_\lambda(x)$$

and hence

$$e_{\lambda\mu} = e_{\mu\lambda}$$

□

The action of a differetial operator on the space of symmetric functions result in an eigenfunction problem.

3.2 Jack symmetric functions and some properties

We now define the Jack symmetric functions. We state here as a theorem

3.2.1 Theorem (Jack symmetric functions)**.** *For each partition λ. there is a unique symmetric function $J_\lambda = J_\lambda(x, \alpha) \in \Lambda_{\mathbb{F}}$ called the Jack symmetric functions such that*

a. $J_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu$ *(Triangularity) where* $u_{\lambda\mu} \in \mathbb{F}$ *and* $u_{\lambda\lambda} = 1$

b. $\langle J_\lambda, J_\mu \rangle = 0$ *if* $\lambda \neq \mu$ *(Orthogonality)*

Proof. (Macdonald (1995)). Let J_λ be the eigenfunctions of the operator E constructed such that J_λ satisfies

$$EJ_\lambda = e_{\lambda\lambda} J_\lambda$$

where $e_{\lambda\lambda}$ is the eigenvalue of E. Then,

$$e_{\lambda\lambda} u_{\lambda\nu} = \sum_{\nu \leq \mu \leq \lambda} e_{\lambda\nu} u_{\mu\nu}$$

for all pairs of partitions ν, λ such that $\nu \leq \lambda$. We have

$$(e_{\lambda\lambda} - e_{\nu\nu}) u_{\lambda\nu} = \sum_{\nu < \mu \leq \lambda} e_{\lambda\nu} u_{\mu\nu}$$

Since the eigenvalues of the operator E are distinct, $e_{\lambda\lambda} \neq e_{\nu\nu}$ if $\nu \neq \lambda$. The equation determines $u_{\lambda\nu}$ uniquely in terms of the $u_{\lambda\nu}$ such that $\nu < \mu \leq \lambda$. Hence a symmetric function J_λ exist satisfying conditions (a) and (b).

By self-adjointness of the E, we have

$$\begin{aligned} E_{\lambda\lambda} \langle J_\lambda, J_\mu \rangle &= \langle EJ_\lambda, J_\mu \rangle = \langle J_\lambda, EJ_\mu \rangle \\ &= e_{\mu\mu} \langle J_\lambda, J_\mu \rangle \end{aligned}$$

Since $e_{\lambda\lambda} \neq e_{\mu\mu}$ if $\lambda \neq \mu$, J_λ satisfy condition (a) and (b).

Finally, we want to show that J_λ are uniquely determined by (a) and (b). Let λ be a partition and assume that J_μ are determined for all $\mu < \lambda$. Then J_λ has the form from (a)

$$J_\lambda = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu} J_\mu$$

$$\begin{aligned} \langle J_\lambda, J_\mu \rangle &= \langle m_\lambda + \sum_{\nu < \lambda} v_{\lambda\nu} J_\nu, J_\mu \rangle \\ 0 &= \langle m_\lambda, J_\mu \rangle + \sum_{\mu < \lambda} v_{\lambda\mu} \langle J_\mu, J_\mu \rangle \\ -\langle m_\lambda, J_\mu \rangle &= \sum_{\mu < \lambda} v_{\lambda\mu} \langle J_\mu, J_\mu \rangle \\ \sum_{\mu < \lambda} v_{\lambda\mu} &= \frac{-\langle m_\lambda, J_\mu \rangle}{\langle J_\mu, J_\mu \rangle}, \ \langle J_\mu, J_\mu \rangle \neq 0 \end{aligned}$$

□

The Jack symmetric function $J_\alpha(x_1, x_2, ...)$ is defined for infinite set of variables $x = (x_1, x_2, ...)$. If for any finite set $x = (x_1, ..., x_n)$ such that $x_{n+1} = x_{n+2} = ... = 0$ then $J_\lambda(x_1, ..., x_n)$ is called the Jack symmetric polynomial [Stanley (1989)].

The following are some properties of the Jack symmetric functions.

Let $x = (i, j) \in \lambda$ and define the hook length of λ at x as [Macdonald (1995)]

$$h(x) = \lambda_i + \lambda_j - i - j + 1$$

Set

$$H_\lambda = \prod_{x \in \lambda} h(x)$$

as the product of the hook-lengths:

3.2.1 Proposition. a. $\alpha = 1$, $J(x; 1) = H_\lambda s_\lambda(x)$ is the Schur's function s_λ

b. $\alpha = 2$, $J(x; 2) = Z_\lambda(x)$ is the Zonal function indexed by λ.

c. $\alpha = \frac{1}{2}$, $J(x; \frac{1}{2})$, then $J_\lambda(x; \alpha)$ occurs naturally as the Zonal function on the homogeneous space G/K, where $G = \mathbb{G}L_n(H)$ and K is the quanternionic unitary group of $n \times n$ matrices [Macdonald (1995), Stanley (1989)]

Also, the Jack functions, $J_\lambda(x; \alpha)$ has a limit as α approaches specific limit.

- $J_\lambda(x; \alpha) \longrightarrow e_\lambda$, the elementary symmetric function as $\alpha \longrightarrow 0$
- $J_\lambda(x; \alpha) \longrightarrow m_\lambda$, the monomial symmetric function as $\alpha \longrightarrow \infty$

3.2.1 Example. Consider the partition $\lambda = (1^n) = (1, 1, ..., 1)$. Then $l(\lambda) = |\lambda| = \sum_{i=1}^n \lambda_i = n$. The $v_{\lambda,1^n} = n!$ and

$$J_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}(\alpha) m_\mu = n! m_{(1^n)} = n! e_n$$

where $u_{\lambda\mu}(\alpha)$ are integer functions depending on α

3.2.2 Proposition. For any $n \geq 0$, the Jack symmetric function $J_{(n)}$ where $(n) = (n, 0, 0, \ldots)$ has the expansion

$$J_{(n)} = \sum_{|\lambda| = n} \alpha^{n - l(\lambda)} n! z_\lambda^{-1} J_\lambda \tag{3.2.1}$$

See Macdonald (1995) for the proof.

3.2.3 Proposition. Let $n \geq 0$. The Jack polynomials $J_\lambda(x_1, \ldots, x_n; \alpha) = 0$ if $l(\lambda) > n$ and are linearly independent if $l(\lambda) \leq n$.

Proof. By Stanley (1989), suppose $l(\lambda) > n$. Let μ and λ be partitions such that $|\mu| = |\lambda|$ and $\mu \leq \lambda$, then $m_\mu(x_1, ..., x_n) = 0$ and so

$$J_\lambda(x; \alpha) = \sum_{\mu \leq \lambda} u_{\lambda\mu}(\alpha) m_\mu = 0$$

On the other hand, if $\lambda^1 \neq \lambda^2 \neq \cdots$ are all distincts partitions of length $\leq n$, then $m_{\lambda^i}(x_1, ..., x_n)$ are all linearly independent. Hence by the orthogonality and triangularity of the Jack symmetric polynomials, the $J_{\lambda^i}(x_1, ..., x_n; \alpha)$ are all linearly independent. □

3.3 The CMS Operator

We define a partial differential operator called the Calogero-Moser-Sutherland (CMS) operator [Sergeev and Veselov (2005)]. The CMS model describes a system of n identical particles of mass lying on a circle of circumference d and interacting pairwise with each other. Sergeev and Veselov deduced that the Jack symmetric polynomials are the eigenfunctions of the CMS operator.

3.3.1 Definition. Let Λ_n be a space of all symmetric polynomials depending on the parameter α. The differential operator $\mathcal{L}_{n,\alpha} : \Lambda_n \longrightarrow \Lambda_n$ is defined by

$$\begin{aligned}\mathcal{L}_{n,\alpha} &= \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2 + \frac{1}{\alpha}\sum_{1\le i<j\le n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right) - \frac{(n-1)}{\alpha}\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2 + \frac{2}{\alpha}\sum_{i\neq j}\frac{x_ix_j}{x_i-x_j}\frac{\partial}{\partial x_i} \end{aligned} \tag{3.3.1}$$

The operator (3.3.1) preserves the ring of symmetric polynomials, Λ_n. Let $p_r(x)$ be the power sum symmetric polynomial defined in (2.6.1). Since the power sum symmetric polynomial generates Λ_n, we can apply the operator $\mathcal{L}_{n,\alpha}$ on $p_r(x)$, that is $\mathcal{L}_{n,\alpha}p_r(x)$.
The first sum on the right hand side of (3.3.1) gives

$$\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2(p_r(x)) = p_r(x)$$

which preserves Λ_n. Similarly, the terms in the second sum of (3.3.1) are of the form;

$$\frac{x_ix_j}{x_i-x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right), \quad \forall \quad i\neq j$$

Then, the polynomial

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)f, \quad \forall \quad i\neq j$$

is antisymmetric under the interchange of x_i and x_j, and hence divisible by $x_i - x_j$. It implies that $\mathcal{L}_{n,\alpha}p_r(x) \in \Lambda_n$. Thus, if $f \in \Lambda_n$, then $\mathcal{L}_{n,\alpha}f \in \Lambda_n$. Also, if f is homogeneous of degree n, then $\mathcal{L}_{n,\alpha}f$ is also homogeneous of degree n.

The CMS operator is stable under restriction to the number of variables [Sergeev and Veselov (2005)]. This allows us to apply the homomorphism (2.2.1) on the differential operator $(\mathcal{L}_n)$ such that for any integers $m \geq n$, we have

$$\rho_{m,n}(\mathcal{L}_m) = \rho_{m,n}(\mathcal{L}_n)$$

The following theorem in the definition of the Jack symmetric polynomials by Sergeev and Veselov establishes the relationship between the Jack symmetric polynomials $J_\lambda(x;\alpha)$ and the CMS operator $L_\alpha^{(n)}$

3.3.1 Theorem. *The Jack symmetric polynomials $J_\lambda(x;\alpha)$ is an eigenfunction of the CMS operator $L_\alpha^{(n)}$ for all $J_\lambda(x;\alpha) \in \Lambda_n$*

Proof. The proof follows from the proof of Theorem 3.1.2 by replacing E by $L_\lambda^{(n)}$, then we have

$$L_\lambda^{(n)} J_\lambda(x;\alpha) = e_{\lambda\lambda} J_\lambda(x;\alpha)$$

where $J_\lambda(x;\alpha)$ is the eigenfunction of $L_\lambda^{(n)}$ and it corresponding eigenvalue is $e_{\lambda\lambda}$. The eigenvalue $e_{\lambda\lambda}$ for the CMS operator is given by Sergeev and Veselov [Sergeev and Veselov (2005)]

$$e_{\lambda\lambda} = \sum_{i=1}^{n} \lambda_i^2 - 2\alpha \sum_{i=1}^{n} (i-1)\lambda_i = 2N(\lambda) - 2\alpha N(\lambda) + |\lambda|$$

where $N(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$ □

The CMS operator is stable under the change of n variables [Sergeev and Veselov (2005)].

The Jack polynomials form a particular basis for the ring of symmetric polynomials by the action of the CMS operators on the Jack polynomials. The stability of the CMS operators with respect to the number of variables implies that for any partition λ,

$$J_\lambda(x_1, \ldots, x_{n-1}, 0) = J_\lambda(x_1, \ldots, x_{n-1})$$

where $J_\lambda(x_1, \ldots, x_{n-1}) = 0$ if $\lambda_n \neq 0$.

4. Super Jack polynomials and the deformed CMS operator

In this chapter, we study a class of functions which preserve symmetry when the number of variables increase. These functions are called supersymmetric functions. We also extend the Jack symmetric functions to super Jack symmetric polynomials depending on two finite variables and a parameter. The super Jack polynomials are supersymmetric polynomials which are doubly symmetric and satisfies a certain differential operation condition. We then state that the super Jack symmetric polynomials are the eigenfunctions of the defomed CMS operator (Sergeev and Veselov (2005)).

4.1 Supersymmetric functions

Let $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$ be a set of n and m independent variables. Let $P_{n,m} = \mathbb{C}[x_1, \ldots, x_2, y_1, \ldots, x_m]$ be a polynomial algebra of $n + m$ variables. Let $\Lambda_{n,m,\theta}$ be a subalgebra of $P_{n,m}$

4.1.1 Definition. A polynomial $f(x, y, \theta)$ of $\Lambda_{n,m,\theta}$ is said to be supersymmetric if it is symmetric separately in the variables x and y and satisfies the following differential operator

$$\left(\frac{\partial}{\partial x_i} + \theta\frac{\partial}{\partial y_j}\right) f\Big|_{x_i=y_j} = 0 \tag{4.1.1}$$

$$\left(x_i\frac{\partial}{\partial x_i} + \theta y_j\frac{\partial}{\partial y_j}\right) f\Big|_{x_i=y_j} = 0 \tag{4.1.2}$$

We want to determine the set of polynomials that generate the subalgebra. We consider the following theorem

4.1.1 Theorem. *The supersymmetric power polynomials are given in terms of the power symmetric polynomials by*

$$p_r(x, y) = p_r(x) + (-1)^{r-1}p_r(y) \tag{4.1.3}$$

for a non-negative integer r.

Proof. We have the following generating function of the power sum symmetric polynomials of the variables x and y

$$H_{x/y} = \frac{\prod_{j=1}^{m}(1 + y_j t)}{\prod_{i=1}^{n}(1 - x_i t)}$$

Differentiating

$$\begin{aligned}
H'_{x/y} &= \frac{d}{dt}\left(\prod_{j=1}^{m}(1+y_jt)\prod_{i=1}^{n}(1-x_it)^{-1}\right)\\
&= \prod_{i=1}^{n}(1-x_it)^{-1}\sum_{k=1}^{m}y_k\prod_{k,j=1,k\neq j}^{m}(1+y_jt)+\prod_{j=1}^{m}(1+y_jt)\sum_{l=1}^{n}\frac{x_l}{(1-x_lt)^2}\prod_{i,l=1,i\neq l}^{n}(1-x_it)^{-1}\\
&= \frac{\prod_{j=1}^{m}(1+y_jt)}{\prod_{i=1}^{n}(1-x_it)}\left[\sum_{l=1}^{n}\frac{x_l}{(1-x_lt)}+\sum_{k=1}^{m}\frac{y_k}{(1+y_kt)}\right]\\
&= H_{x/y}\left(\sum_{l=1}^{n}\frac{x_l}{(1-x_lt)}+\sum_{k=1}^{m}\frac{y_k}{(1+y_kt)}\right)\\
\frac{H'_{x/y}}{H_{x/y}} &= \sum_{l=1}^{n}\frac{x_l}{(1-x_lt)}+\sum_{k=1}^{m}\frac{y_k}{(1+y_kt)}
\end{aligned}$$

Expanding the generating function for the symmetric power sums, gives us that

$$p(t)=\sum_{r\geq 1}p_rt^{r-1}=\sum_{r\geq 1}\sum_{i=1}^{n}x_i^rt^{r-1}=\sum_{i=1}^{n}\frac{x_i}{1-x_it}$$

$$\begin{aligned}
\frac{H'_{x/y}}{H_{x/y}}=P_{x/y}(t) &= \sum_{r\geq 1}p_r(x)t^{r-1}+(-1)^{r-1}\sum_{r\geq 1}p_r(y)t^{r-1}\\
&= \sum_{r\geq 1}\left(p_r(x)+(-1)^{r-1}p_r(y)\right)t^{r-1}=\sum_{r\geq 1}p_r(x/y)t^{r-1}
\end{aligned}$$

□

In fact, as shown in (Sergeev and Veselov (2005)), for generic value θ, the ring $\Lambda_{m,n,\theta}$ is generated by the deformed power sum polynomial in the variables x and y called the deformed power sums defined by

$$p_r(x,y,\theta)=\sum_{i=1}^{n}x_i^r-\frac{1}{\theta}\sum_{j=1}^{m}y_j^r=p_r(x)-\frac{1}{\theta}p_r(y) \tag{4.1.4}$$

for non-negative integer r.Observe that $p_r(x,y,\theta)\in\Lambda_{n,m,\theta}$.

4.1.1 Lemma. Equation (4.1.4) satisfies (4.1.1).

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x_l}p_r(x,y,\theta)&=\frac{\partial}{\partial x_l}\left(\sum_{i=1}^{n}x_i^r-\frac{1}{\theta}\sum_{j=1}^{m}y_j^r\right)=rx_l^{r-1}\\
\frac{\partial}{\partial y_q}p_r(x,y,\theta)&=\frac{\partial}{\partial y_q}\left(\sum_{i=1}^{n}x_i^r-\frac{1}{\theta}\sum_{j=1}^{m}y_j^r\right)=-\frac{1}{\theta}ry_q^{r-1}\\
\left(\frac{\partial}{\partial x_l}p_r(x,y,\theta)-\theta\frac{\partial}{\partial y_q}p_r(x,y,\theta)\right)\Big|_{x_l=x_q}&=\left(rx_l^{r-1}-\frac{\theta}{\theta}ry_q^{r-1}\right)\Big|_{x_l=x_q}=0
\end{aligned}$$

□

Elements of $\Lambda_{n,m,\theta}$ are called supersymmetric functions. Sergeev and Veselov deduced that every deformed differential operator of the CMS, $\mathcal{L}_{n,m,\theta}$ preserves the algebra $\Lambda_{n,m,\theta}$.

4.2 Super Jack Polynomials

In this section, we want to define a homomorphism map that maps the homogeneous elements of the algebra of the Jack symmetric functions to the supersymmetric polynomials.

Let Λ be an algebra of the Jack symmetric functions. Let $p_r(z) = z_1^r + z_2^r + \ldots$ be the r-order power sum symmetric functions and let $J_\lambda(z,\theta)$ be the Jack symmetric function.

4.2.1 Definition. A homomorphism φ defined by

$$\varphi\colon \Lambda \longrightarrow \Lambda_{n,m,\theta} \tag{4.2.1}$$

such that

$$p_r(z) \mapsto \varphi(p_r(z)) = p_r(x,y,\theta) \tag{4.2.2}$$

The homomorphism φ is called the Kerov's map

The definition shows that any symmetric function can be expressed in terms of the Jack symmetric functions. It is observed from theorem 3.2.1 that the Jack symmetric functions can be epressed as a linear combination of the monomial symmetric polynomial, m_λ which in turn can be expressed in terms of the power sum symmetric polynomial p_λ. It follows that the Jack functions can be expressed as polynomials in the p_λ, with coefficients that are rational functions in the extra parameter θ.

The fact that we can express the Jack symmetric functions, $J_\lambda(z,\theta)$ as a linear combination of the power sum symmetric functions p_λ allows us to apply the Kerov's map φ on $J_\lambda(z,\theta)$. The action of φ on $J_\lambda(z,\theta)$ produces the the following results.

4.2.2 Definition. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition for a generic variable θ (indeterminate) and $z = (z_1, z_2, \ldots)$ be an infinite sequence of variables. Let $J_\lambda(z,\theta)$ be the Jack symmetric function. Then we have the polynomial

$$SJ_\lambda(x,y,\theta) = \varphi(J_\lambda(z,\theta)) \tag{4.2.3}$$

The polynomial $SJ_\lambda(x,y,\theta)$ is called the super Jack polynomial.

The Kerov's map φ maps the elements of the space of symmetric functions into the space of the supersymmetric polynomials. There is a duality relationship between the Jack symmetric functions and the super Jack polynomials athough there is no such analogous relation for the Jack polynomials.

4.2.1 Remark. The super Jack polynomials form a basis of $\Lambda_{n,m,\theta}$ (Desrosiers et al., 2012).

4.3 Super Jack polynomial and the deformed CMS Operator

In Sergeev and Veselov paper [Sergeev and Veselov (2004)], the algebra $\Lambda_{n,m,\theta}$ generate a certain differential operator called the deformed Calogero-Moser-Sutherland operators denoted by $\mathcal{L}_{n,m,\theta}$. The

operator $\mathcal{L}_{n,m,\theta}$ is an extension of the operator in (3.1.2) to infinite number of variables depending on the parameter θ.

4.3.1 Definition. Let $x = x_1, \ldots, x_n$ and $y = (y_1, \ldots, y_m)$ be infinite set of variables. Let $\mathcal{L}_{n,m,\theta}$ be a map

$$\mathcal{L}_{n,m,\theta} \colon \Lambda_{n,m,\theta} \longrightarrow \Lambda_{n,m,\theta} \tag{4.3.1}$$

such that

$$\begin{aligned}\mathcal{L}_{n,m,\theta} =& \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} \right)^2 - \theta \sum_{j=1}^{n} \left(y_j \frac{\partial}{\partial y_j} \right)^2 + \theta \sum_{1 \le i < j \le n} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right)^2 \\ & - \sum_{1 \le i < j \le m} \frac{y_i + y_j}{y_i - y_j} \left(y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right)^2 - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{x_i + y_j}{x_i - y_j} \left(x_i \frac{\partial}{\partial x_i} + \theta y_j \frac{\partial}{\partial y_j} \right)^2 \\ & - (\theta(n-1) - m) \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} y_j \frac{\partial}{\partial y_j} \right)\end{aligned} \tag{4.3.2}$$

The super Jack symmetric polynomials are eigenfunctions of the $\mathcal{L}_{n,m,\theta}$. Sergeev and Veselov showed that the super Jack symmetric polynomials are the eigenfunctions of any differential operator. In particular, the $SP_\lambda(z, \theta)$ are the eigenfunctions of (4.3.2) in the algebra $\Lambda_{n,m.\theta}$.

For any homogeneous function $f \in \Lambda_{n,m,\theta}$, $\mathcal{L}^f_{n,m,\theta} \in \Lambda_{n,m,\theta}$ and we have

$$\mathcal{L}^f_{n,m,\theta} SP_\lambda(z, \theta) = f(\lambda) SP_\lambda(z, \theta)$$

Where $f(\lambda)$ are the eigenvalues of the differential operator $\mathcal{L}_{n,m,\theta}$

4.3.1 Remark. The homomorphism φ is uniquely determined by $p_r(z)$, the free generators of Λ.

5. Conclusions

In this work, we have reviewed the ring of symmetric polynomials and its extension to the ring of symmetric functions. We have shown that the symmetric functions have interesting connection with mathematical physics. That is, the Jack polynomials are the eigenfunctions of the CMS operators. The supersymmetric version of the Jack polynomials, the super Jack polynomials, turned out to be the eigenfunctions of the deformed CMS operators. We showed that the CMS operator preserved the ring of symmetric functions. Finally, we have shown that the Kerov's map translates the ring of symmetric functions to the ring of super symmetric polynomials.

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Gloire à Dieu

Dedication

I dedicate this project to my mother, Helena Kumah Boadi and father Joseph Boadi and all my family

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